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A.A.Arkhipov, V.I.Savrin

LSZ ASYMPTOTIC CONDITION AND DYNAMIC EQUATIONS IN QUANTUM FIELD THEORY

1. BETHE-SALPETER EQUATION

Serpukhov 1983

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Abstract

Arkhipov A.A., Savrin V.I.

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In our work we consider some techniques that may be appropriate for the derivation of dynamic equations in quantum field theory. A new method of deriving equations based on the use of LSZ asymptotic condition is described and it is proved that with the help of this method it becomes possible to obtain equations for wave functions both of scattering and bound states. Our work is described in several papers under the same title. The first paper is devoted to the Bethe-Salpeter equation. In the second paper we examine dynamic equations for a three-particle system. In the third paper we formulate the asymptotic condition for elementary, as well as for composite particles. It also contains a rigorous derivation of formulae for amplitudes of physical processes in a three-particle system. In our fourth paper we show, how the LSZ asymptotic condition leads to the universal technique of deriving dynamic equations in quantum field theory. Besides, here iterative schemes that can be applied for calculating basic physical quantities, are discussed.

Аннотация

Архипов А.А., Саврин В.И.

Асимптотическое условие LSZ и динамические уравнения в квантовой теории поля. 1. Уравнение Бете-Солпитера. Серпухов, 1983.

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В работе рассматривается ряд методов, которые можно использовать для вывода динамических уравнений в квантовой теории поля. Описан новый метод вывода уравнений, основанный на использовании асимптотического условия LSZ, и показано, что с помощью этого метода можно получать уравнения как для волновых функций состояний рассеяния, так и для волновых функций связанных состояний. Содержание работы изложено в нескольких статьях под одним общим названием. Первая статья посвящена уравнению Бете-Солпитэра. Во второй статье будут рассмотрены динамические уравнения для системы трех частиц. Формулировке асимптотического условия как для элементарных, так и для составных частиц будет посвящена третья статья, где также будет изложен строгий вывод формул для амплитуд физических процессов в системе трех частиц. В четвертой статье будет показано, каким образом асимптотическое условие LSZ приводит к универсальному методу вывода динамических уравнений в квантовой теории поля. Там же будут рассмотрены итерационные схемы, которые можно применять для вычисления основных физических величин.

INTRODUCTION

This paper is devoted to the description of some techniques that can be used to derive dynamical equations in quantum field theory. Dynamical equations are quite important in the theory. In the early fifties a four-dimensional purely relativistic equation for the bound state wave function of two Dirac particles with an arbitrary interaction was derived by Salpeter and Bethe with the help of Feynman diagrams/1/. Soon after Gell-mann and Low proposed a formal derivation of the Bethe-Salpeter equation in quantum field theory/2/. In the later papers some other techniques of deriving the Bethe-Salpeter equations were proposed*). In this paper we consider some of them. All the methods of deriving the Bethe-Salpeter equations familiar nowadays could mainly be divided into two groups. The first group contains methods that are applied while deriving the Bethe-Salpeter equation for the scattering state wave function. In this case they usually proceed from the wave function defined in terms of matrix element of field operators in the interaction representation. The second group includes methods that are used in the derivation of the Bethe-Salpeter equation for the bound state wave function. These methods proceed from the expression for the bound state wave function in terms of Heisenberg field operators and are based on the investigation of the singularities of four-point Green functions in the invariant mass of a two-particle system.

In this paper we describe a new universal method applicable for the derivation of the Bethe-Salpeter equation for the wave function both of scattering and bound states. Furthermore, this method proves to be quite convenient for the derivation of expressions for scattering amplitudes of elementary particles in composite systems.

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*) Review/3/ is devoted to the investigation of the Bethe-Salpeter equation. It contains a numerous bibliography on the subject.

