Nonadiabatic Particle Motion in Magnetic Mirror Traps

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Abstract: By numerical integration of the equation of single particle motion, the basic features of the actual nonadiabatic escape of particles are studied. The results are compared with the predictions of two existing theoretical models: "diffusion" model derived by B. V. Chirikov and "tunneling" model introduced by R. K. Varma.

In a strong magnetic field, a charged particle gyrates around a line of magnetic field, and simultaneously drifts across and over the magnetic surface. We restrict ourselves to the case of axially-symmetric and constant magnetic field, then the drift motion occurs around the axis of symmetry and can be removed. When a magnetic field slowly (adiabatically) varies during the gyration motion of a particle, the magnetic moment, defined by $\mu = v_\perp^2/2\Omega$, is approximately constant, where $v_\perp$ is the velocity component perpendicular to the magnetic field $\vec{B}$, $\Omega = |\vec{B}|$ is the gyration frequency of the particle and we set the mass and the charge as unity for simplicity. Because the magnetic moment vector is directed opposite to $\vec{B}$ and its strength μ is an adiabatic invariant, a particle in a non-uniform magnetic field is pushed along a magnetic line into the area of weak field. A magnetic mirror whose field strength has minimum in the mirror is a plasma confinement scheme which utilizes the nature of this adiabatic invariance of particle magnetic moment.

Let us consider a single charged-particle motion in an axi-symmetric magnetic mirror without electric field. The Hamiltonian is written in the cylindrical coordinates (r, θ, z) as follows:

$$H(p_z, p_r, z, r) = \frac{1}{2} \{p_z^2 + p_r^2 + [\frac{p_\theta}{r} - A_\theta(z, r)]^2\}, \quad (1)$$

where p_θ is the angular momentum which is constant, and A_θ is the vector potential which produces a magnetic mirror whose field strength has minimum, Ω_{min}, at z = 0 and maximum, Ω_{max}, at z = ±L/2 along the field line.

If the adiabatic approximation is valid, the particle that

has initially a magnetic moment μ_0 larger than the critical value $\mu_c = E/\Omega_{max}$ can be trapped and bounces with the bounce frequency, ω_b, in the mirror, where E is the particle energy which is a constant of motion. This feature can be seen from the expression: $H = v_{\shortparallel}^2/2 + \mu\Omega = E$. The condition of $\mu_0 > \mu_c$ is equivalent to that of $\psi_0 < \psi_c$ where ψ is the pitch angle in the velocity space being defined by $\psi \equiv \sin^{-1}(v_{\shortperp}/v)$, and ψ_0 and ψ_c are the initial and the critical values, respectively.

In actual situations, however, the magnetic moment μ is not a strictly constant during the particle motion, because the ratio of the gyration period to the bouncing period is finite, that is, the magnetic field strength varies during the gyration period. Then the particle that adiabatically trapped in the mirror, i.e. $\mu_0 > \mu_c$ can escape from the mirror when the moment μ becomes less than the critical value μ_c due to its nonadiabatic change.

There exist two theoretical models which describe the non-adiabatic escape. One is a diffusion model in μ-space derived by B. V. Chirikov.[1] This theory makes use of the fact that the magnetic moment rapidly changes by an amount $\Delta\mu = \xi\sin\phi$ everytime the particle crosses the median plane of the mirror, where ϕ is the gyrophase of the particle at the median plane and ξ is a function of μ . From this fact Chirikov uses a standard mapping in (μ, ϕ) at median plane of the mirror. This mapping is linearized with respect to μ and is characterized by only one parameter, the so-called stability parameter, S. When S is less than unity, there is only bounded oscillation of μ around each resonance value which is evaluated by the resonance between the gyration oscillation and the bouncing oscillation of the particle. If S is much larger than unity, there occur resonance overlaps and stochastic changes of μ . The diffusion in μ-space can be observed by use of the standard mapping in the case of S >> 1. When μ decreases to a value less than the critical value μ_c, the particle escapes from the mirror.

The other theoretical model is based on the ensemble view point introduced by R. K. Varma.[2] According to this theory, the probability density, F(x, t), at the coordinate x parallel to the magnetic field and at the time t is expressed as a sum of the contribution of a set of wave functions $\Psi_n(x, t)$ which are distinguished by the ensemble mode number n, $F(x, t) = \sum\limits_{n=1} |\Psi_n(x, t)|^2$.

