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THERMO FIELD THEORY VERSUS IMAGINARY TIME FORMALISM *

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ABSTRACT

We calculate a two-loop diagram at finite temperature to compare Thermo Field Theory (=Th.F.Th.) with the conventional imaginary time formalism (=Im.T.F.). The summation over the Matsubara frequency in Im.T.F. is carried out at two-loop level, and the result is shown to coincide with that of Th.F.Th.. We confirm that in Im.T.F. the temperature dependent divergences cancell out at least in the calculation of effective potential of φ^4 theory, as in Th.F.Th..

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§1. Introduction

There exist two kinds of acceptable finite temperature quantum field theories which allow the use of conventional Feynmann rules. One is called imaginary time formalim (=Im.T.F.) which was originally established by Matsbara,[1] elaborated by Abrikosov et al.[2] and later extended to the relativistic case by A.D. Linde and others.[3] The other is a real time formalism, what we call Thermo-Field Theory (=Th.F.Th.) which was introduced by Y. Takahashi and H. Umezawa[4] and developed by H. Matsumoto and others.[5-13] The difference of these two formalisms lies in the manner the analytic continuation in time is perfomed. In the former it is directly along the imaginary time axis. In the latter, it is first done along the real time axis back and forth, and in the end one reaches the same point as in the former. Due to this difference, "time" is lost in Im.T.F., while it is present in Th.F.Th.. (For the details of analytic continuation and other possible types of real time formalisms, readers are referred to Refs. 9) and 14).) In the practical calculations the difference is striking. For instance, to calculate a scalar one-loop tadpole diagram we evaluate,

1) Im.T.F.

$$\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\ \frac{i}{(\frac{2\pi n}{\beta})^2+\vec{k}^2+m^2}\ , \tag{1}$$

2) Th.F.Th.

$$\int\frac{d^4k}{(2\pi)^4}\left[\frac{i}{k^2-m^2+i\varepsilon}+2\pi\,\delta(k^2-m^2)\,\frac{1}{e^{\beta|k_0|}-1}\right] \tag{2}$$

[illegible] zero and finite temperature part and also the statistical factor are all contained in the summation over integer, while they are clearly separated in Th.F.Th.. Further, integral is a conventional one in Th.F.Th., but in Im.T.F. we must perform the summation which appears difficult in higher orders. We would think it is due to this difficulty that $T \neq 0$ two-loop calculation has not been done in Im.T.F.. In fact, it was by the use of Th.F.Th. that the recent $T \neq 0$ two-loop integrals were carried out.[8,12,13] In Refs. 8) and 13) it is explicitly shown that the temperature dependent divergences cancell with those coming from the counter terms prepared at zero temperature, and thus the perturbation scheme of Th.F.Th. is satisfactory. Also as shown in Ref.13), the penperature divergent term can be treated by the dimensional regularization due to the fact that Th.F.Th. is formulated in a covariant manner. To see the same cancellation of divergences and reassure ourselves that Im.T.F. is as good, we must perform the two-loop calculation. We should like to carry it out in the present paper.

§2. Two-Loop Calculation

As an example we calculate a typical scalar two-loop diagram (Fig.1) in the Im.T.F. and compare it with that of Th.F.Th..

1) Th.F.Th.[8),13)]

$$I=\frac{(-if)^2}{6}\int\frac{d^4k}{(2\pi)^4}\frac{d^4\ell}{(2\pi)^4}\left\{\frac{i}{k^2-m_a^2+i\varepsilon}\ \frac{i}{(k-\ell)^2-m_b^2+i\varepsilon}\ \frac{i}{\ell^2-m_c^2+i\varepsilon}\right. \tag{3-1}$$

