ABSTRACT

In this paper we study the nonoscillatory behaviour of even order functional differential inequalities and equations with continuous distributed retarded and advanced arguments. For arguments we give some conditions under which these inequalities only have nonoscillatory solutions of degree 0 or n.
1. INTRODUCTION

In this paper we are concerned with nonoscillatory behaviour caused by retarded and advanced arguments for the following differential inequalities and equations:

\[ y^{(n)}(t) + p(t)[y(g(t))]^n y(g(t)) \geq 0, \quad (1.1) \]
\[ \{ y^{(n)}(t) + p(t)[y(g(t))]^n y(g(t)) \geq 0 \}. \quad (1.2) \]
\[ \{ y^{(n)}(t) + p(t)[y(g(t))]^n y(g(t)) \geq 0 \}. \quad (1.3) \]
\[ \{ y^{(n)}(t) + p(t)[y(g(t))]^n y(g(t)) \geq 0 \}. \quad (1.4) \]

where \( n \) is even.

Below the following conditions are assumed to hold:

1. \( p > 0, g(t) \leq t, h(t) \geq t \), \( g(t) \) and \( h(t) \) are continuous, \( g(t, \xi) \leq t \), \( h(t, \xi) \geq t \) are continuous, \( g(t) \to \infty \), \( h(t) \to \infty \) for \( t \in [a, b] \).
2. \( \sigma = [a, b] \) is a nondecreasing function.
3. Integrals in (1.3) and (1.4) are Stieltjes integrals.
4. \( g(t, \xi) \) and \( h(t, \xi) \) are continuous nondecreasing functions with \( t \) or \( \xi \) respectively.
5. There exists \( \varphi(t, \xi) \), \( \chi(t, \xi) \) such that \( \varphi(\varphi(\varphi(t, \xi), \xi), \xi) = g(t) \), \( \chi(\chi(\chi(t, \xi), \xi), \xi) = h(t) \), \( \lim \varphi(t, \xi) = \lim \chi(t, \xi) = \infty \), \( \varphi(t, \xi) \) and \( \chi(t, \xi) \) are nondecreasing functions with \( t \) or \( \xi \) respectively.

Definition 1: If \( y(t) \) is a nonoscillatory solution of (1.1) — (1.4), also there are an even integer \( k \in \{ 0, 2, \ldots, n \} \) and a number \( t_0 > 0 \) such that

\[ y(t)y^{(k)}(t) > 0, \text{ on } [t_0, \infty) \text{ for } 0 \leq k < l \]
\[ (-1)^{-k}y(t)y^{(k)}(t) > 0, \text{ on } [t_2, \infty) \text{ for } k \in \mathbb{N} \]

then such a \( y(t) \) is said to be a nonoscillatory solution of degree \( k \), and the set of all solutions of someone of (1.1) — (1.4) is denoted by \( \mathcal{N} \).

Suppose the set of all nonoscillatory solutions of someone of (1.1) — (1.4) is denoted by \( \mathcal{N} \).

Lemma 1: We have

\[ \mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1 \cup \ldots \cup \mathcal{N}_k \]

The proof of this Lemma is similar to the proof in [2].

In this paper we give some sufficient conditions under which \( \mathcal{N}_0 = \{ \text{i.e. } \mathcal{N} = \mathcal{N}_0 \} \) for (1.1), (1.3), and \( \mathcal{N}_0 = \{ \text{i.e. } \mathcal{N} = \mathcal{N}_0 \} \) for (1.2), (1.4). For variatory classes of \( g(t) \), \( g(t, \xi) \) or \( h(t) \), \( h(t, \xi) \) we obtain variatory nonoscillatory criteria for (1.3) — (1.4). We generalized results in [1]. On the other hand, we also give some nonoscillatory criteria for general superlinear inequalities:

\[ \{ y^{(n)}(t) - f(y(g(t))) \} \leq 0, \text{ on } [t_0, \infty) \text{ for } 0 \leq k < l \]
\[ \{ y^{(n)}(t) - f(y(g(t))) \} \leq 0, \text{ on } [t_2, \infty) \text{ for } k \in \mathbb{N} \]
\[ \{ y^{(n)}(t) + p(t)[y(g(t))]^n y(g(t)) \} \leq 0, \text{ on } [t_0, \infty) \text{ for } 0 \leq k < l \]
\[ \{ y^{(n)}(t) + p(t)[y(g(t))]^n y(g(t)) \} \leq 0, \text{ on } [t_2, \infty) \text{ for } k \in \mathbb{N} \]

