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ABSTRACT

A practical method is described for establishing action-angle variables for a Hamiltonian system. That is, a given nearly integrable Hamiltonian is divided into an exactly integrable system plus a perturbation in action-angle form. The transformation of variables, which is carried out using a few short trajectory integrations, permits a rapid determination of trajectory properties throughout a phase space volume.

I. Introduction

The action-angle formulation of Hamiltonian mechanics and the associated perturbation theory provide a powerful method for investigating the long-term behavior of trajectories. In particular one can determine if the trajectories are constrained by additional conservation laws than just the energy. Unfortunately, the application of the action-angle formulation has been severely limited by the difficulty of performing the transformation from the usual variables of the problem to action-angle variables. In other words, in many practical applications there is no obvious neighboring, integrable, Hamiltonian problem which can be used to study the actual problem of interest.

Recently, techniques have been developed for studying the structure of magnetic fields using so-called magnetic coordinates [1-4]. This problem is fundamentally identical to a one degree of freedom, time-dependent Hamiltonian. In this paper, a version of this technique is given which is applicable to arbitrary one degree of freedom, time-periodic Hamiltonians. The extension to higher dimensional, time-independent Hamiltonians is also outlined.

The technique developed in this paper permits the evaluation of the Hamiltonian in action-angle variables using relatively short integrations of a few particle trajectories. That is, a given nearly integrable Hamiltonian is divided into an exactly integrable Hamiltonian and a perturbation in standard action-angle form. The transformation equations to the ordinary phase space from the action-angle space are also obtained. Using standard Hamiltonian perturbation theory, one can establish a number of features of the trajectories without further analysis.

The Fourier analysis of particle trajectories, which plays a fundamental role in this technique, has been used in the study of molecules [5] and in the

study of star motions in galaxies [6]. However, the evaluation of the Hamiltonian in action-angle variables by this method has, to the author's knowledge, only been carried out in the study of magnetic fields.

The purpose of this paper is to establish a transformation from ordinary canonical coordinates q,p,t to action-angle coordinates θ,J,t in order to simplify the study of nearly integrable Hamiltonian systems. A formal method for finding a Fourier series representation of $q(\theta,J,t)$ and $p(\theta,J,t)$ on each action surface of an integrable Hamiltonian is given in Sec. II. The practical numerical procedures are briefly discussed in Sec. III. In practice, only a limited number of Fourier terms are retained in the representations of q and p. Consequently, it may be necessary to modify these representations to have a canonical transformation. The method for making the transformation canonical is discussed in Sec. IV and the expression for the Hamiltonian in θ,J,t coordinates is given. The simplest application of this procedure, which is the study of one degree of freedom, time-dependent, nearly integrable, Hamiltonian systems, is given in Sec. V. The extension to higher dimensional systems is discussed in Sec. VI.

II. Action-Angle Evaluation

Let $H(q,p,t)$ be a one degree of freedom, integrable, time-dependent Hamiltonian which is periodic in time with period 2π. We wish to find transformation equations $q(\theta,J,t)$ and $p(\theta,J,t)$ such that J is constant and θ advances uniformly in time along a trajectory. That is

$$\frac{dJ}{dt} = 0 \ , \tag{1}$$

$$\frac{d\theta}{dt} = \upsilon(J) \ . \tag{2}$$

If the Jacobian of the transformation has the value unity,

$$\frac{\partial(q,p)}{\partial(\theta,J)} = 1 \ , \tag{3}$$

then θ, J are the action-angle variables of the problem.

The integrability assumption is just that coordinates θ,J exist and that q and p are periodic functions of θ and t. In other words, q and p can be given as a Fourier sum

$$q(\theta,J,t) = \sum_{n,m} q_{nm}(J) \exp[i(nt-m\theta)] \ ,$$
$$p(\theta,J,t) = \sum_{n,m} p_{nm}(J) \exp[i(nt-m\theta)] \ . \tag{4}$$

Suppose a trajectory is followed from an arbitrary initial point. We can assume without loss of generality that the initial time and initial θ are zero. Along the trajectory J is constant and $\theta = \upsilon t$; so the trajectory q(t), p(t) must be of the form

$$q(t) = \sum q_{nm} \exp[i(n-\upsilon m)t] \ ,$$
$$p(t) = \sum p_{nm} \exp[i(n-\upsilon m)t] \ . \tag{5}$$