Each of the wavefunctions obeys a one-dimensional (along x)
Schrödinger-like equation where the role of $\hbar$ is replaced by μ_o/n
with initial magnetic moment μ_o. This model describes the non-
adiabatic escape as the escape due to "tunneling-like effect",
and predicts that the lifetime of adiabatically trapped particles
in a magnetic mirror with given energy and given magnetic moment
μ_o is approximately determined by the inverse of the product of
the number of hit on the both walls per unit time, ω_b/π, and the
transmission rate. If the WKB approximation can be used for the
calculation of the transmission rate, the lifetime of the n-th
ensemble mode becomes

$$\tau_n = \frac{\pi}{\omega_b} \exp\{n \frac{2}{\mu_o} \int_a^b \overline{\sqrt{2[\mu_o\Omega(x)-E]}} dx\} , \qquad (2)$$

where the integral takes place over the region of the potential
barrier, $\mu_o\Omega(x) > E$, from one zero of the integrand(at $x = a$) to
the other (at $x = b$). The argument of the exponential is pro-
portional to n/ε where $\epsilon^{-1} \equiv L\Omega_o/v$, Ω_o is the typical gyrofre-
quency, so there exist <u>multiple lifetimes</u>.

<u>Numerical Calculation</u> The purpose of the present work is to
examine the basic features of the actual nonadiabatic escape of
the particles by a numerical calculation, and to compare the
results with the predictions of these two existing theories.

The basic equations are the single particle equation of
motion,

$$\frac{d\vec{r}}{dt} = \vec{v} , \quad \frac{d\vec{v}}{dt} = \vec{v} \times \vec{\Omega} \quad (\vec{\Omega} = \vec{B}) . \qquad (3)$$

We numerically integrated these equations by the 6-th order Runge-
Kutta method. Two cases are studied.

Case (A) In this case, we consider particles at various values of
stability parameter S by using a simple magnetic field configura-
tion which is divergence free but not rotation free. It is
expressed as follows:

$$\Omega_r = -\Omega_o \frac{\pi r}{L} a \sin(2\pi z/L), \quad \Omega_\theta = 0 , \qquad (4)$$
$$\Omega_z = \Omega_o[1-a \cos(2\pi z/L)].$$

We take the parameter a as 0.2, the corresponding mirror ratio
being $R = (1+a)/(1-a) = 1.5$. We chose the initial conditions such
that $\cos \psi_o = 0.550 < \cos \psi_c = 0.577$ ($\mu_o > \mu_c$) and 500 particles
with the same energy and the same magnetic moment μ_o (or the same

pitch angle ψ_0) are uniformly distributed on a circle around the guiding center at $r_{go} = 0.05L$. These are distinguished each other by their initial gyrophases ϕ_0. Initial z-coordinates are the same, i.e. $z_0 = 0$. The energy is varied as $\varepsilon^{-1} \equiv L\Omega_0/v = 4$ through 20, the corresponding stability parameter being varied as $S = 1$ through 14. The stability parameter has the form

$$S = \frac{\alpha}{\varepsilon^2} e^{-\kappa/\varepsilon} , \qquad (5)$$

where, in the present case, $\alpha \cong 2.84$ and $\kappa \cong 0.332$. The particle that reaches at $z = \pm L$ is considered as escaped. In Fig. 1 the stability parameter is ploted by the solid curve. The crosses and the dashed curve denote the survival particles in the mirror after 200 times the gyration period. The crosses are the numerical values and the dashed curve is the theoretical one which is derived by the consideration of diffusion in μ-space with a constant diffusion coefficient $D = \omega_b \xi^2 (1-1/S)^2/4\pi$ and with the absorbing boundary at $\mu = \mu_c$. The number of survival particles is calcula-ted by this diffusion model with absorbing boundary as[3]

$$N(t) = \frac{N_0}{\sqrt{\pi}} \gamma(\frac{1}{2}, \frac{(\mu_0 - \mu_c)^2}{4Dt}) , \qquad (6)$$

where $\gamma(\nu, x)$ $(\equiv \int_0^x t^{\nu-1} e^{-t} dt)$ is the incomplete gamma function.

We can clearly see a threshold of the stability parameter in order to have a substantial particle loss.