In the derivation of dynamical equations we do not use some concrete model of a field theory but just exploit the facts that serve as a basis for axiomatic formulations of quantum field theory. This means that from the very beginning we refuse the discussions connected with the divergencies in the theory and the methods of their elimination, i.e. renormalization procedures. Besides, we additionally assume that there exist vacuum expectations of Heisenberg field operators (or Green functions) and matrix elements defining the Bethe-Salpeter wave functions, without justifying rigorously this assumption. Our technique becomes universal due to the application of LSZ asymptotic condition/4/, whose validity is also assumed.

Our work is described in several papers under the same title. In the first paper we consider some methods that are appropriate for the derivation of the Bethe-Salpeter dynamical equation.

2. BETHE-SALPETER EQUATION FOR WAVE FUNCTION OF TWO-PARTICLE SYSTEM IN QFT

The Bethe-Salpeter two-particle wave function for scattering states can be defined through the matrix element

$$\Phi_{ab}(x_1 x_2) = \langle 0 | T(\Phi_a(x_1)\,\Phi_b(x_2)) | \Phi_{ab}; in \rangle, \qquad (1)$$

where $\Phi_a(x_1)$ and $\Phi_b(x_2)$ are Heisenberg field operators of particles a and b, $|\Phi_{ab}; in\rangle$ is a state vector corresponding to the asymptotic (when $t \to -\infty$) configuration of two free particles in in-basis. Vector $|\Phi_{ab}; in\rangle$ can be represented as a result of the action of creation operators on the vacuum state vector

$$|\Phi_{ab}; in\rangle = a^+_{in} b^+_{in} |0\rangle .$$

In the axiomatic formulation of quantum field theories/4/ asymptotic in-and out-fields are defined in terms of Heisenberg fields with the help of Yang-Feldman equations

$$\Phi_a(x) = \phi^{in}_a(x) + \int dy\, D^{ret}_a(x-y) j_a(y) =$$
$$= \phi^{out}_a(x) + \int dy\, D^{adv}_a(x-y) j_a(y),$$

where the current operator $j_a(x)$ is defined in terms of Heisenberg field by the equation $j_a(x) = \hat{K}^a_x \Phi_a(x)$; $\hat{K}^a_x$ is a differential operator (the Klein-Gordon operator in the case of scalar particles, the Dirac operator for spinor particles, etc.) satisfying the equation

$$\hat{K}^a_x D^{ret}_a(x) = \hat{K}^a_x D^{adv}_a(x) = \delta^4(x), \qquad \text{with } D^{ret}_a(x) = 0 \text{ at } x^0 < 0,\ D^{adv}_a(x) = 0 \text{ at } x^0 > 0.$$

It is clear from the Yang-Feldman equations that asymptotic in- and out-fields satisfy the equation

$$\hat{K}^a_x \phi^{in}_a(x) = 0, \quad \hat{K}^a_x \phi^{out}_a(x) = 0.$$

Creation and annihilation operators are defined in terms of asymptotic in- and out-fields and smooth normalized solutions of the equation $\hat{K}^a_x f_a(x) = 0$ /4/. Below we give explicit formulae for these operators.

We can elegantly derive a dynamical equation for wave function(1) if apply the Bogolubov formulation/5,6/ of axiomatic quantum field theory. In this case the Heisenberg field can be expressed in terms of S-operator and asymptotic fields as/6/:

$$\Phi_a(x) = T(\phi_a^{ex}(x) S) S^+ . \qquad (2)$$

where T-product should be treated as follows*): S-operator is assumed to be a functional of asymptotic fields that is representable as an expansion in normal products of asymptotic fields, and then T-product is defined by the Wick theorem about the expansion of T-product in normal products. Depending on the representation of S-operator the index **ex** may take **in** or **out**. Henceforth we use the out-representation, so, in (2) the asymptotic field index **out** is omitted. It can be readily proved that (2) entails the Yang-Feldman equations if we identify the current operator $j_a(x)$ and the radiation operator of the first order/6/

$$j_a(x) = i \frac{\delta S}{\delta \phi_a(x)} S^+ .$$

Applying (2), we obtain

$$T(\Phi_a(x_1)\Phi_b(x_2)\dots) = T(\phi_a(x_1)\phi_b(x_2)\dots S) S^+ . \qquad (3)$$