+ (next page)

$$+ \frac{i}{k^2 - m_a^2 + i\varepsilon} \frac{i}{(k-\ell)^2 - m_b^2 + i\varepsilon} 2\pi\delta(\ell^2 - m_c^2) \frac{1}{e^{\beta|\ell_0|} - 1}$$

$$+ \frac{i}{k^2 - m_a^2 + i\varepsilon} \frac{i}{\ell^2 - m_c^2 + i\varepsilon} 2\pi\delta((k-\ell)^2 - m_b^2) \frac{1}{e^{\beta|k_0-\ell_0|} - 1}$$

$$+ \frac{i}{\ell^2 - m_c^2 + i\varepsilon} \frac{i}{(k-\ell)^2 - m_b^2 + i\varepsilon} 2\pi\delta(k^2 - m_a^2) \frac{1}{e^{\beta|k_0|} - 1} \qquad (3\text{-}2)$$

$$+ \frac{i}{\ell^2 - m_c^2 + i\varepsilon} 2\pi\delta((k-\ell)^2 - m_b^2)\, 2\pi\delta(k^2 - m_a^2) \frac{1}{e^{\beta|k_0-\ell_0|} - 1} \frac{1}{e^{\beta|k_0|} - 1}$$

$$+ \frac{i}{(k-\ell)^2 - m_b^2 + i\varepsilon} 2\pi\delta(k^2 - m_a^2)\, 2\pi\delta(\ell^2 - m_c^2) \frac{1}{e^{\beta|\ell_0|} - 1} \frac{1}{e^{\beta|k_0|} - 1}$$

$$+ \frac{i}{k^2 - m_a^2 + i\varepsilon} 2\pi\delta((k-\ell)^2 - m_b^2)\, 2\pi\delta(\ell^2 - m_c^2) \frac{1}{e^{\beta|k_0-\ell_0|} - 1} \frac{1}{e^{\beta|\ell_0|} - 1} \Bigg] \qquad (3\text{-}3)$$

(The term with three statistical factors which is pure imaginary relative to other terms has been discarded.*))

2) Im.T.F.

$$I = \frac{(-if)^2}{6} \frac{i^2}{\beta^2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{d^3\ell}{(2\pi)^3} \frac{i}{x^2 + a^2} \frac{i}{(x-y)^2 + b^2} \frac{i}{z^2 + c^2}, \qquad (4)$$

where

$$x = \frac{2\pi n}{\beta}, \quad y = \frac{2\pi m}{\beta}, \quad a^2 = \vec{k}^2 + m_a^2, \quad b^2 = (\vec{k} - \vec{\ell})^2 + m_b^2, \quad c^2 = \vec{\ell}^2 + m_c^2. \qquad (5)$$

*) There is an argument (Y. Nakano and H. Umezawa, private communication) based on the analysis in Ref.9 that the imaginary part should be absent when summed up to all orders.

After various attempts we find the following procedure for the comparison is the simplest one. First we note a formula [15]

$$\frac{1}{\beta} \sum_{n=-\infty}^{\infty} f(\omega_n^{\pm}) = \frac{1}{4\pi} \int_C dz \frac{f(z)(e^{i\beta z/2} \pm e^{-i\beta z/2})}{e^{i\beta z/2} \mp e^{-i\beta z/2}}$$

$$= \frac{1}{4\pi} \left[\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty} dz + \int_{i\varepsilon+\infty}^{i\varepsilon-\infty} dz \right] \frac{f(z)(e^{i\beta z/2} \pm e^{-i\beta z/2})}{e^{i\beta z/2} \mp e^{-i\beta z/2}}, \qquad (6)$$

where

$$\omega_n^{+} = \frac{2\pi n}{\beta} \ \text{(for Boson)}, \qquad \omega_n^{-} = \frac{(2n+1)\pi}{\beta} \ \text{(for Fermion)}.$$

We make use of it for y, (Note $x = \frac{2\pi n}{\beta}$ is an integer while performing the y-integration.) and get *)

$$I = \frac{-if^2}{6} \sum_{n=-\infty}^{\infty} \frac{1}{\beta} \int_C \frac{dy}{4\pi} \frac{1}{x^2 + a^2} \frac{1}{(x-y)^2 + b^2} \frac{1}{y^2 + c^2} \frac{e^{i\beta y/2} + e^{-i\beta y/2}}{e^{i\beta y/2} - e^{-i\beta y/2}} \qquad (7)$$