When ε^{-1} is equal to 4, the Störmer radius is greater than the radial coordinate of the guiding center, therefore no particles escape.[4]

Case (B)   We found multiple lifetimes in the case of particles having stability parameter below threshold. The magnetic field, which is taken to be both divergence free and rotation free, is of the following form

$$\Omega_r = -\Omega_0 a \sin(2\pi z/L) I_1 (2\pi r/L), \qquad \Omega_\theta = 0,$$
$$\Omega_z = \Omega_0 [1 - a \cos(2\pi z/L) I_0 (2\pi r/L)] , \qquad (7)$$

where I_0 and I_1 are the modified Bessel functions and we take the parameter a as 0.2. Cosine of initial pitch angle is larger than before and is $\cos\psi_0 = 0.570$ which is still smaller than the critical one, $\cos\psi_c = 0.577$. The initial guiding center coordinate divided by the mirror length is $r_{go}/L = 0.177$. In this case, all the particles have nearly the same guiding center and nearly the

same magnetic moment μ_o which is measured at their guiding center.
The parameter ϵ^{-1} is taken from 14 to 16.3. The corresponding
stability parameter is from 0.4 to 0.8. Figure 2 shows the number
of survival particles N(t) on logarithmic scale versus time. We
can identify two lifetimes which are defined by the inverse of the
slope. In Fig. 3, the escaping time on the initial gyrophases are
ploted. Figure 3 shows that in some regions of the gyrophase the
dispersion of the escaping time is large, while in other regions
it is small. Figure 4 shows the logarithm of two lifetimes for
each value of ϵ^{-1}. One can find that the two slopes are nearly
proportional to ϵ^{-1} in qualitative agreement with the WKB approxi-
mation for the tunneling model of R. K. Varma. The stability
parameter, in this case, is depicted by the solid line in Fig. 4,
so that Chirikov's model predicts only bounded oscillations of the
magnetic moment.

Summary of the Results
1) For particle of pitch angles not very close to the critical
value, substantial nonadiabatic loss occurs when stability para-
meter exceeds a critical value nearly equal to 3.
2) Dependence of the number of remaining particles at given time
on stability parameter is in good agreement with the diffusion
model with absorbing boundary.
3) Even if Chirikov's stability parameter is below unity, particles
of pitch angle very close to the critical value can escape from the
magnetic mirror by nonadiabatic effects.
4) In this case, multiple lifetimes (at least two) are observed in
accord with Varma's ensemble theory.

References
1) B. V. Chirikov: Sov. J. Plasma Phys. <u>4</u> (1978) 289, and refer-
   ences cited therein.
2) R. K. Varma: Phys. Rev. Lett. <u>26</u> (1971) 417.
3) T. Tange: Private communication.
4) O. V. Serdyuk, Sov. Phys. Tech. Phys. <u>21</u> (1976) 1252.

Figure Captions
Fig. 1 Stability parameter (solid curve) and the number of survival
particles N(t = 200 $2\pi/\Omega_o$) in the mirror (crosses: numerical
results, dashed curve: diffusion model with absorbing boundary)
versus ϵ^{-1}.

Fig. 2   The number of survival particles in the mirror in logarithmic scale versus time for $\epsilon^{-1} = 14.4$.

Fig. 3   Dependence of the escaping time of each particle on the initial gyrophase ϕ_0.
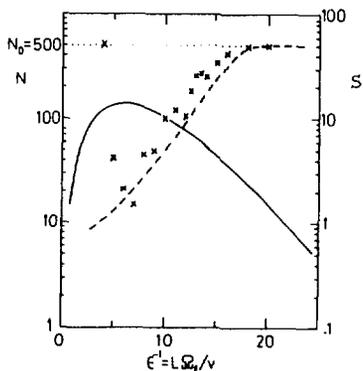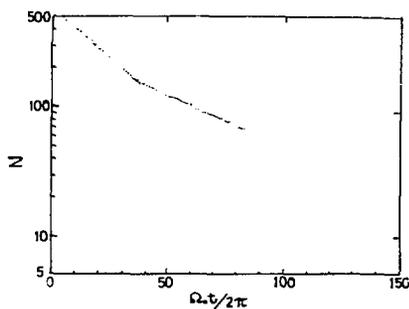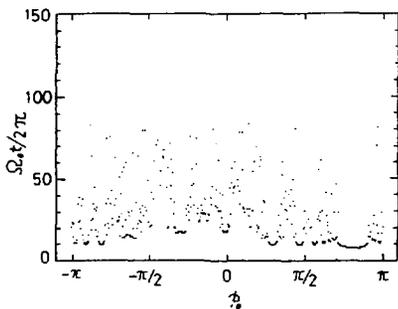
Fig. 4   Lifetimes τ for each ϵ^{-1}.

Fig. 1

Fig. 2

Fig. 3

Fig. 4