For T-product of asymptotic fields we have a standard representation

$$T(\phi_a(x)\bar{\phi}_a(y)) = :\phi_a(x)\bar{\phi}_a(y): + \frac{1}{i} D_a(x-y), \qquad (4)$$

with $D_a(x)$ being a Green causality function satisfying the equation $\hat{K}^a_x D_a(x) = \delta^4(x)$.

With account of (3) we can rewrite expression (1) for the Bethe-Salpeter wave function as:

$$\Phi_{ab}(x_1 x_2) = \langle 0 | T(\phi_a(x_1)\phi_b(x_2) S) | \Phi_{ab}; out \rangle , \qquad (5)$$

as far as $|\Phi_{ab}; in\rangle = S|\Phi_{ab}; out\rangle$. If now use the Bogolubov reduction formula

$$[F(\phi), a^+] = \int dx \frac{\delta F(x)}{\delta \phi(x)} f_a(x),$$

*) Here we do not go into niceties of the distinction between the above-defined T-product and the conventional Dyson's definition of T-product. For the discussion of these niceties refer to monograph/6/. Our terminology also comes from there.

where $F(\phi)$ is some functional of asymptotic fields, $f_a(x)$ is a smooth normalized solution (of the wave packet type) of the equation $\hat{K}^a_x f_a(x) = 0$, then from (5) we can obtain for the wave function

$$\Phi_{ab}(x_1 x_2) = \int dy_1 dy_2 < 0 \left| \frac{\delta T(\phi_a(x_1)\phi_b(x_2)S)}{\delta\phi_a(y_1)\delta\phi_b(y_2)} \right| 0 > \Phi^{(o)}_{ab}(y_1 y_2), \qquad (6)$$

with $\Phi^{(o)}_{ab}(x_1 x_2) = f_a(x_1) f_b(x_2)$ being the initial state wave function of two non-interacting particles.

The two-particle (four-point)Green function defined as

$$G_{ab}(x_1 x_2; y_1 y_2) = i^2 < 0 \mid T(\Phi_a(x_1)\Phi_b(x_2)\bar{\Phi}_a(y_1)\bar{\Phi}_b(y_2)) \mid 0 > =$$

$$= i^2 < 0 \mid T(\phi_a(x_1)\phi_b(x_2)\bar{\phi}_a(y_1)\bar{\phi}_b(y_2)S)S^+ \mid 0 >,$$

after partial transformation of the time-ordered product can be represented in the form:

$$G_{ab}(x_1 x_2; y_1 y_2) = \int dz_1 dz_2 < 0 \left| \frac{\delta T(\phi_a(x_1)\phi_b(x_2)S)}{\delta\phi_a(z_1)\delta\phi_b(z_2)} S^+ \right| 0 > \times$$

$$\times D_a(z_1 - y_1) D_b(z_2 - y_2).$$

With the equation obtained it is easy to see that linear equation (6) is equivalent to

$$\Phi_{ab}(x_1 x_2) = [(G_{ab} * D_a^{-1} D_b^{-1}) * \Phi^{(o)}_{ab}](x_1 x_2). \qquad (7)$$

Operation "*" means the convolution of functions in configuration space.