The q_{nm}, the p_{nm}, and υ are constant along the trajectory; so a Fourier decomposition of q(t) and p(t) will determine these quantities. More precisely, the Fourier transform of q(t) is defined by

$$q_F(\omega) \equiv \int_{-\infty}^{\infty} q(t) \exp(-i\omega t) \frac{dt}{2\pi} . \tag{6}$$

Letting $\delta(x)$ be the Dirac delta function, Eqs. (5) and (6) imply

$$q_F(\omega) = \sum q_{nm} \delta(\omega - \omega_{nm}) \tag{7}$$

with

$$\omega_{nm} \equiv n - \upsilon m \tag{8}$$

the Fourier spectra of $q(t)$ and $p(t)$ consist of lines. The amplitudes of the lines determine the q_{nm} and the p_{nm}. The locations of the lines determine υ and the integers n and m associated with each line.

Once the q_{nm} and p_{nm} associated with a trajectory have been evaluated, the action J of the trajectory can be easily determined from

$$J = \frac{1}{2\pi} \int_0^{2\pi} p \frac{\partial q}{\partial \theta} d\theta \tag{9}$$

with t set equal to a constant in the expressions, for p and q, Eq. (4). The form for J follows from the unit Jacobian condition, Eq. (3). Explicitly,

$$\frac{\partial(q,p)}{\partial(\theta,J)} = \frac{\partial q}{\partial \theta} \frac{\partial p}{\partial J} - \frac{\partial q}{\partial J} \frac{\partial p}{\partial \theta} . \tag{10}$$

Integrating this equation with respect to J and θ and using the unit Jacobian condition

$$J = \frac{1}{2\pi} \int dJ \int_0^{2\pi} d\theta \left(\frac{\partial q}{\partial \theta} \frac{\partial p}{\partial J} - \frac{\partial q}{\partial J} \frac{\partial p}{\partial \theta}\right) . \tag{11}$$

Equation (9) can be obtained by integrating the second term by parts in both J and θ.

By following trajectories, which have an initial point q_o, p_o either interior or exterior to the torus defined by $q(\theta,Jt)$, $p(\theta,J,t)$; the J dependence of the q_{nm}, the p_{nm}, and v can be evaluated at as many points as desired. In this way, a formal transformation is established, action surface by action surface, between q,p,t and θ,J,t coordinates.

III. Numerical Evaluation

Let us consider how the results of the last section can be implemented in a practical numerical procedure, which is essentially identical to that used in the magnetic field problem [1,2]. A trajectory is followed from an arbitrary starting point for a time T both forward and backward in time. This means q(t) and p(t) are known functions for $|t| < T$ along that trajectory. To evaluate the Fourier transform let

$$Q(\omega) \equiv \frac{1}{2\pi} \int_{-T}^{T} q(t) \left[\frac{\eta}{T} (2\pi)^{1/2} \exp\left(-\frac{\eta^2}{2}\frac{t^2}{T^2}\right)\right] \exp(-i\omega t)dt \ . \tag{12}$$

The term in square brackets is a window function which reduces the white noise in the Fourier transform due to the step functions at $\pm$ T. The parameter η is adjusted to give optimal accuracy as discussed below. The Fourier transform can be quickly evaluated by using the Fast Fourier Transform technique. If Eq. (5) is inserted into Eq. (12), one finds to an accuracy $\exp(-\eta^2/2)$,

$$Q(\omega) = \sum q_{nm} \exp\left[-\frac{1}{2}\left(\frac{\omega_{nm}-\omega}{\omega_*}\right)^2\right] \tag{13}$$

with ω_{nm} defined by Eq. (8) and $\omega_* = \eta/T$. If η and T are chosen appropriately, $Q(\omega)$ consists of distinct Gaussian spikes with the amplitude of the spike giving the appropriate q_{nm}.

The conditions on the required length of trajectory integration 2T and the value of the white noise reduction parameter η can now be formulated. To identify distinct lines, ω_* must be smaller than the difference between neighboring integer values of ω_{nm}. This difference $\Delta\omega$ is either one or υ depending on which is smaller. To have a clear resolution we set $\omega_* = \Delta\omega/\eta$. Since $\omega_* = \eta/T$ this means $\eta^2 = T\Delta\omega$. The trajectory integration time 2T must be sufficiently long that the residual white noise, $\exp(-\eta^2/2)$, is smaller than Fourier amplitudes of interest. Four figure accuracy is equivalent to $2T \approx 40/\Delta\omega$ or $\eta \approx 4.5$ which means the full trajectory integration is equivalent to about six poloidal or toroidal circuits, which ever takes longer, of the torus. The accuracy improves exponentially with additional trajectory integration.