The proof of this Lemma is similar to the proof in [2].
Definition: Nonlinear function \( f(x) \) is said to be superlinear if

\[
\lim_{x \to 0} \frac{f(x)}{|x|} = \infty.
\]

We shall make use of the following results of one-order inequalities which were obtained by the author in [3].

Set

\[
\begin{align*}
&\{ z^{(i+1)} + \sum_{k=1}^{n} z^{(k)} \dot{y}^{(k)}(t) + \sum_{k=1}^{n} y^{(k)}(t) \dot{z}^{(k)}(t) \} \geq 0, \\
&\{ z^{(i+1)} + \sum_{k=1}^{n} z^{(k)} \dot{y}^{(k)}(t) + \sum_{k=1}^{n} y^{(k)}(t) \dot{z}^{(k)}(t) \} \geq 0, \\
&\sum_{k=1}^{n} z^{(k)} \dot{y}^{(k)}(t) + \sum_{k=1}^{n} y^{(k)}(t) \dot{z}^{(k)}(t) \geq 0.
\end{align*}
\]

\[
\begin{align*}
&\lim_{x \to 0} \int_{a}^{x} \frac{P(s) ds}{r(s)} > \frac{1}{e}, \\
&\lim_{x \to 0} \int_{a}^{x} \frac{P(s) ds}{r(s)} = 0, \\
&\lim_{x \to 0} \int_{a}^{x} \frac{P(s) ds}{r(s)} > \frac{1}{e}, \\
&\lim_{x \to 0} \int_{a}^{x} \frac{P(s) ds}{r(s)} > 0.
\end{align*}
\]

Lemma 2. If conditions (H1) and (H2) are satisfied, then (1.10) has no nonoscillatory solution.

Lemma 3. If conditions (H1') and (H2') are satisfied, then (1.11) has no nonoscillatory solution.

2. RETARDED INEQUALITIES AND EQUATIONS

Here we are interested in the situation in which there is no nonoscillatory solution of degree 0 of (1.1) or (1.3) \( \omega_0 = 0 \) or all nonoscillatory solutions of (1.1) or (1.3) are of degree \( n \).

Theorem 1. If \( g(t) \leq kt \), for some \( k \in (0,1) \) and all \( t \geq T \), then there is no nonoscillatory solution of degree 0 of (1.1) \( \omega_0 = 0 \).

Proof. Case 1: In the case \( g(t) = kt \), for some \( k \in (0,1) \) and all \( t \geq T \), let \( y(t) \) be a nonoscillatory positive solution of degree 0 of (1.1). So \( y(t) \geq 0 \), \( y^{(n)}(t) < 0 \), \( y^{(n)}(t) > 0 \), ..., \( y^{(n)}(t) > 0 \), for \( t \geq T \). Set

\[
\begin{align*}
\kappa &= \kappa^*, \\
p &= p^* K - \frac{a+1}{2}, \\
\gamma(t) &= y^{(n)}(t) - p^* u_n y^{(n-1)}(t) + p^* u_{n-1} y^{(n-2)}(t) + \cdots + p^* u_0 y^{(n)}(t) - p^* u_{n-2} y^{(n-1)}(t) \cdots p^* u_1 y(t).
\end{align*}
\]

We have

\[
\gamma'(t) = y^{(n)}(t) - p^* u_n y^{(n-1)}(t) + p^* u_{n-1} y^{(n-2)}(t) + \cdots + p^* u_0 y^{(n)}(t) - p^* u_{n-2} y^{(n-1)}(t) \cdots p^* u_1 y(t).
\]