Completely transforming the time-ordered product in the four-point Green function by the generalized Wick theorem/5/, we obtain

$$G_{ab} = G^{(o)}_{ab} + G^{(o)}_{ab} * R_{ab} * G^{(o)}_{ab}, \qquad (8)$$

where $G^{(o)}_{ab} = D_a D_b$ is a free Green function of two particles, function R_{ab} has the structure

$$R_{ab} = R^{(2)}_a D_b^{-1} + R^{(2)}_b D_a^{-1} + R^{(4)}_{ab}, \qquad (9)$$

functions

$$R^{(2)}_i(x; y) = \frac{1}{i} < 0 \left| \frac{\delta^2 S}{\delta\bar{\phi}_i(x)\delta\phi_i(y)} S^+ \right| 0 > \qquad (10)$$

$i = a, b$

are vacuum expectations of radiation operators of the second order, and function

$$R^{(4)}_{ab}(x_1x_2;y_1y_2) = \frac{1}{i^2}\left< 0 \left| \frac{\delta^4 S}{\delta\bar{\phi}_a(x_1)\delta\bar{\phi}_b(x_2)\delta\phi_a(y_1)\delta\phi_b(y_2)} S^+ \right| 0 \right> \qquad (11)$$

is vacuum expectation of the radiation operator of the forth order.

After substituting (8) for a two-particle Green function into linear relation (7) we find

$$\Phi_{ab}(x_1x_2) = \Phi^{(o)}_{ab}(x_1x_2) + (G^{(o)}_{ab} * R_{ab} * \Phi^{(o)}_{ab})(x_1x_2) = \\ = \Phi^{(o)}_{ab}(x_1x_2) + (G^{(o)}_{ab} * R^{(4)}_{ab} * \Phi^{(o)}_{ab})(x_1x_2). \qquad (12)$$

The second equation of relation (12) is the consequence of the stability of one-particle states. Note, that this stability entails

$$D_i * R^{(2)}_i * f_i = 0, \quad i = a, b. \qquad (13)$$

Hence, linear relation (7) can be written as

$$\Phi_{ab} = (\bar{G}_{ab} * D_a^{-1} D_b^{-1}) * \Phi^{(o)}_{ab}, \qquad (14)$$

where

$$\bar{G}_{ab} = G^{(o)}_{ab} + G^{(o)}_{ab} * R^{(4)}_{ab} * G^{(o)}_{ab} = \\ = G_{ab} - G^{(o)}_{ab} * (R^{(2)}_a D_b^{-1} + R^{(2}_b D_a^{-1}) * G^{(o)}_{ab}. \qquad (15)$$

Define function V_{ab} by relation

$$R^{(4)}_{ab} = V_{ab} + V_{ab} * G^{(o)}_{ab} * R^{(4)}_{ab}. \qquad (16)$$

Substituting relation (16) into (12), we come to the dynamical equation for the Bethe-Salpeter wave function:

$$\Phi_{ab}(x_1x_2) = \Phi^{(o)}_{ab}(x_1x_2) + (G^{(o)}_{ab} * V_{ab} * \Phi_{ab})(x_1x_2). \qquad (17)$$

The inhomogeneous term in (17) is a wave function of the system of two free non-interacting particles. It corresponds to the boundary condition of the scattering problem at $x_1^o \to -\infty$, $x_2^o \to -\infty$. The function $R^{(4)}_{ab}$ can easily be proved to be directly related to the elastic scattering amplitude of two particles. In fact, using the Bogolubov reduction formulae for the matrix element of an S-operator corresponding to the elastic scattering process of two particles, we obtain

$$<\Phi_{ab}; out | S - 1 | \Phi_{ab}; out> = i^2 \int dx_1 dx_2 dy_1 dy_2 \times \\ \times \overset{*}{f}_a(x_1) \overset{*}{f}_b(x_2) R^{(4)}_{ab}(x_1x_2;y_1y_2) f_a(y_1) f_b(y_2).$$