IV. Alteration to Canonical Transformation

In practice, only a limited number of terms in the Fourier series for q and p will be retained. Yet, we would like the transformation from q,p,t to θ,J,t coordinates to be canonical. As long as the transformation is canonical, the θ,J,t coordinates are the action-angle coordinates for a neighboring Hamiltonian with the two Hamiltonians related by a small perturbation. In this section a method is given for altering a near canonical transformation to be canonical.

The traditional method for defining the canonical transformation to action-angle variables [7] uses the generating function S. The transformation

is canonical if and only if a function S exists which has the differential form

$$dS = pdq + \theta dJ + \left(\frac{\partial S}{\partial t}\right)_{q,J} dt \ . \tag{14}$$

In one dimension, the existence of S and the unit Jacobian condition can be shown to be equivalent.

Suppose q and p are given as some functions of $\hat{\theta},J,t$. The notation, $\hat{\theta}$, implies that the angular coordinate is not necessarily the canonical angle θ. Transforming the differential form for dS into $\hat{\theta},J,t$ coordinates, one finds

$$\theta = \frac{\partial S}{\partial J} - p\,\frac{\partial q}{\partial J} \ , \tag{15}$$

$$p\,\frac{\partial q}{\partial \hat{\theta}} = \frac{\partial S}{\partial \hat{\theta}} \ , \tag{16}$$

$$\left(\frac{\partial S}{\partial t}\right)_{q,J} = \frac{\partial S}{\partial t} - p\,\frac{\partial q}{\partial t} \ . \tag{17}$$

The first of these equations, Eq. (15), implies that S is not a single valued function of position. Instead, if the angle θ goes to infinity, then S/θ equals J. The second equation, Eq. (16), gives an indefinite integral for S

$$S = \int^{\hat{\theta}} p\,\frac{\partial q}{\partial \hat{\theta}}\,d\hat{\theta} \ . \tag{18}$$

Of course, an arbitrary function of J and t could be added to the integral. This abitrary function would adjust the location of $\theta = 0$.

One would like the action to be given by

$$J = \frac{1}{2\pi} \int_0^{2\pi} p \frac{\partial q}{\partial \hat{\theta}} d\hat{\theta} \ . \tag{19}$$

If this equation were true, then we would find the simple relation between θ and $\hat{\theta}$

$$\theta = \hat{\theta} + \lambda (\hat{\theta}, J, t) \tag{20}$$

with $\lambda(\hat{\theta},J,t)$ periodic in $\hat{\theta}$ and t. Otherwise, an advance by 2π in $\hat{\theta}$ would not imply an advance by 2π in θ. The condition that the action be given by an integral over $\hat{\theta}$ is just the condition that $\hat{\theta},J,t$ coordinates be canonical when viewed from a single action surface.

Suppose p and q are given as functions of $\hat{\theta}$ and t on an action surface. It is then easy to modify these functions so that they satisfy the action equation, Eq. (19). Let the equations $\hat{p}(\hat{\theta},t)$ and $q(\hat{\theta},t)$ be given and let

$$p(\hat{\theta},t) = [1 + \xi(t)] \hat{p}(\hat{\theta},t) \ . \tag{21}$$

The equation for the action becomes

$$J = \frac{1+\xi(t)}{2\pi} \int_0^{2\pi} \hat{p} \frac{\partial q}{\partial \hat{\theta}} d\theta \ . \tag{22}$$

The function $\xi(t)$ can be easily chosen to cancel the time dependence of the integral. If $\xi(t)$ is required to have zero average, then Eq. (21) also specifies the value of the action on that surface.

Let us assume q and p are given functions of $\hat{\theta},t$ on an action surface and satisfy the action equation, Eq. (19). Then one can write

$$S = \hat{\theta}\, J + \tilde{S} \tag{23}$$

with $\tilde{S}$ periodic in θ and t. That is, $\tilde{S}$ is a single valued function of position. The difference between $\hat{\theta}$ and θ is given by

$$\lambda = \frac{\partial \tilde{S}}{\partial J} - p\,\frac{\partial q}{\partial J}\,. \tag{24}$$

This implies λ is determined by the choice of q and p as functions of $\hat{\theta}$ and t on the neighboring action surfaces.