We choose positive constants \( u_0, u_1, \ldots, u_{n-2} \), such that

\[
\begin{align*}
(1 - \kappa^* u_{n-1}) &= 0, \\
\kappa^* u_{n-1} - u_{n-2} &= 0, \\
\kappa^* u_{n-2} - u_{n-3} &= 0, \\
&\vdots \\
\kappa^* u_1 - u_0 &= 0, \\
u_1 - u_0 &= 0
\end{align*}
\]

that is, \( u_{n-2} = 1/\kappa^*, \ u_{n-3} = 1/\kappa^3, \ldots, \ u_0 = 1/\kappa^\frac{n-1}{2} \).

Notice

\[
y^{(n)}(t) - py(t) > 0. \tag{2.2}
\]
and
\[ y(t) = y(t) - p(t)x(t) \geq 0. \]  

Then we can obtain that \( y(t) \) is a negative solution of (2.3). But i.e. Conditions (H_1) and (H_2) are satisfied. So using Lemma 2 we can see that there is no negative solution of (2.3). This led to a contradiction. The proof for \( y(t) < 0 \) being similar.

Case 2. In the case \( g(t) < k \) for some \( k \in (0,1) \) and all \( t \in J \).
If there is a positive solution of degree 0 \( y(t) > 0 \) for (1.1), then
\[ y(t) < 0, \text{ i.e. } y(t) < y(kt) \]
so we have
\[ 0 \leq y(t) - p(t)x(t) \leq y(t) - p(t)x(t) \]

From the proof in Case 1, we are also led to a contradiction. The proof for \( y(t) < 0 \) being similar. The proof of Theorem 1 is complete.

**Theorem 2**
If \( g(t) < t - k \) for some \( k > 0 \) and all \( t > t_0 \), then

\[ \lim_{t \to \infty} \frac{\ln t}{n} e > 1. \]  

implies that (1.1) has no nonoscillatory solution of degree 0 \( (\mathcal{N}_0 = \emptyset) \).

**Proof.** Case 1. If \( g(t) = t - k \) for some \( k > 0 \) and all \( t > t_0 \), then Kunamo [1] has proved this result.

Case 2. If \( g(t) < t - k \) for some \( k > 0 \) and all \( t > t_0 \), then we can suppose there is a nonoscillatory positive solution of degree 0, \( y(t) > 0 \) of (1.1). So we have \( y(t) < 0 \), \( y(t) > 0 \), and
\[ 0 \leq y(t) - p(t)x(t) \leq y(t) - p(t)x(t) \]

From the proof of Case 1 we can deduce that the condition (2.4) implies the inequality
\[ y(t) - p(t)x(t) > 0, \]
has no nonoscillatory positive solution. This led to a contradiction. The proof for \( y(t) < 0 \) being similar. We proved this Theorem.

**Remark:** Using Theorems 1, 2, we have the following results. If \( g(t) \) is taken forms as \( t^v \) for any \( v \in (0,1) \), then for all \( p \)
(1.1) has no nonoscillatory solution of degree 0. Also if \( g(t) \) are taken as \( t - k t^x \), then for some \( p \), which satisfies \( \int_{t_0}^{t_0 + \varepsilon} t^x \leq 1 \) (1.1) has no nonoscillatory solution of degree 0. But if \( g(t) \) is taken as \( t - e^{-t} \), then we need the condition as follows.

\[ \limsup_{t \to \infty} \frac{\ln t}{n} e > 1. \]

for some \( k \in (0,1, ..., n-1) \), to assume that (1.1) has no nonoscillatory solution of degree 0.

**Theorem 3.** For (1.3) if \( \mathcal{N} = \emptyset \) if in (1.3) one of following conditions is satisfied

\[ g(x,y) > 0 \text{ for all } x \in \mathbb{R}, y \in (a,b), \]  

then for (1.3) \( \mathcal{N} = \emptyset \). For (2.2) \( \mathcal{N} = \emptyset \).

