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M.H.A. Hassan

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M.K. Wallis

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M.H.A. Hassan

International Centre for Theoretical Physics, Trieste, Italy.

and

M.K. Wallis

Department of Applied Mathematics and Astronomy,

University College Cardiff, Wales, United Kingdom.

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** Permanent address: School of Mathematical Science, University of Khartoum, Khartoum, Sudan.

ABSTRACT

The effects of the sectored Interplanetary Magnetic Field on charged dust grains orbiting around the Sun under radiation pressure and Poynting - Robertson drag forces are examined for initially circular and non-inclined orbits. The distribution function of the charged grains satisfies a Fokker- Planck equation in which the sectored field is taken as a source of stochastic impulses. By adopting the integrals of the impulse-free motion as variable parameters, the Fokker- Planck equation can be properly treated as a diffusion equation.

Analytic solutions of the resulting diffusion equation show that dust grains injected near the ecliptic plane are scattered strongly to high helio -latitudes. The scattering is more pronounced for small grains injected at large distances from the Sun.

INTRODUCTION

It is well known that Kepler orbits of small interplanetary dust grains moving under the gravitational attraction of the Sun are modified by radiation pressure and the velocity dependent Poynting- Robertson drag force (for a recent review see Burns et al., 1979). The motion of grains with density ρ and radii s, such that $\rho s > 5.7 \times 10^{-5}$ is dominated by the force of gravity and the effect of radiation pressure is simply to modify the gravitational constant. Under Poynting -Robertson forces the orbits of the grains are predicted to become more circular and spiral in towards the Sun. Several authors (Parker 1964; Belton 1966; and Wyatt 1969) noted that solar particles and solar radiation may charge interplanetary dust grains to a positive potential of up to 10 V. They also recognized that the Stochastic Interplanetary Magnetic Field (IMF) carried by the solar wind exerts an additional force on these charged grains which may significantly perturb their orbits.

In order to analyse the effects of fluctuating electromagnetic forces on charged dust particles following Keplerian orbits, two methods appear to have been employed. The first method (Parker 1964; Morfill and Grün 1979) assumes the principal perturbing electromagnetic force to be perpendicular to the plane of the orbital motion. The evolution of the particles distribution function is assumed to satisfy a simple one-dimensional diffusion equation in which all non-fluctuating forces as well as dynamical friction terms are neglected. The analysis is restricted to non-inclined and circular orbits and the results show significant changes in inclinations only for particles with radii equal to one micron or less. The other more recent approach examines the effects of the stochastic electromagnetic force on the osculating orbital elements of the dust grains (Consolmagno 1979; Barge et al. 1982) in the absence of Poynting-Robertson forces. The distribution function is shown to satisfy a Fokker-Planck diffusion equation in which the orbital elements (Consolmagno 1979) or the constants of unperturbed motion (Barge et al. 1982) are used as variable parameters . However, no solution of the diffusion equation is given.

the results of Consolmagno show significant perturbations of the orbital elements of grains smaller than 3 microns, while the results of Barge et al. give significant spread in inclination only for grains smaller than 0.5 microns.

In a previous paper we have obtained a general solution of the Fokker- Planck equation describing small stochastic changes to particles in Kepler orbits and we have applied the solution to the problem of inter-planetary gas. In this report we wish to apply this general solution to the problem of charged dust particles moving under gravity, radiation pressure and Poynting-Robertson forces, while subjected to a stochastic electro-magnetic force due to the sectored IMF. The stochastic model is described in Section 2. The Fokker-Planck equation dscribing the evolution of the distribution function is reduced to a diffusion-type equation by adopting the constants of the impulse- free motion as variable parameters. Because of Poynting-Robertson forces, the energy and angular momentum are not conserved quantities and the constants of motion derived here are different from those used by Barge et al. (1982). In section 3 we evaluate the Fokker-Planck coefficient in terms of the power spectrum of the fluctuating IMF. For simplicity and for purposes of illustration of our procedure, the final expression for the dispersion matrix is obtained for the case in which the initial particles orbits are non-inclined and nearly circular. However the development can be readily extended to cover the cases of inclined and elliptic orbits. The result will be reported elsewhere. In section 4 we examine in some detail variations in orbit inclinations induced by the stochastic electromagnetic force. The mean spread in inclination is found to be proportional to $\frac{r_s}{r}$ (where r_s is the initial distance of grains and r is the distance after time t). This is different from the result of Morfill and Grün (1979) which is proportional to $\frac{r_s}{r} \ln \frac{r_s}{r}$. Finally a summary of our results is given in section 5.