These are two ways to view Eq. (24) for λ. First, if the transformation equations are evaluated on only a few action surfaces, then one can take $\lambda = 0$ on these surfaces by imagining that the q and p equations on the action surfaces, which are not evaluated, have the appropriate form. That is, if transformation equations $q(\theta,t)$ and $p(\theta,t)$ are given on a limited set of action surfaces, then these transformation equations are consistent with some global canonical transformation $q(\theta,J,t)$ and $p(\theta,J,t)$ provided they satisfy Eq. (19) for the action. In other words, they yield a time-independent action. This interpretation is tempting since the effects of perturbations to a Hamiltonian in action-angle coordinates are primarily determined by the variation of the perturbation in a surface of constant action. The second view of Eq. (24) for λ arises if transformation equations $q(\hat{\theta},J,t)$ and $p(\hat{\theta},J,t)$ are specified over a finite volume of phase space. For example, if the transformation is analytic, each Fourier term has the form

$$q_{nm}(J) = J^{(m/2)}[a_o + a_1 J + a_2 J^2 + \ldots] \tag{25}$$

so the Fourier terms can be specified as $J^{m/2}$ times a polynomial in J. In this case, one can evaluate λ. Given λ and $q(\hat{\theta},J,t)$ and $p(\hat{\theta},J,t)$, one can evaluate the Hamiltonian in θ,J,t coordinates to whatever accuracy is desired.

Once one has transformation equations $q(\theta,J,t)$ and $p(\theta,J,t)$, one would like to know the Hamiltonian in θ,J,t coordinates, which we denote by $h(\theta,J,t)$. Canonical transformation theory says

$$h(\theta,J,t) = H(q,p,t) + \left(\frac{\partial S}{\partial t}\right)_{q,J} . \tag{26}$$

This can also be written using Eqs.(17) and (18)

$$h = H(q,p,t) - p\,\frac{\partial q}{\partial t} + \frac{\partial}{\partial t}\left(\int^{\hat{\theta}} p\,\frac{\partial q}{\partial \hat{\theta}}\,d\hat{\theta}\right) \tag{27}$$

with $\hat{\theta},J,t$ the implied variables. In practice we would like the Fourier decomposition of $h(\theta,J,t)$

$$h = h_o(J) + \sum_{n,m}{}' h_{nm}(J)\exp[i(nt - m\theta)] , \tag{28}$$

which can be easily evaluated if $q(\theta,J,t)$ and $p(\theta,J,t)$ are given. The prime on the sum implies $n = 0$, $m = 0$ is omitted. By definition of h, the Hamiltonian equations

$$\dot{\theta} = \frac{\partial h}{\partial J} \quad , \quad \dot{J} = -\frac{\partial h}{\partial \theta} \tag{29}$$

are satisfied. If one carried out the transformation $q(\theta,J,t)$, $p(\theta,J,t)$ exactly, then h would clearly equal $h_o(J)$ and the rotation number ν would give dh_o/dJ. If the transformation is carried out with good but finite accuracy,

then one expects the h_{nm} to be small compared to h_o and they can be viewed as a perturbation of the unperturbed system h_o.

V. Nearly Integrable Hamiltonians

With a procedure established for obtaining the action-angle transformation for integrable Hamiltonians, let us consider the actual subject of the paper, nearly integrable Hamiltonians. By nearly integrable, we mean the Hamiltonian is near enough to being integrable that a trajectory can be followed for about six poloidal and toroidal circuits without straying far from some torus. This condition does not involve the accuracy of the procedure, just its utility. For a nearly integrable system, one can follow the procedure of the previous sections and establish transformation equations $q(\theta,t)$ and $p(\theta,t)$ on a number of action surfaces. The non-integrability serves only to widen the Gaussian spikes in the Fourier spectrum. Since these spikes have significant intrinsic width for practical trajectory integration times, the Fourier spectrum does not give particularly strong evidence as to integrability.