**Proof.** In the case (1.3). Otherwise, let \( y(t) \) be a nonoscillatory solution of (1.3) of degree \( \ell \). We may assume that \( y(t) > 0 \) for \( t > t_0 \).
Take $t_2$ such that 

\[ \max_{t \leq t_2} g(t) \leq l \]

is bounded. In fact, we may choose $t_2$ such that 

\[ \max_{t \leq t_2} g(t) \leq l \]

For $t \geq t_2$, we have 

\[ y^{(k)}(t) + \frac{p^k}{\lambda^k} y^{(k-1)} + \sum_{j=0}^{l} g_i(t) \leq 0 \]

As in Theorem 2, if $y(t)$ is a nonoscillatory positive solution of degree 0 of the equation 

\[ y^{(n)}(t) + \frac{p^n}{\lambda^n} y(t) + \sum_{j=0}^{l} g_j(t) \leq 0 \]

then we have $y(t) > 0$, $y''(t) > 0$, $y^{(n)}(t) > 0$, that is, $y(t)$ is a nonoscillatory negative solution of 

\[ y^{(n)}(t) + \frac{p^n}{\lambda^n} y(t) + \sum_{j=0}^{l} g_j(t) \leq 0 \]

Using the condition $n \sqrt{p} > 1$ from Lemma 2, this fact led to a contradiction. Secondly, suppose $g(t, \xi) = kt \xi$ for $\xi \in [a, b]$, $t \geq t_0$, we can obtain 

\[ y^{(n)}(t) - \frac{p^n}{\lambda^n} y(t) + \sum_{j=0}^{l} g_j(t) \leq 0 \]

where $y(t)$ is a nonoscillatory positive solution of degree 0 of (1.3), i.e. 

\[ y(t) > 0, \quad y''(t) > 0, \quad y^{(n)}(t) > 0 \]

As in Theorem 2, we can easily obtain a contradiction.

2. Suppose condition (2.6) is satisfied. Firstly, suppose $g(t, \xi) = kt \xi$ for some $k$, which satisfied $0 < k < 1$, $\xi \in [a, b]$, $b > a > 0$, $t \geq t_0$. Set 

\[ y^{(n)}(t) - \frac{p^n}{\lambda^n} y(t) + \sum_{j=0}^{l} g_j(t) \leq 0 \]

where $y(t)$ is a nonoscillatory positive solution of degree 0 of (1.3), i.e. 

\[ y(t) > 0, \quad y''(t) > 0, \quad y^{(n)}(t) > 0 \]

As in Theorem 2, we can easily obtain a contradiction.
From Lemma 2, we can obtain a contradiction. Secondly, suppose \( g(t, c^k) <ct^l \) for some \( k \), which satisfies \( 0<k<l \), \( t \in [a,b] \), and all \( t > t_0 \), then we can easily obtain that

\[
y(t) = y_{0} + pt \int_{a}^{b} y(kx^{l}) \, dx + \cdots + pt \int_{a}^{b} y(kx^{l} \cdot \cdots \cdot x) \, dx + \cdots + pt \int_{a}^{b} y(kx^{l} \cdot \cdots \cdot x) \, dx + \cdots + pt \int_{a}^{b} y(kx^{l} \cdot \cdots \cdot x) \, dx
\]

where \( y(t) \) is a positive solution of degree 0 of (1.3), i.e.,

\[
y(t) = y_{0} + pt \int_{a}^{b} y(kx^{l}) \, dx + \cdots + pt \int_{a}^{b} y(kx^{l} \cdot \cdots \cdot x) \, dx + \cdots + pt \int_{a}^{b} y(kx^{l} \cdot \cdots \cdot x) \, dx + \cdots + pt \int_{a}^{b} y(kx^{l} \cdot \cdots \cdot x) \, dx
\]