2. FORMULATION OF THE STOCHASTIC MODEL

Consider a distribution of non-interacting charged dust particles [illegible] an inclination w_3 at an initial distance r_s from the Sun. The particles move under the gravitation attraction of the Sun while subjected to radiation pressure, the Poynting-Robertson drag force and a stochastic electromagnetic force due to the sectored Interplanetary Magnetic Field.

In the absence of the stochastic force, the equation of motion of an individual grain is given by

$$\ddot{\underline{r}} = -\frac{\mu}{r^2}\hat{\underline{r}} - \frac{\bar{\mu}}{r^2}\left(2\dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}\right) , \qquad (1)$$

where r and θ define the plane of the orbital motion;

$\mu = GM(1-\beta)$ is the solar gravitational constant reduced by radiation pressure and $\bar{\mu} = \beta/c$.

There are two constants of motion satisfying (1):

$$w_1 = V_r^2 + V_\theta^2 - \frac{2\mu}{r} + g(t) \qquad (2)$$

$$w_2 = rV_\theta + \bar{\mu}\theta . \qquad (3)$$

where

$$g(t) = 2\bar{\mu}\int\left(2V_r^2 + V_\theta^2\right)\frac{dt}{r^2} . \qquad (4)$$

The equation of the orbit can readily be obtained from (1)

$$\frac{d^2u}{d\theta^2} + \frac{\bar{\mu}}{\bar{w}_2}\frac{du}{d\theta} + u = \frac{\mu}{\bar{w}_2^2} . \qquad (5)$$

where

$u = \frac{1}{r}$, $\bar{w}_2 = w_2(1-2\theta\gamma)$ and $\gamma = \frac{\bar{\mu}}{2w_2}$.

The solution of (5) for small Poynting-Robertson effect ($\gamma \ll 1$) was found by Robertson (1937)

$$r = \frac{w_2^2}{\mu} \left\{ 1 + e \cos\theta + \gamma\theta \left(4 - e \cos\theta \right) \right\}^{-1} . \qquad (6)$$

Let us now assume that the stochastic electromagnetic force carried by the sectored Interplanetary Magnetic Field causes small perturbations to the constants of motion w_1, w_2 and w_3. Adopting these constants as variable parameters instead of the velocity variables in the Fokker-Planck equation, the distribution function of the charged dust grains is found to satisfy the diffusion-type equation (Hassan and Wallis 1983)

$$\frac{df}{dt} = - c f - a_i \frac{\partial f}{\partial w_i} + \frac{1}{2} b_{ij} \frac{\partial^2 f}{\partial w_i \partial w_j} , \qquad (7)$$

where

$$c = \frac{\partial}{\partial w_i} \frac{\langle \Delta w_i \rangle}{\tau} - \frac{1}{2} \frac{\partial^2}{\partial w_i \partial w_j} \frac{\langle \Delta w_i \Delta w_j \rangle}{\tau}$$

$$a_i = \frac{\langle \Delta w_i \rangle}{\tau} - \frac{\partial}{\partial w_j} \frac{\langle \Delta w_i \Delta w_j \rangle}{\tau} \qquad (8)$$

$$b_{ij} = \frac{\langle \Delta w_i \Delta w_j \rangle}{\tau}$$

Δw_i are the stochastic changes caused by the perturbing force. The left-hand side of (7) has to be evaluated along the unperturbed orbit satisfying (6). τ is an arbitrary time interval which is larger than the time scale of fluctuations.

The solution of (7) is given by Hassan and Wallis (1983) and is found to be the general normal distribution

$$f(w_i, \tau) = (2\pi)^{-1} |B_{ij}|^{-\frac{1}{2}} \exp\left\{ - \lambda - \tfrac{1}{2} (w_i - A_i) B_{ij}^{-1} (w_j - A_j) \right\} \qquad (9)$$

in which

$$\lambda = \int_{t_0}^{t} c \, dt , \quad A_i = A_i^0 + \int_{t_0}^{t} a_i \, dt , \quad B_{ij} = B_{ij}^0 + \int_{t_0}^{t} b_{ij} \, dt$$

and A_i^0, B_{ij}^0 determine the initial value of f.