Once one has canonical transformation equations on an action surface, the Hamiltonian on that surface $h(\theta,J,t)$ can be evaluated. Identifying, for the moment $\hat{\theta}$ and θ, the Hamiltonian h is given by Eq. (27). Generally, the most useful form for $h(\theta,J,t)$ is the Fourier decomposed form, Eq. (28), with $h_o(J)$ defining a neighboring integrable Hamiltonian. Assuming enough Fourier terms are retained in $q(\theta,t)$ and $p(\theta,t)$, say the torus is approximated to one percent accuracy, the derivative of $h_o(J)$ is accurately given by the rotation number ν, which is evaluated as part of the procedure for obtaining $q(\theta,t)$ and $p(\theta,t)$. With a non-integrable Hamiltonian, the Fourier coefficients, h_{nm} represent not only approximations in the transformation equations, but give

the locations and size of the islands and stochastic regions of the trajectories.

Suppose one constructs $q(\theta,t)$ and $p(\theta,t)$ on a number of action surfaces and the sequence of sets $[q(\theta,t), p(\theta,t)]$ form nested tori. As noted earlier, one can construct analytic functions $q(\theta,J,t)$ and $p(\theta,J,t)$ which define a canonical transformation throughout the volume of phase space enclosed by the largest torus. This can be done so that $q(\theta,J,t)$ and $p(\theta,J,t)$ equal $q(\theta,t)$ and $p(\theta,t)$ on the action surfaces for which the transformation was explicity evaluated. Assuming $H(q,p,t)$ is analytic, the Hamiltonian $h(\theta,J,t)$ derived from an analytic transformation is analytic. This implies the Fourier coefficients h_{nm} and the rotation number ν can be smoothly interpolated over the entire volume of the largest torus.

The important consequences of the analyticity of the h_{nm} and ν is that one can evaluate the location and magnitude of trajectory islands and stochastic regions throughout a large volume of phase space after having evaluated only a few action surfaces. Actually this can be done by a trivial inspection of the h_{nm} and ν. That is, one looks for rational numbers N/M which equal ν somewhere in the phase space volume of interest. The magnitude of the islands and stochastic regions are then determined by the magnitude of the resonate Fourier terms h_{NM}.

To make these statements more explicit, let us review a few basic results of Hamiltonian perturbation theory. Suppose one has the Hamiltonian

$$h = h_o(J) + h_{NM}\cos(Nt - M\theta) \tag{30}$$

with $\delta = |h_{NM}/h_o| << 1$. Then, the importance of the perturbation depends critically on whether the derivative of h_o, which is the rotation number ν,

equals N/M. If ν differs from N/M significantly, then the perturbation slightly distorts the tori on which trajectories lie. This distortion is roughly $\delta J/J \sim \delta$. On the other hand, if $\nu = N/M$ at some value of the action J_o, then there is a much larger distortion of the trajectories, $\delta J/J_o \sim \delta^{1/2}$ and a new set of nested tori, called an island, is created about one of the unperturbed trajectories on the J_o surface. One can solve the perturbation problem analytically. The answer is that the location of the perturbed trajectories in the θ,J,t coordinate system is

$$J = J_o \pm \Delta \{s^2 - \sin^2[Nt - M\theta)/2]\}^{1/2} \tag{31}$$

with Δ the island half-width

$$\Delta = \left|\frac{4h_{NM}}{d\nu/dJ}\right| \tag{32}$$

and s a label for the various trajectories. For $s < 1$, the trajectories lie inside the island and for $s > 1$, they lie outside. If there are at least two Fourier terms h_{nm}, which resonate with different values of ν, then the trajectories can become stochastic. That is, wander over a finite volume and not just a surface in phase space. Cherikov's island overlap criterion [8] says that if the sum of the half-widths of two islands equals the separation between the islands, then the trajectories become stochastic in essentially all of the volume of phase space covered by the islands.

At this point, a few statements about accuracy may be useful. First, so long as $q(\theta,J,t)$ and $p(\theta,J,t)$ are canonical, the Hamiltonian $h(\theta,J,t)$ provides an exact description of the trajectories. Consequently, errors of description arise only to the extent $q(\theta,J,t)$ and $p(\theta,J,t)$ do not define a canonical

transformation. Second, if the transformation is not canonical, the error arises from the difference between the Fourier components of the Hamiltonian, h_{nm}, and those given by a neighboring canonical transformation. Such errors can give fictitious island widths or regions of stochasticity. Generally, this occurs when the resonant Fourier components, h_{nm}, have errors of order unity.