(The contour C encircles the poles on the real axis.) Hence

$$I = \sum_{n=-\infty}^{\infty} \frac{1}{2\beta} \left[\frac{1}{2c} \frac{1}{x^2 + a^2} \left\{ \frac{1}{(x-ic)^2 + b^2} + \frac{1}{(x+ic)^2 + b^2} \right\} f(c) \right.$$

$$\left. + \frac{1}{2b} \frac{1}{x^2 + a^2} \left\{ \frac{1}{(x-ib)^2 + c^2} + \frac{1}{(x+ib)^2 + c^2} \right\} f(b) \right], \qquad (8)$$

where we have picked up the poles along the contours C_1 and C_2 (Fig.2) and

$$f(c) = (e^{+\beta c/2} + e^{-\beta c/2}) \,/\, (e^{-\beta c/2} - e^{+\beta c/2}) = -\coth(\beta c/2). \qquad (9)$$

*) The momentum integration $\frac{d^3k}{(2\pi)^3} \frac{d^3\ell}{(2\pi)^3}$ is understood in the expressions (7), (8), (10), (11), (13) - (15).

After rearrangement of terms we have

$$I = \frac{-if^2}{6}\,\frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\Bigg[\frac{f(c)}{2c}\,\frac{b-c}{b}\left\{\frac{1}{x^2+(b-c)^2}-\frac{1}{x^2+a^2}\right\}\frac{1}{a^2-(b-c)^2}$$

$$+\frac{f(c)}{2c}\,\frac{b+c}{b}\left\{\frac{1}{x^2+(b+c)^2}-\frac{1}{x^2+a^2}\right\}\frac{1}{a^2-(b+c)^2}$$

$$+\frac{f(b)}{2b}\,\frac{c-b}{c}\left\{\frac{1}{x^2+(c-b)^2}-\frac{1}{x^2+a^2}\right\}\frac{1}{a^2-(c-b)^2}$$

$$+\frac{f(b)}{2b}\,\frac{c+b}{c}\left\{\frac{1}{x^2+(c+b)^2}-\frac{1}{x^2+a^2}\right\}\frac{1}{a^2-(c+b)^2}\Bigg]. \qquad (10)$$

We next use (6) again and integrate over x, and then perform another rearrangement of terms to end up with

$$I = \frac{-if^2}{6}\,\frac{1}{4}\Bigg[\frac{1}{ac}\,\frac{b^2-(a^2+c^2)}{\{b^2-(a+c)^2\}\{b^2-(a-c)^2\}}\,f(c)f(a)$$

$$+\frac{1}{ab}\,\frac{c^2-(a^2+b^2)}{\{c^2-(a+b)^2\}\{c^2-(a-b)^2\}}\,f(a)f(b)$$

$$+\frac{1}{bc}\,\frac{a^2-(b^2+c^2)}{\{a^2-(b+c)^2\}\{a^2-(b-c)^2\}}\,f(b)f(c)$$

$$+\frac{2}{\{a^2-(b+c)^2\}\{a^2-(b-c)^2\}}\Bigg]. \qquad (11)$$

Noting

$$f(a) = -\coth(\beta a/2) = -\left(1+\frac{1}{e^{\beta a}-1}\right), \qquad (12)$$

we can separate zero- and finite-temperature parts in (11), as follows:

$$I_0\ (\text{= zero temperature part})\ /\ (-if^2/6) = \frac{1}{abc}\,\frac{1}{a+b+c}, \qquad (13)$$

I_1 (= terms with one statistical factor) / $(-if^2/6)$

$$= \frac{-2}{abc}(b+c)\left[a^2-(b-c)^2\right]\frac{1}{A}\,\frac{1}{e^{\beta a}-1}$$

$$+\frac{-2}{abc}(c+a)\left[b^2-(c-a)^2\right]\frac{1}{A}\,\frac{1}{e^{\beta b}-1}$$

$$+\frac{-2}{abc}(a+b)\left[c^2-(a-b)^2\right]\frac{1}{A}\,\frac{1}{e^{\beta c}-1}, \qquad (14)$$