3. EVALUATION OF THE FOKKER - PLANCK COEFFICIENTS

The first order changes Δw_i are related to the velocity changes by the relations

$$\Delta w_1 = 2 \left(V_r \Delta V_r + V_\theta \Delta V_\theta \right)$$

$$\Delta w_2 = r \Delta V_\theta \qquad (10)$$

$$\Delta w_3 = \frac{r \Delta V_n}{w_3} ,$$

where V_θ and V_r are readily obtained from (3) and (6)

$$V_\theta = \frac{w_2}{r} \left(1 - 2\gamma\theta \right) \qquad (11)$$

$$V_r = \frac{\mu}{w_2} \left(1 - 4\gamma\theta \right) \left\{ e \sin\theta - \gamma \left(4 + e\,\theta \sin\theta - e \cos\theta \right) \right\}$$

and ΔV_n is the instantaneous velocity increment in the direction normal to the plane of the orbit.

The velocity changes ΔV_r, ΔV_θ and ΔV_n are calculated from the Lorentz equation

$$\frac{d\underline{V}}{dt} = \frac{q}{m} \left(\underline{E} + \frac{\underline{V} \times \underline{B}}{c} \right) , \qquad (12)$$

where q is the charge on the dust grain, $\underline{U}$ is the velocity of the solar wind in the radial direction ($U \sim 400$ km s^{-1}), $\underline{V}$ is the orbital velocity and $\underline{B}$ is the stochastic Interplanetary Magnetic Field.

Since $V \ll U$, we shall thereafter replace (12) by the approximate result

$$\frac{dV}{dt} = -\frac{q}{mc}\, U \times B = -\frac{qU}{mc}\left(B_\theta \hat{n} - B_n \hat{\theta} \right) . \tag{13}$$

It should be noted that although the omission of the term $\frac{q}{mc} V \times B$ simplifies the algebra considerably; its inclusion is a straightforward extension of the development below. For purposes of illustration, however, it will sufficient to restrict our attention to nearly circular orbits for which (13) is a good approximation (Barge et al., 1982).

It is then readily seen from (13) that

$$\Delta V_r = 0$$

$$\Delta V_\theta = \frac{qU}{mc} \int_0^\tau B_n \, dt \tag{14}$$

$$\Delta V_n = -\frac{qU}{mc} \int_0^\tau B_\theta \, dt .$$

If the initial inclination of the orbits is small enough we may assume that the components B_θ and B_n of the IMF have the form (Consolmagno 1979; Barge et al., 1982):

$$B_\theta = \frac{B_0 r_0}{r} h(t) \quad , \qquad B_n = \varepsilon B_\theta \quad , \tag{15}$$

where $B_0 \simeq 3 \times 10^{-5}$ G, $r_0 \simeq$ IAU and $\varepsilon \simeq 10^{-2}$.

The term $\frac{B_0 r_0}{r}$ describes the average IMF at a distance r from the Sun (Parker 1958) and the function $h(t)$ describes the stochasticity of the field due to the sectored IMF.

The calculations of the coefficients $\frac{\langle \Delta w_i \rangle}{\tau}$ and $\frac{\langle \Delta w_i \Delta w_j \rangle}{\tau}$ are now easily accomplished in terms of the power spectrum of the fluctuating field. Since the power spectrum is related to the auto-correlation function through the relation (Ichimaru 1973):

$$\langle B_i(t) B_j(t-s) \rangle = \int_{-\infty}^{\infty} d\omega \, \langle | B_{ij}(\omega) |^2 \rangle \, e^{i\omega s} , \tag{16}$$

where $B_1 = B_\theta$, $B_2 = B_n$ and $|B_{ij}(w)|^2 = B_i(w) B_j(-w)$.