It is particularly advantageous to use the method outlined in this paper to study stochasticity induced by the overlap of small relatively high order islands, say $n \sim m \gtrsim 5$. In this case, the resonant h_{nm} can be quite small and still have important effects on the trajectories. Generally, the h_{nm} converge exponentially. This means, that for high n or m, the Fourier coefficients are roughly approximated by

$$|h_{nm}| \sim a^{|n|} \, b^{|m|} \tag{33}$$

with a and b positive constants which are significantly less than one. Consider two types or error. First, assume the correction $\xi(t)$ was not made to obtain a proper time-independent action integral, see Eq. (22). The correct $\xi(t)$ would have Fourier components which could generally be approximated by

$$\xi_n \sim \varepsilon^{|n|} \, . \tag{34}$$

The effect of these coefficients would be to couple the h_{nm} of the same m but different n. Therefore, an adequate approximation for the h_{nm} requires $a > \varepsilon$. Similarly, if $\hat{\theta}$, defined in Sec. IV, were related to the canonical angle θ by the relation

$$\theta = \hat{\theta} + \lambda \sin(\hat{\theta}) \tag{35}$$

but $\hat{\theta}$ were used as if it were canonical, then an adequate approximation for the h_{nm} would require $b > \lambda$. The simple formulas, $a > \varepsilon$ and $b > \lambda$, can be used to define the required accuracy of the calculation once the relevant amplitudes of the resonant h_{nm} are known. The relevant amplitudes of the h_{nm} are generally some fraction of that required for island overlap.

VI. Generalization

In this section, the generalization to multidimensional Hamiltonian systems is outlined. For simplicity of notation, a two-dimensional Hamiltonian $H(q,p,Q,P)$ is considered. However, the generalization to a higher number of dimensions is obvious.

Assuming as before that the Hamiltonian is nearly integrable, we seek transformation equations for q,p,Q, and P of the form

$$q(\theta,J,\phi,I) = \sum_{n,m} q_{nm}(J,I) \exp[i(n\phi - m\theta)] \quad . \tag{36}$$

Along the trajectories, we would like the actions J and I to be constant and the angles to advance uniformly,

$$\dot{\phi} = \omega(J,I) \quad ,$$

$$\dot{\theta} = \nu(J,I) \quad . \tag{37}$$

The Fourier decomposition of a trajectory $q(t)$ should establish definite

frequencies ω_{nm} and amplitudes q_{nm} just as in the one-dimensional case. The characteristic frequencies ω_{nm} now satisfy

$$\omega_{nm} = n\omega - m\upsilon \ , \tag{38}$$

so one must find two rotation numbers, υ and ω. In a manner which is essentially identical to the one-dimensional case, one can find transformation equations on a surface of constant I and J. That is $q(\theta,\phi)$, ..., $P(\theta,\phi)$.

In the one-dimensional case, a canonical transformation could be carried out action surface by action surface provided q and p have a time-independent action integral. In multiple dimensional systems, the analagous condition that a transformation be canonical on the surface of constant actions is that the action integrals be independent of the angles. As in one dimension, this can be ensured by the existence of a generating function S.

In two dimensions, a function S which satisfies

$$dS = pdq + PdQ + \theta dJ + \phi dI \tag{39}$$

establishes a canonical transformation between the variables q,..., P and θ ..., I. To make the transformation equations canonical on a surface of constant actions, we will retain $q(\theta,\phi)$ and $Q(\theta,\phi)$ and modify $p(\theta,\phi)$ and $P(\theta,\phi)$. We will let $\hat{p}(\theta,\phi)$ and $\hat{P}(\theta,\phi)$ be the transformation equations derived from the Fourier decomposition of the trajectory and $p(\theta,\phi)$ and $P(\theta,\phi)$ the canonical transformation equations. For simplicity, we will consider only the problem of making the transformation canonical in the surface of constant actions. The considerations relative to making the transformation canonical as functions of the actions are similar to the one-dimensional case. In

practice, it may not be necessary to make the transformation canonical. A discussion of this point was given at the end of the last section.

There are a number of ways to make a transformation canonical in the surface of constant actions. One could proceed by analogy with one dimension and let

$$p(\theta,\phi) = [1 + \xi(\phi)][1 + \eta(\theta)]\ \hat{p}(\theta,\phi) ,$$

$$P(\theta,\phi) = [1 + \xi(\phi)][1 + \eta(\theta)]\ \hat{P}(\theta,\phi) . \tag{40}$$

Unfortunately, the equations for ξ and η are coupled and a solution may require iteration. Therefore, for variety, we will assume the Jacobian $\partial(q,Q)/\partial(\theta,\phi)$ is finite and give an alternative method for obtaining p and P. Although extensions are possible to the case in which the Jacobian vanishes along a line, these extensions will not be considered.