I_2 (= terms with two statistical factors) / $(-if^2/6)$

$$= \frac{4}{ab}\left[c^2-(a^2+b^2)\right]\frac{1}{A}\,\frac{1}{e^{\beta a}-1}\,\frac{1}{e^{\beta b}-1}$$

$$+\frac{4}{bc}\left[a^2-(b^2+c^2)\right]\frac{1}{A}\,\frac{1}{e^{\beta b}-1}\,\frac{1}{e^{\beta c}-1}$$

$$+\frac{4}{ca}\left[b^2-(c^2+a^2)\right]\frac{1}{A}\,\frac{1}{e^{\beta c}-1}\,\frac{1}{e^{\beta a}-1}, \qquad (15)$$

where the quantity A is defined by

$$A = \left[b^2-(a+c)^2\right]\left[b^2-(c-a)^2\right] = \left[c^2-(a+b)^2\right]\left[c^2-(a-b)^2\right]$$

$$= \left[a^2-(b+c)^2\right]\left[a^2-(b-c)^2\right]. \qquad (16)$$

One can easily check that (13) - (15) agree with (3-1) - (3-3) by performing k_0 and ℓ_0 integrations on the latter. This is how we manage to calculate a two-loop diagram in Im.T.F.. It involves some recombination of terms. The use of of formula (6) is not limited to two-loop case, therefore we can use it repeatedly to show the equivalence of formalisms at higher orders.[16] However, the forms such as (13) and (14) are not easy to handle. It is more convenient if (13) and (14) are expressed as in (3-1) and (3-2), since we are then able to perform the ordinary Feynman integrals and also make use of the dimensional regularization.

The result is [13] (for $m_a = m_b = m_c = m$)

$$
\begin{aligned}
I = & -\frac{if^2m^2}{8(4\pi)^4}\left(-\frac{4}{\varepsilon}\ln m^2 + 2\ln^2 m^2 - 6\ln m^2 + 4\gamma\ln m^2\right) \\
& + \frac{if^2m^2}{128\pi^4}\left[\frac{2}{\varepsilon} - \gamma - \ln m^2 + \left(2-\frac{\pi}{\sqrt{3}}\right)\right]\int_1^\infty dx\,\frac{\sqrt{x^2-1}}{e^{\beta m x}-1} \\
& + \frac{-if^2m^2}{128\pi^4}\int_1^\infty dx\int_1^\infty dy\,\frac{1}{e^{\beta m x}-1}\,\frac{1}{e^{\beta m y}-1}\,\ln\left|\frac{X_+}{X_-}\right| ,
\end{aligned}
\tag{17}
$$

where

$$X_\pm = (1 \pm 2\sqrt{x^2-1}\sqrt{y^2-1})^2 - 4x^2y^2$$

and γ is Euler's constant.

As regard to the temperature dependent divergences, let us suppose we are dealing with the effective potential in φ^4 theory. All integrals other than (7) are one-loop integrals, which can easily be calculated by both methods with identical results. Furthermore we have just seen the agreement for the integral of eq.(7). Therefore, the proof that the temperature dependent divergences cancell out with each other in Th.F.Th.[13] applies to Im.T.F. as well.

§3. Summary and concluding remarks

We have calculated a two-loop diagram at finite temperature in Im.T.F., and shown that it does agree with the result in Th.F.Th.. At the same time we have seen a great advantage of Th.F.Th. in actual higher order calculations. Although in this note we studied one special case, we are convinced that the agreement also holds in other cases.[16]

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Fig.1 The two-loop scalar diagram of Eqs.(3) and (4).

Fig.2 The y-integration contour of Eq.(7).

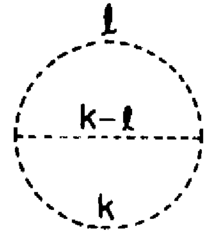

FIG.1

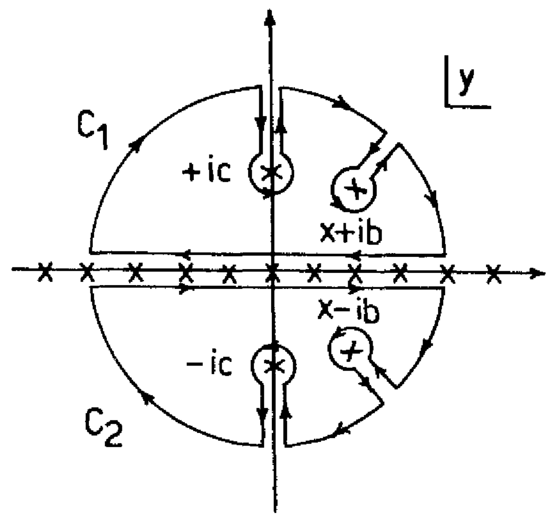

FIG. 2