It is readily found that

$$\langle \Delta w_i \rangle / \tau = 0$$

and

$$\frac{\langle \Delta w_i \Delta w_j \rangle}{\tau} = \begin{bmatrix} \frac{4 \varepsilon^2 V_\theta^2}{r^2} & 2\frac{\varepsilon^2 V_\theta}{r} & -\frac{2\varepsilon}{r w_2} \\ \frac{2 \varepsilon^2 V_\theta}{r} & \varepsilon^2 & \frac{\varepsilon}{w_2} \\ -\frac{2\varepsilon}{r w_2} & \frac{\varepsilon}{w_2} & \frac{1}{w_2^2} \end{bmatrix} \frac{q^2 U^2 B_0^2 r_0^2}{m^2 c^2} J \tag{17}$$

where

$$J = \int_{-\infty}^{\infty} \frac{| e^{i\omega\tau} - 1 |^2}{\omega^2 \tau} \langle | \bar{h}(\omega) |^2 \rangle \, d\omega \tag{18}$$

The evaluation of the orbital integrals $\int_{t_0}^{t} b_{ij} \, dt \equiv \int_{\theta_0}^{\theta} \frac{r^2}{w_2} b_{ij} \, d\theta$

can be achieved upon using (6) and (11) to express r and V_0 in terms of θ . Thus using (17) leads to

$$B_{ij} = B_{ij}^0 + \begin{bmatrix} 4 \delta^2 \lambda_1 & 2 \delta^2 \lambda_2 & -2 \delta \lambda_3 \\ 2 \delta^2 \lambda_2 & \delta^2 \lambda_4 & \delta \lambda_4 \\ -2 \delta \lambda_3 & \delta \lambda_4 & \lambda_4 \end{bmatrix} \frac{4 q^2 U^2 B_0^2 r_0^2}{m^2 c^2 w_2^2} J \tag{19}$$

in which

$$\lambda_1 = \frac{\mu^2}{w_2^3} \left\{ \theta \left(1 + \frac{e^2}{2}\right) + 2e \sin\theta + \frac{e^2}{4} \sin 2\theta + \delta \left[\theta^2 (3 - e^2) + 2e\theta \sin\theta + 2e \cos\theta - e^2 \theta \sin 2\theta - \frac{1}{2} e^2 \cos 2\theta \right] \right\} - \frac{\mu^2}{w_2^3} \{ \theta \longleftrightarrow \theta_0 \}$$

$$\lambda_2 = \theta - \theta_0$$

$$\lambda_3 = \frac{1}{\sqrt{1 - e^2}} \left\{ \cos^{-1} \left(\frac{e + \cos\theta}{1 + e\cos\theta} \right) - \cos^{-1} (\theta \longleftrightarrow \theta_0) \right\}$$

$$\lambda_4 = t - t_0 \tag{20}$$

and

$$\delta = \varepsilon w_1 \quad .$$

4. DIFFUSION IN INCLINATION

To compare our results with those of Morfill and Grün (1979) and Consolmagno (1979), we consider in this section the evaluation of the root mean square spread in the orbital inclination. Thus in view of (9), (19) and (20) it is easily seen that

$$\langle w_3^2 \rangle^{\frac{1}{2}} = \frac{q U B_0 r_0}{2^{\frac{1}{2}} m c w_2} \, J^{\frac{1}{2}} (t - t_0)^{\frac{1}{2}} \quad . \tag{21}$$

If the Poynting-Robertson effect is neglected, $w_2 = \sqrt{\mu a(1 - e^2)}$ (where a is the semi-major axis of the elliptical orbit) and (21) agrees with the result of Consolmagno (1979) for the case of initially non-inclined orbits.

For nearly circular orbits, the use of (5) in (3) gives the long-established result (Robertson 1937; Wyatt et al. 1950):

$$t - t_0 \simeq \frac{r_s^2}{4\bar{\mu}} \left(1 - \frac{r^2}{r_s^2} \right) \tag{22}$$

Thus substituting (22) with $w_2 = (\mu r)^{1/2}$ in (21) gives

$$\langle w_3^2 \rangle^{\frac{1}{2}} = \frac{q U B_0 r_0^{\frac{3}{2}}}{2\sqrt{2}\, m c \sqrt{\mu \bar{\mu}}} \left(\frac{r_s}{r_0}\right)^{\frac{1}{2}} \left(\frac{r_s}{r}\right)^{\frac{1}{2}} \left(1 - \frac{r^2}{r_s^2}\right)^{\frac{1}{2}} J^{\frac{1}{2}} \tag{23}$$

In Figs.1-5 we plot $\langle w_3^2 \rangle^{1/2}$ as a function of $\frac{r}{r_s}$ for various values of r_s and s taking the density, $\rho = 3.9$ cm^{-3}, the voltage V = 10V, the solar wind speed $U = 4 \times 10^7$ cm sec^{-1} and the power spectrum [illegible]. This value of the power spectrum agrees with observations (Consolmagno 1979; Jokippi and Coleman 1968).