The form of the generating function S, Eq. (39), implies the canonical momenta are given by

$$p = \left(\frac{\partial S}{\partial q}\right)_{Q,J,I} . \tag{41}$$

However, we wish to express S in the action-angle coordinates. One can easily show that

$$p = \frac{1}{\partial(q,Q)/\partial(\theta,\phi)}\ \frac{\partial(S,Q)}{\partial(\theta,\phi)} \tag{42}$$

with the actions held constant and

$$P = - \frac{1}{\partial(q,Q)/\partial(\theta,\phi)} \frac{\partial(S,q)}{\partial(\theta,\phi)} . \tag{43}$$

These equations yield the obvious relation

$$\frac{\partial S}{\partial \theta} = p \frac{\partial q}{\partial \theta} + P \frac{\partial Q}{\partial \theta} \tag{44}$$

with the convention that the variables are θ,J,ϕ,I unless explicitly stated otherwise. This relation implies S is not a single valued function of position. The form for S is

$$S = \tilde{S} + \theta J + \phi I \tag{45}$$

with $\tilde{S}$ single valued and therefore periodic in θ and ϕ.

To evaluate the canonical momenta, the Fourier decomposition of $\tilde{S}$ will be evaluated. First, consider the Fourier terms with $m \neq 0$. Now

$$\tilde{S}_{nm} = \frac{1}{(2\pi)^2} \int_0^{2\pi} (S-\theta J-\phi I) \exp\,[i(n\phi-m\theta)]d\theta d\phi . \tag{46}$$

An integration by parts in θ and the use of Eq. (44) for $\partial S/\partial\theta$ gives

$$\tilde{S}_{nm} = \frac{i}{m} \frac{1}{(2\pi)^2} \int_0^{2\pi} (p \frac{\partial q}{\partial \theta} + P \frac{\partial Q}{\partial \theta}) \exp[i(n\phi-m\theta)]d\theta d\phi . \tag{47}$$

Similarly, for $n \neq 0$, one can show

$$\tilde{S}_{nm} = - \frac{i}{n} \frac{1}{(2\pi)^2} \int_0^{2\pi} (p \frac{\partial q}{\partial \phi} + P \frac{\partial Q}{\partial \phi}) \exp[i(n\phi-m\theta)]d\theta d\phi . \tag{48}$$

The condition that the transformation be canonical reduces to the condition

that $\hat{S}_{nm}$ be the same Fourier components when both n and m are nonzero. The $\hat{S}_{oo}$ term will not play a role in our analysis and is therefore ignored.

The procedure for modifying non-canonical momentum transformation equations $\hat{p}(\theta,\phi)$ and $\hat{P}(\theta,\phi)$ into canonical equations is obvious. Evaluate $\hat{S}_{nm}$ using $\hat{p}$ and $\hat{P}$ in both Eqs. (47) and (48). The two forms for the $\hat{S}_{nm}$ will generally not be equal for n and m both nonzero. However, it is easy to make a sensible choice for all the $\hat{S}_{nm}$. Any choice will establish a canonical transformation.

The momenta equations, Eqs. (42) and (43), cannot be evaluated until the two actions are given. The actions are given by

$$J = \frac{1}{2\pi} \int_0^{2\pi} \left(p \frac{\partial q}{\partial \theta} + P \frac{\partial Q}{\partial \theta} \right) d\theta \tag{49}$$

$$I = \frac{1}{2\pi} \int_0^{2\pi} \left(p \frac{\partial q}{\partial \phi} + P \frac{\partial Q}{\partial \phi} \right) d\phi \tag{50}$$

However, Eq. (44) implies the periodic part of the generating function, $\hat{S}$, plays no role in the determination of the actions when $\partial(q,Q)/\partial(\theta,\phi)$ is finite. That is, if I and J in Eq. (45) are given any value whatsoever, then the actions evaluated using Eqs. (42), (43), (49), and (50) will agree with these values. The obvious choice is to use $\hat{p}$ and $\hat{P}$ in Eqs. (49) and (50) to give values to J and I. These values will generally depend somewhat on the ϕ and the θ surface on which the integrals are performed due to $\hat{p}$ and $\hat{P}$ not being proper canonical transformation equations.