If we choose \( U_n \), \( U_{n-1} \), \ldots, \( U_1 \) such that

\[
U_n = (k^n)^{-1}, \quad U_{n-1} = (k^n)^{-1}, \quad \ldots, \quad U_1 = (k^n)^{-1},
\]

i.e.,

\[
U_n = k^n, \quad U_{n-1} = k^n, \quad \ldots, \quad U_1 = k^n,
\]

then we have

\[
y(t) = y_{0} + pt \int_{a}^{b} y(kx^{l}) \, dx + \cdots + pt \int_{a}^{b} y(kx^{l} \cdot \cdots \cdot x) \, dx + \cdots + pt \int_{a}^{b} y(kx^{l} \cdot \cdots \cdot x) \, dx + \cdots + pt \int_{a}^{b} y(kx^{l} \cdot \cdots \cdot x) \, dx
\]

Suppose \( y(t) \) is a positive solution of the following inequality:

\[
y(t) > \frac{y_0}{-p} \int_{a}^{b} y(kx^{l}) \, dx + \cdots + \frac{y_0}{-p} \int_{a}^{b} y(kx^{l} \cdot \cdots \cdot x) \, dx + \cdots + \frac{y_0}{-p} \int_{a}^{b} y(kx^{l} \cdot \cdots \cdot x) \, dx + \cdots + \frac{y_0}{-p} \int_{a}^{b} y(kx^{l} \cdot \cdots \cdot x) \, dx
\]

Then we can obtain that \( \mathcal{N}_n = \mathcal{N}_0 \) for (1.2) or (1.4).
Theorem 5. If \( h(t) > \omega t \) for some \( \omega > 1 \) and all \( t \geq T \), then all nonoscillatory solutions of (1.2) are of degree 0. (\( N = N_0 \) i.e. \( N = 0 \)).

Proof. (1) In case \( h(t) = \omega t \), for some \( \omega > 1 \) and all \( t \geq T \). Let \( y(t) \) be a nonoscillatory positive solution of degree \( n > 0 \) of (1.2). Proceeding exactly as in the proof of Theorem 3, we see that \( z = n \), that is, \( y(t) > 0 \), \( y'(t) > 0 \), \( \cdots \), \( y^{(n-1)}(t) > 0 \) for all large \( t \), say \( t > T \). Notice

\[
y^{(n)}(t) - p y'(aw^t) \geq 0.
\]

Define

\[
z(t) = y^{(n)}(t) + p^n u_{n-1} y^{(n-1)}(aw^t) + p^n u_{n-2} y^{(n-2)}(aw^t) + \cdots + p^n u_0 y^{(0)}(aw^t) \geq 0,
\]

where

\[
p^n = \frac{\omega^n - 1}{\omega^n - \omega}, \quad u_i (i = 1, \ldots, n-2)
\]

are some constants.

Then we have

\[
z(t) = y^{(n)}(t) + p^n u_{n-1} y^{(n-1)}(aw^t) + p^n u_{n-2} y^{(n-2)}(aw^t) + \cdots + p^n u_0 y^{(0)}(aw^t) \geq 0,
\]

If we choose positive constants \( U_0, U_1, \ldots, U_{n-2} \) which satisfy the following conditions

\[
\omega^k u_{k+1} - 1 = 0, \quad \omega^k u_{k+2} - u_k = 0, \quad \ldots, \quad \omega^k u_n - u_{k-1} = 0
\]

that is

\[
u_0 = 1/\omega^0, \quad u_1 = 1/\omega^1, \quad \ldots, \quad u_{n-2} = \omega^{n-2} - 1/\omega^{n-2}
\]

So \( z(t) \) is a positive solution of (3.2). But

\[
\lim_{t \to \infty} p(t) = \infty, \quad \lim_{t \to \infty} p(t) = \infty > 0.
\]

That is, conditions \( (H_1') \) and \( (H_2') \) are satisfied.

So using Lemma 3 we can see that there is no positive solution of (3.2). This led to a contradiction.