In general the spread in inclination increases as the ratio $\frac{r}{r_s}$ approaches the limiting value $\frac{R_\odot}{r_s}$. This is in disagreement with the result of Morfill and Grün (1979), which gives smaller values in this limit. The source of the disagreement can be traced to the fact that non-stochastic external forces are not incorporated correctly in the simple diffusion equation used in their analysis.

5. SUMMARY

As demonstrated by our general approach (Hassan and Wallis 1983), the Fokker-Planck equation describing the evolution of charged dust grains moving around the Sun under radiation pressure and the Poynting-Robertson force, and subjected to stochastic impulses due to the sectored IMF, can be properly treated as a diffusion equation by adopting the integrals of impulse-free motion as variable parameters . Analytic solutions of the resulting diffusion equation show that the mean spread in particle inclinations, for initially non-inclined and circular orbits, depends strongly on the size of particles and is found to be proportional to the power spectrum of the random IMF, the voltage on the grains, the initial distance from the Sun and the ratio $\frac{r}{r_s}$.

In general, the mean spread in inclination increases with the initial distance from the Sun and with the ratio $\frac{r}{r_s}$, and is more pronounced for small grains. As an example, the magnitude of this scattering for grains ejected near the ecliptic plane at a distance r_s = 10 AU, after reaching, under the Poynting-Robertson force, a distance r = 0.1 AU from the Sun, is about 6° for 5×10^{-4} cm grains, 8° for 4×10^{-4} cm grains, 13° for 3×10^{-4} cm grains, 24° for 2×10^{-4} cm grains and 70° for 10^{-4} cm grains.

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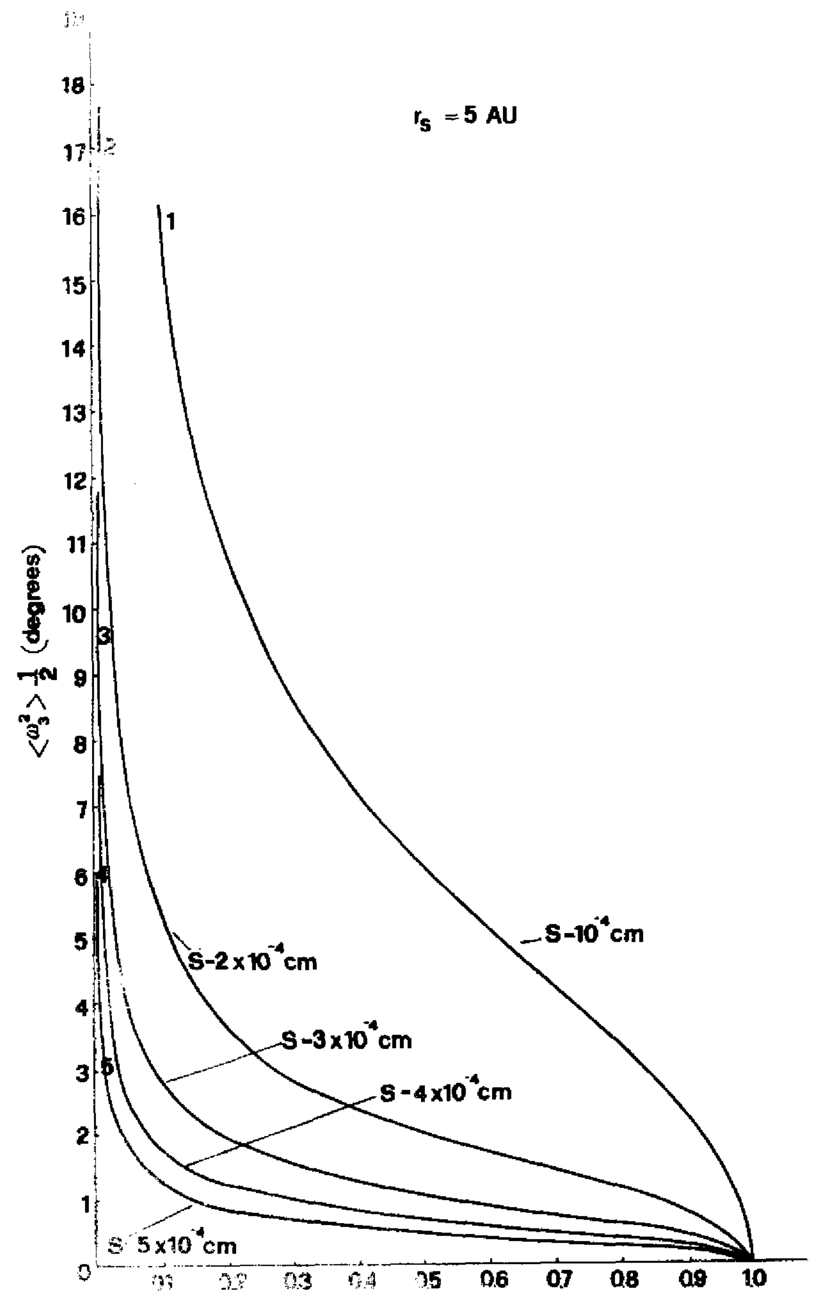
FIGURE CAPTIONS