Once canonical transformation equations have been established, it is easy to evaluate the Hamiltonian on the surface of constant actions. Indeed, since the generating function is independent of time, the action-angle Hamiltonian is obtained by substituting the transformation equations, $q(\theta,J,\phi,I)$, ...,

$P(\theta,J,\phi,I)$, into the Hamiltonian $H(q,p,Q,P)$. This Hamiltonian is most conveniently written in Fourier decomposed form

$$H(\theta,J,\phi,I) = H_o(J,I) + \sum_{n,m} H_{nm}(I,J) \exp[i(n\phi - m\theta)]. \tag{51}$$

As in one dimension, the primary interest usually involves the location and width of islands. Again, this question can be answered by perturbation theory. Let

$$H = H_o(I,J) + H_{NM} \cos(N\phi - M\theta) \tag{52}$$

with $\nu = \partial H_o/\partial J$ and $\omega = \partial H_o/\partial I$. Let I_o and J_o define a surface on which

$$N\omega = M\nu \ . \tag{53}$$

One then finds an island about the surface I_o, J_o. Since the energy, H, is conserved along a trajectory, it is not surprising that the structure in I and J are related by

$$I = I_o - \frac{\nu}{\omega} (J - J_o). \tag{54}$$

So, we need only give the island structure in J space. The determinant of H_o is defined by [7]

$$D_o = \begin{vmatrix} \frac{\partial^2 H_o}{\partial J^2} & \frac{\partial^2 H_o}{\partial J \partial I} & \frac{\partial H_o}{\partial J} \\ \frac{\partial^2 H_o}{\partial I \partial J} & \frac{\partial^2 H_o}{\partial I^2} & \frac{\partial H_o}{\partial I} \\ \frac{\partial H_o}{\partial J} & \frac{\partial H_o}{\partial I} & 0 \end{vmatrix} . \tag{55}$$

The width of the island in J space is

$$\Delta = \left| \frac{4\omega^2}{D_o} H_{NM} \right|^{1/2} \tag{56}$$

and the equation for J is

$$J = J_o \pm \Delta \{s^2 - \sin^2[(n\phi - m\theta)/2]\}^{1/2} . \tag{57}$$

VII. Discussion

Many studies of Hamiltonian systems are based on equations which are nearly integrable. It is of considerable importance in such studies to identify the regions of phase space in which trajectories lie on nested tori and the regions in which trajectories cover a finite volume of the total phase space. When the trajectories lie on tori, the actions, associated with the torus are conserved quantities. It is not surprising, therefore, that action-angle coordinates are particularly useful in the study of the existence of invariants and the general topological properties of trajectories.

Unfortunately, the use of action-angle coordinates has only been possible in the study of systems for which there was an obvious, neighboring, integrable Hamiltonian. Nonetheless, considerable mathematical insight into nearly integrable Hamiltonians has been gained from this perturbative

approach. In particular, it is found that the topology of the trajectories is especially sensitive to the variation of the Hamiltonian in the surface of constant actions. That is the dependence of the Hamiltonian on the angle-like variables. A topology change occurs when there is a resonance between the characteristic frequencies with which the angular coordinates advance along a trajectory and the angular dependence of the Hamiltonian. Since the characteristic frequencies and the Fourier coefficients which describe the angular dependence are often smooth functions of the actions over large volumes of phase space, the examination of the Hamiltonian on a few surfaces of constant actions can often yield the desired information about the trajectories.

In this paper a practical method is developed for carrying out a canonical transformation to action-angle variables, action surface by action surface. An angular plot of the canonical transformation equations gives a torus in the space of the original canonical coordinates. If the tori of different action surfaces form a nested sequence for some range of action variation, then one expects analytic behavior of the Hamiltonian in action-angle variables. The method which has been developed for carrying out the canonical transformation surface by surface yields not only the variation of the Hamiltonian on a surface of constant actions but also the characteristic frequencies of the trajectories in that surface.

The Fourier decomposition of trajectories and the identification of characteristic frequencies in the decomposition are the fundamental techniques of the method. This procedure has already seen use in the study of molecules [5], galaxies [6] and magnetic fields [1-4]. However, the use of the procedure to establish action-angle variables has only been carried out in the magnetic field line problem.

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