(2). In the case \( h(t) > \omega t, \) for some \( \omega > 1 \) and all \( t \geq T \). If there is a positive solution \( y(t) > 0 \) of degree \( n > 0 \) for (1.2), then \( y'(t) > 0 \) i.e. \( y[h(t)] > y(t - T) \). Hence we have

\[
0 \leq y^{(n)}(t) - p y'(at^t) \leq y^{(n)}(a^t) - p y'(at^t)
\]

From the proof in the case \( h(t) = \omega t \), we can obtain a contradiction. The proof of Theorem 5 is complete.

Theorem 6. If \( h(t) > \omega t + \tau \) for some \( \tau > 0 \) and all \( t \geq T \) then the condition

\[
\int_{T}^{t} \frac{z(t)}{z(t)} \geq 1
\]

implies that (1.2) has no nonoscillatory solution of degree \( n. (N = N_0). \)

Proof. In the case \( h(t) = \omega t + \tau \) for some \( \tau > 0 \) and all \( t \geq T \). Suppose (3.3) is satisfied. Otherwise, suppose there is a nonoscillatory positive solution \( y(t) > 0 \) of degree \( n \) of (1.2). So we have \( y'(t) > 0 \) i.e. \( y[h(t)] > y(t - \tau) \) and

\[
0 \leq y^{(n)}(t) - p y'[h(t)] \leq y^{(n)}(t) - p y'[t + \tau]
\]

From the proof in case (1) of Theorem 5, we can deduce that condition (3.3) implies the inequality

\[
y^{(n)}(t) - p y'[t + \tau] \geq 0
\]
has no nonoscillatory positive solution. This led to a contradiction. As in case 2 of Theorem 5, for $h(t, t)$ we are led to a contradiction. We proved Theorem 6.

Theorem 7: For (1.4), $N = N_0 = V$. If in (1.4) one of following conditions is satisfied

1) $h(t, t) > t^k$ 
for all $t \in [a, b]$, $b > a$

and $n(p^*) > 1$ (3.4)

2) $h(t, t) > k^t$ 
for some $k$, which satisfies

$k > 1$, $t \in [a, b]$, and all $t > T$;

then for (1.4) $N = N_0$.

For (3.1), $V = V$.

Proof: (1) In the case (1.4) notice that in the proof of Theorem 3 we only need that

$$\lim_{t \to \infty} g(t, \xi) = 0$$
for $\xi \in [a, b]$.

If $g(t, t)$ is changed by $h(t, t)$, then results can also be proved.

(2) In the case (1.4), Suppose condition (3.4) is satisfied. Firstly suppose $h(t, t) = t + \xi$ for

$t \in [a, b]$, $b > a > 0$.

Set

$$Z(t) = y^{(n)}(t) + \frac{p^*}{n} \int_a^b y^{(n-1)}(t + \xi) d\sigma(t) + \frac{p^*}{n} \int_a^b y^{(n-1)}(t + \xi) d\sigma(t) + \cdots + \frac{p^*}{n} \int_a^b y^{(n-1)}(t + \xi) d\sigma(t)$$

where $p^* = \sqrt{n} > 0$.

So we have

$$Z(t) = y^{(n)}(t) + \frac{p^*}{n} \int_a^b y^{(n-1)}(t + \xi) d\sigma(t) + \frac{p^*}{n} \int_a^b y^{(n-1)}(t + \xi) d\sigma(t) + \cdots + \frac{p^*}{n} \int_a^b y^{(n-1)}(t + \xi) d\sigma(t)$$

As in Theorem 6, if $y(t)$ is a nonoscillatory positive solution of degree $n$ of

$$y(t) = y(t) + \frac{p^*}{n} \int_a^b y^{(n-1)}(t + \xi) d\sigma(t)$$

then we have $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$, ..., $y^{(n-1)}(t) > 0$, $y^{(n)}(t) > 0$, that is, $y(t)$ is a nonoscillatory positive solution of

$$y(t) - p^* \int_a^b y^{(n)}(t + \xi) d\sigma(t) > 0$$

Using the condition $\frac{p^*}{n} > 1$, from Lemma 3 this led to a contradiction.