3. BETHE-SALPETER WAVE FUNCTION FOR BOUND STATES

The Bethe-Salpeter wave function for the bound state of two particles is defined through the matrix element

$$\Phi_{ab}^{A}(x_1 x_2) = \langle 0 | T(\Phi_a(x_1)\Phi_b(x_2)) | \Phi_{ab}^{A} \rangle , \tag{18}$$

where $\Phi_a(x_1)$, $\Phi_b(x_2)$ are again the Heisenberg field operators of particles a and b, $|\Phi_{ab}^{A}\rangle$ is the bound state vector of these particles. For the bound state vector we also use the notation $|M_A;\vec{P},\sigma\rangle$, where M_A is the bound state mass, $\vec{P}$ the full momentum of the bound system, σ is a set of continuous and discrete quantum numbers, so that together with the mass and the momentum these numbers completely characterize the bound state. The method used in the derivation of the Bethe-Salpeter dynamical equation (17) is inapplicable in the case of function (18), though, by intuition, the wave function of the bound state must satisfy the homogeneous equation

$$\Phi_{ab}^{A}(x_1 x_2) = (G_{ab}^{(o)} * V_{ab} * \Phi_{ab}^{A})(x_1 x_2). \tag{19}$$

This result can be understood in the following way. Introduce into (17) variables $X = (x_1 + x_2)/2$, $x = x_1 - x_2$ and proceed to the Fourier-image of the wave function over variable X:

$$\Phi_{ab}(x | P) = \int dX \exp(iPX)\Phi_{ab}(Xx).$$

Then, taking into account the translation invariance of the theory, we can rewrite (17) in the form

$$\Phi_{ab}(x|P) = \Phi_{ab}^{(o)}(x|P) + \int dy\, dz\, G_{ab}^{(o)}(P|x;y)V_{ab}(P|y;z)\Phi_{ab}(z|P), \tag{20}$$

where

$$G_{ab}^{(o)}(Xx;Yy) = G_{ab}^{(o)}(X-Y|x;y) = (2\pi)^{-4}\int dP \exp[-iP(X-Y)]G_{ab}^{(o)}(P|x;y); \tag{21}$$

$$V_{ab}(Xx;Yy) = V_{ab}(X-Y|x;y) = (2\pi)^{-4}\int dP \exp[-iP(X-Y)]V_{ab}(P|x;y). \tag{22}$$

If $\sqrt{P^2} = M_A < m_a + m_b$, then the inhomogeneous term of (20) will, apparently, vanish and thus we obtain the homogeneous equation for the wave function of the bound state

$$\Phi_{ab}^{A}(x|\vec{P}) = \int dy\, dz\, G_{ab}^{(o)}(P_A|x;y)V_{ab}(P_A|y;z)\Phi_{ab}^{A}(z|\vec{P})$$

$$P_A^{o} = E_A = E(\vec{P}, M_A) = \sqrt{\vec{P}^2 + M_A^2},\quad P_A^2 = M_A^2. \tag{23}$$

On the other hand, it is clear that the transition to the homogeneous equation is also caused by imposing correct boundary conditions, proceeding from the physical formulation of the problem of describing a bound system. The rigorous derivation of (23) is based on the investigation of the singularities of a two-particle Green function over the invariant mass of a two-particle system. Here we shortly describe this method.

Define the Fourier-image of a two-particle Green function:

$$(2\pi)^4 \delta^4(P-Q)\bar{G}_{ab}(P \mid x; y) = \int dX\, dY \exp(iPX - iQY)\bar{G}_{ab}(Xx; Yy). \qquad (24)$$

The formula representing the inverse transformation is

$$\bar{G}_{ab}(Xx; Yy) = \bar{G}_{ab}(X-Y \mid x; y) =$$
$$= (2\pi)^{-4} \int dP \exp[-iP(X-Y)]\bar{G}_{ab}(P \mid x; y). \qquad (25)$$