Figs.1-5 The root mean-square-spread in inclination of particles injected at various distances from the Sun, plotted as a function of the distance from the Sun.



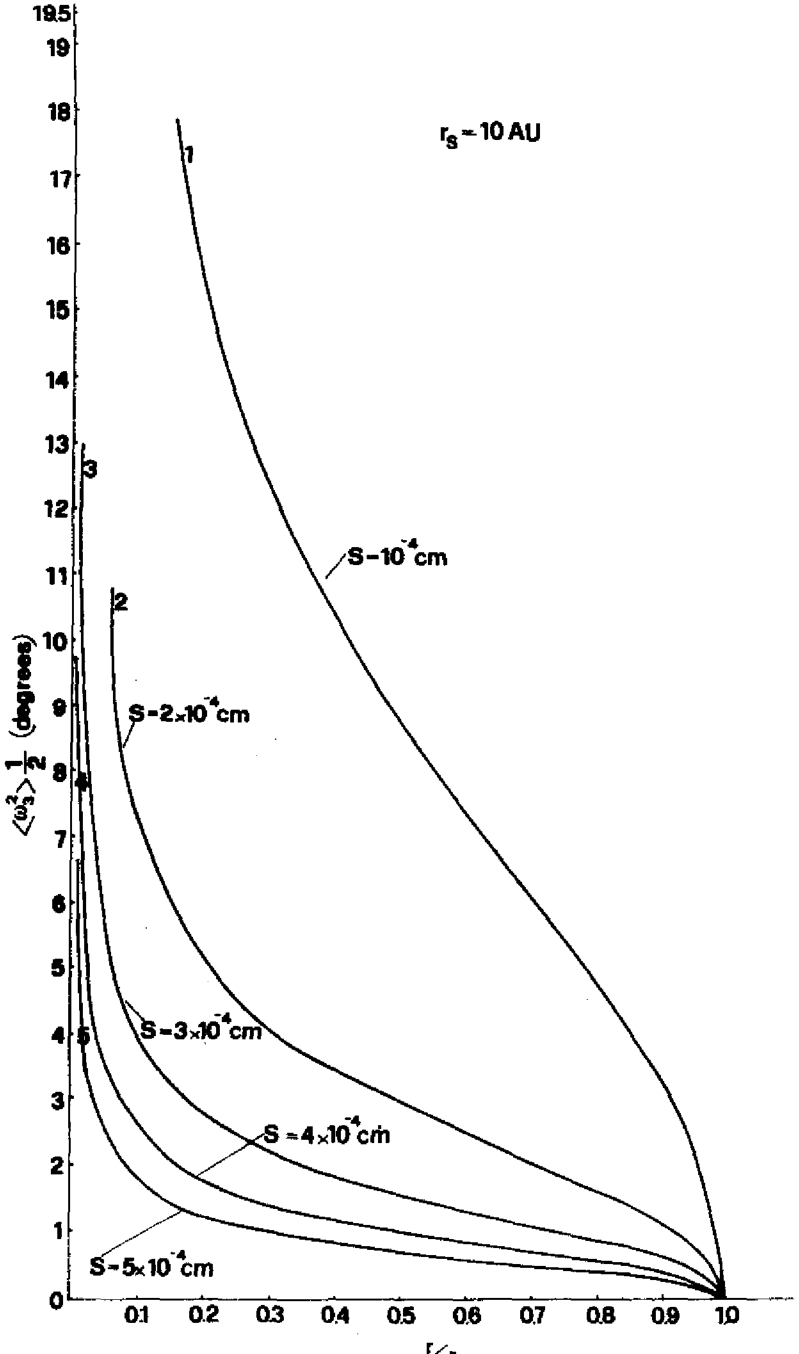