Secondly, suppose $h(t, t) > t^k$ for $t \in [a, b]$, $b > a > 0$, $t > T$.

Then we can easy obtain

$$0 < y^{(n-1)}(t) + \frac{p^*}{n} \int_a^b y^{(n-1)}(t + \xi) d\sigma(t) + \cdots + \frac{p^*}{n} \int_a^b y^{(n-1)}(t + \xi) d\sigma(t)$$

where $y(t)$ is a nonoscillatory positive solution of degree $n$ of (1.4) i.e. $y(t) > 0$, $y'(t) > 0$, ..., $y^{(n)}(t) > 0$.

So we have

$$y(t) = y(t) - \frac{p^*}{n} \int_a^b y^{(n-1)}(t + \xi) d\sigma(t)$$

As in Theorem 6, we can easily obtain a contradiction.

2. Suppose (3.5) is satisfied. Firstly, suppose $h(t, t) = ut^k$ for some $u$, which satisfies $u > 1$, $t \in [a, b]$, $t > T$.

Set

$$Z(t) = y^{(n)}(t) + \frac{u^*}{n} \int_a^b y^{(n-1)}(t + \xi) d\sigma(t) + \frac{u^*}{n} \int_a^b y^{(n-1)}(t + \xi) d\sigma(t) + \cdots + \frac{u^*}{n} \int_a^b y^{(n-1)}(t + \xi) d\sigma(t)$$

As in Theorem 6, if $y(t)$ is a nonoscillatory positive solution of degree $n$ of

$$y(t) - p^* \int_a^b y^{(n)}(t + \xi) d\sigma(t) > 0$$

where $p^* = \sqrt{n} > 0$.

So we have

$$y(t) = y(t) - \frac{u^*}{n} \int_a^b y^{(n-1)}(t + \xi) d\sigma(t)$$

As in Theorem 6, we can easily obtain a contradiction.
If we choose $u_{n-2}, u_{n-3}, \ldots, u_n, u_0$, which satisfies

$$u_{n-2} = \frac{1}{n \cdot \omega}, \quad u_{n-1} = \frac{1}{(n-1) \cdot \omega}, \quad \ldots, \quad u_n = \frac{1}{(n + 1) \cdot \omega},$$

i.e.,

$$u_{n-2} = \frac{\omega}{n}, \quad u_{n-1} = \frac{\omega}{n-1}, \quad \ldots, \quad u_n = \frac{\omega}{n + 1},$$

then we have

$$y^{(n)} - p(t) \int_{t-h}^{t} y^{(n)}(s) \, ds \geq 0,$$

so we deduce that $z(t)$ is a nonoscillatory positive solution of the following inequality.

$$\int_{t-h}^{t} y^{(n)} - p(t) \int_{t-h}^{t} y^{(n)}(s) \, ds \, ds \geq 0.$$

Using Lemma 3, we can obtain a contradiction.

Secondly, suppose $h(t, x) = x^2$, for some $w$, which satisfies $w^2 > 1$, for $t \geq a$, and all $t \geq t_0$, then we can easily obtain that

$$0 \leq y^{(n)} - p(t) \int_{t-h}^{t} y^{(n)}(s) \, ds \, ds \int_{t-h}^{t} y(s) \, ds \leq y^{(n)}(t) - p(t) \int_{t-h}^{t} y^{(n)}(s) \, ds \, ds \int_{t-h}^{t} y(s) \, ds,$$

where $y(t)$ is a positive solution of degree $n$ of (1.4), i.e.,

$$y(t) > 0, \quad y(t) > 0,$$

$$y(t) > y(t), \quad y(t) > y(t).$$

As in Theorem 5, we can easily obtain a contradiction.

(3) In the case (1.2). The proof of the Theorem is obvious. Proof of Theorem 7 is complete.

Theorem 8. For the general superlinear inequality (1.8)', if (3.4) or (3.5) is satisfied, then we have $\mathcal{N} = \emptyset$, $\mathcal{N} = \emptyset$.