From the expression for a two-particle Green function in terms of the vacuum expectation of the Heisenberg field operators it can be easily seen that the quantity $\bar{G}_{ab}(P|x; y)$ contains a pole singularity at the energy of the bound state. More precisely, for this quantity we can write down the following representation

$$\bar{G}_{ab}(P \mid x; y) = \frac{1}{i} \frac{\sum_\sigma \Phi^A_{ab}(x|\vec{P},\sigma)\bar{\Phi}^A_{ab}(y|\vec{P},\sigma)}{2E(\vec{P}\ M_A)[P^0 - E(\vec{P}, M_A) + i0]} + \bar{G}^{Reg}_{ab}(P \mid x; y), \qquad (26)$$

where $\bar{G}^{Reg}_{ab}(P|x; y)$ does not contain the pole singularity any more at $P^0 = E(\vec{P}, M_A) = \sqrt{\vec{P}^2 + M_A^2}$. From representation (26) near the pole corresponding to the bound state, we obtain

$$\bar{G}_{ab}(P \mid x; y) \cong [i(P^2 - M_A^2 + i0)]^{-1} \sum_\sigma \Phi^A_{ab}(x|\vec{P},\sigma)\bar{\Phi}^A_{ab}(y|\vec{P},\sigma). \qquad (27)$$

Wave functions $\Phi^A_{ab}(x|\vec{P},\sigma)$ are related to the matrix element (18) as follows

$$\langle 0 \mid T(\Phi_a(x_1)\Phi_b(x_2)) \mid M_A; \vec{P}, \sigma\rangle = (2\pi)^{-3/2} \exp(-iPX)\Phi^A_{ab}(x \mid \vec{P}, \sigma),$$

$$\langle M_A; \vec{P}, \sigma \mid T(\bar{\Phi}_a(x_1)\bar{\Phi}_b(x_2)) \mid 0\rangle = (2\pi)^{-3/2} \exp(iPX)\bar{\Phi}^A_{ab}(x \mid \vec{P}, \sigma).$$

Now note, that substitution of (16) into (15) leads to the equation for a two-particle Green function

$$\bar{G}_{ab} = G^{(0)}_{ab} + G^{(0)}_{ab} * V_{ab} * \bar{G}_{ab}. \qquad (28)$$

Carrying out the Fourier transformation in this equation and using (21), (22) and (25), we get:

$$\bar{G}_{ab}(P \mid x; y) = G^{(0)}_{ab}(P \mid x; y) +$$
$$+ \int dx'dy' G^{(0)}_{ab}(P|x; x')V_{ab}(P|x'; y')\bar{G}_{ab}(P|y'; y). \qquad (29)$$

Substitute representation (26) for a two-particle Green function into eq. (29), then multiply both sides of the obtained equation by $(P^2 - M_A^2)$ and proceed to the limit $P^2 \to M_A^2$. As a result we find

$$\Phi_{ab}^{A}(x \mid \vec{P}, \sigma) = \int dx' dy' G_{ab}^{(o)}(P_A \mid x; x') V_{ab}(P_A \mid x'; y') \, \Phi_{ab}^{A}(y' \mid \vec{P}, \sigma), \qquad (30)$$

$$P_A^2 = M_A^2 .$$

In the derivation of eq.(30) it is necessary to take into account the linear independence of wave functions $\Phi_{ab}^{A}(x \mid \vec{P}, \sigma)$, corresponding to different quantum numbers σ.

Eq. (30) is the Bethe-Salpeter equation of interest that is valid for the wave function of the bound state. The rigorous derivation of (30) given can serve as a justification of heuristic arguments concerning the omission of the inhomogeneous term in (17) when proceeding to the description of bound states of a two-particle system. The train of thought can also be reversed. e.g. after equation (30) is obtained, we can state that in order to describe scattering states, it is necessary to proceed from (30) to an equation with the inhomogeneous term that would correctly define the boundary conditions at $x_1^o, x_2^o \to -\infty$ In this case the first method applied in our derivation of (17) could be an argument for the correctness of such a transition.

In our next papers we shall show that both these methods are applicable in the investigation of a three-particle system in quantum field theory.

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А.А.Архипов, В.И.Саврин

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Институт физики высоких энергий, 144284, Серпухов, Московской обл.