The proof of Theorem 8 is similar to Theorem 4.

Remark. From Theorem 4 and Theorem 8 we can see that if deviating arguments are taken forms as $t^a, t^a, t^a, t^a$, then for superlinear inequality (2.4), $N_0 = 0$, $N = N_0$, and for (3.4)

$$N_0 = 0, \quad N = N_0.$$ Notice that for the superlinear case we do not need such conditions, as (2.4), (3.3) to have $N_0 = \emptyset$, $N = N_0$. 4. \textbf{EQUATIONS WITH ADVANCED AND RETARDED ARGUMENTS.}

In this section we are concerned with the differential equations with both retarded and advanced arguments,

$$y^{(n)} - p(t) \int_{t-h}^{t} y^{(n)}(s) \, ds \, ds \geq 0,$$

where $p > 0, q > 0, a_1(t)$ and $a_2(t)$ are continuous nondecreasing functions.

Theorem 9. For the equation (4.1), if one of the following conditions is satisfied:

(1) $g(t) \leq 1$, for $t \geq a$, $t \geq a$, all $t \geq T$ and $\frac{p(t)}{a_1(t)} > 1$. 

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\[ \text{\textbf{Theorem 9}} \]

We consider the following inequality:

\[ \left\{ \sum_{\nu=0}^{n-1} (x_{\nu+1} - x_{\nu})^2 \right\} \geq 0. \]  \[ (5.1) \]

If \( n = 2 \), then we have

\[ \left\{ \sum_{\nu=0}^{n-1} (x_{\nu+1} - x_{\nu})^2 \right\} \geq 0. \]

where \( a = \frac{1}{2} \), \( b = \frac{1}{2} \). For any \( n \) we have

\[ \left\{ \sum_{\nu=0}^{n-1} (x_{\nu+1} - x_{\nu})^2 \right\} \geq 0. \]

Notice that

\[ \left\{ \sum_{\nu=0}^{n-1} (x_{\nu+1} - x_{\nu})^2 \right\} \geq 0. \]

So we can obtain that if \( p > 0, b > a > 0 \), \( \sqrt{p/n} \rightarrow e > 1 \), then for \( (5.1) \) \( \mathcal{N}^n \).

\[ \text{\textbf{Example 2}} \]

\[ \left\{ \sum_{\nu=0}^{n-1} (x_{\nu+1} - x_{\nu})^2 \right\} \geq 0. \]  \[ (5.2) \]

where \( p > 0, b > a > 0 \). If \( k \) is such that \( kb < 1 \), then for \( (5.2) \) \( \mathcal{N}^n \). If \( k \) is such that \( ka > 1 \), then for \( (5.2) \) \( \mathcal{N}^n \).

\[ \text{\textbf{Example 3}} \]

\[ \left\{ \sum_{\nu=0}^{n-1} (x_{\nu+1} - x_{\nu})^2 \right\} \geq 0. \]  \[ (5.3) \]

If \( \sqrt{p/n} \rightarrow e > 1 \), then for \( (5.3) \) \( \mathcal{N}^n \) \( (p > 0, b > a > 0) \).
where \( p > 0, q > 0, b_i > a_i > 0 \) \((i=1,2)\). \( 0 < \frac{b_1}{a_1} < 1, \frac{b_2}{a_2} > 1 \). Then (5.4) oscillates.

**Example 5.**

\[
\phi^{(\alpha)}(x) = p f_1^{(\alpha)} \int_{a_1}^{b_1} f_2(y \xi - \xi - \cdots - \xi_1) \, d\xi \cdots \, d\xi_n + q f_1^{(\alpha)} \int_{a_1}^{b_1} f_2(y \xi - \xi - \cdots - \xi_1) \, d\xi \cdots \, d\xi_n \quad \tag{5.5}
\]

where \( f_1 \) and \( f_2 \) are superlinear. Then (5.5) oscillates.

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**REFERENCES**