r_s = 10 AU
S = 10^{-4} cm
S = 2×10^{-4} cm
S = 3×10^{-4} cm
S = 4×10^{-4} cm
S = 5×10^{-4} cm
$\langle\omega_3^2\rangle^{\frac{1}{2}}$ (degrees)
r/r_s

Fig.2

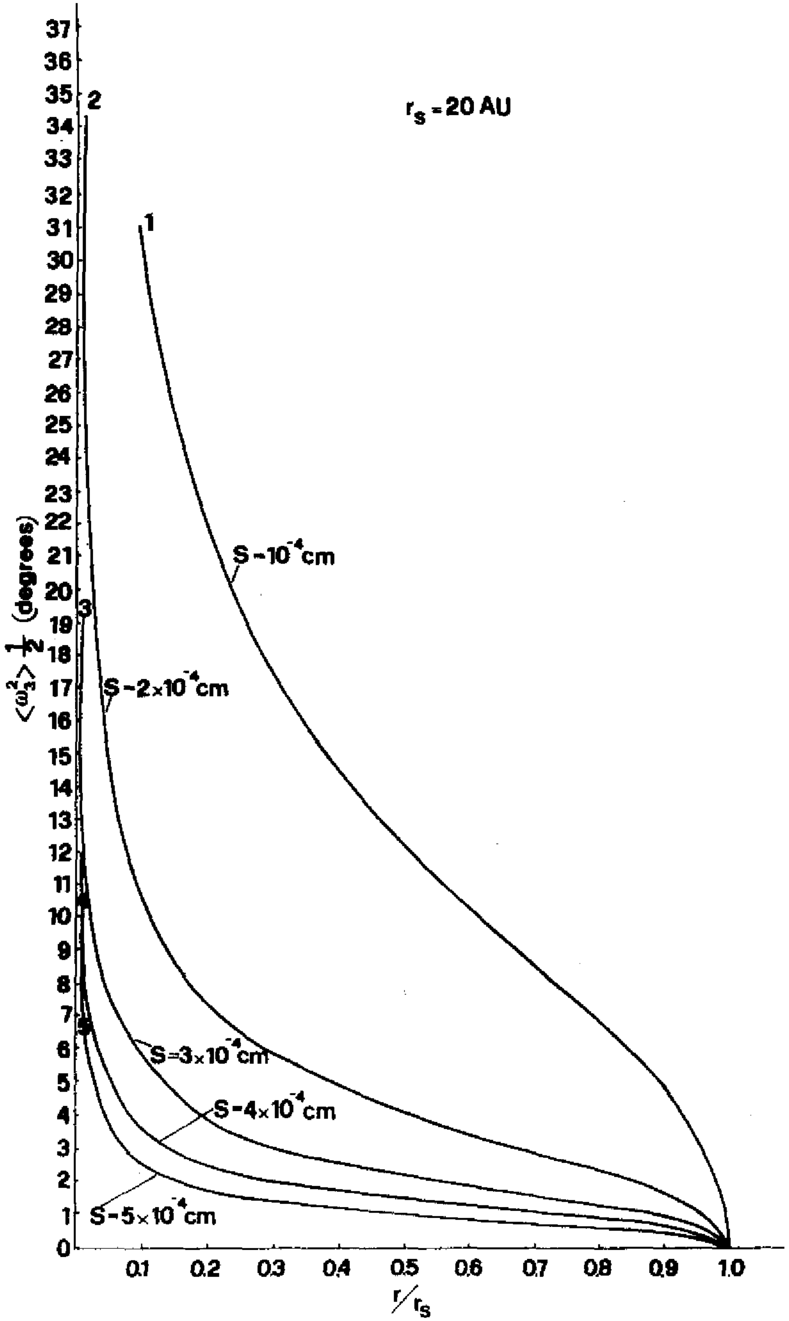
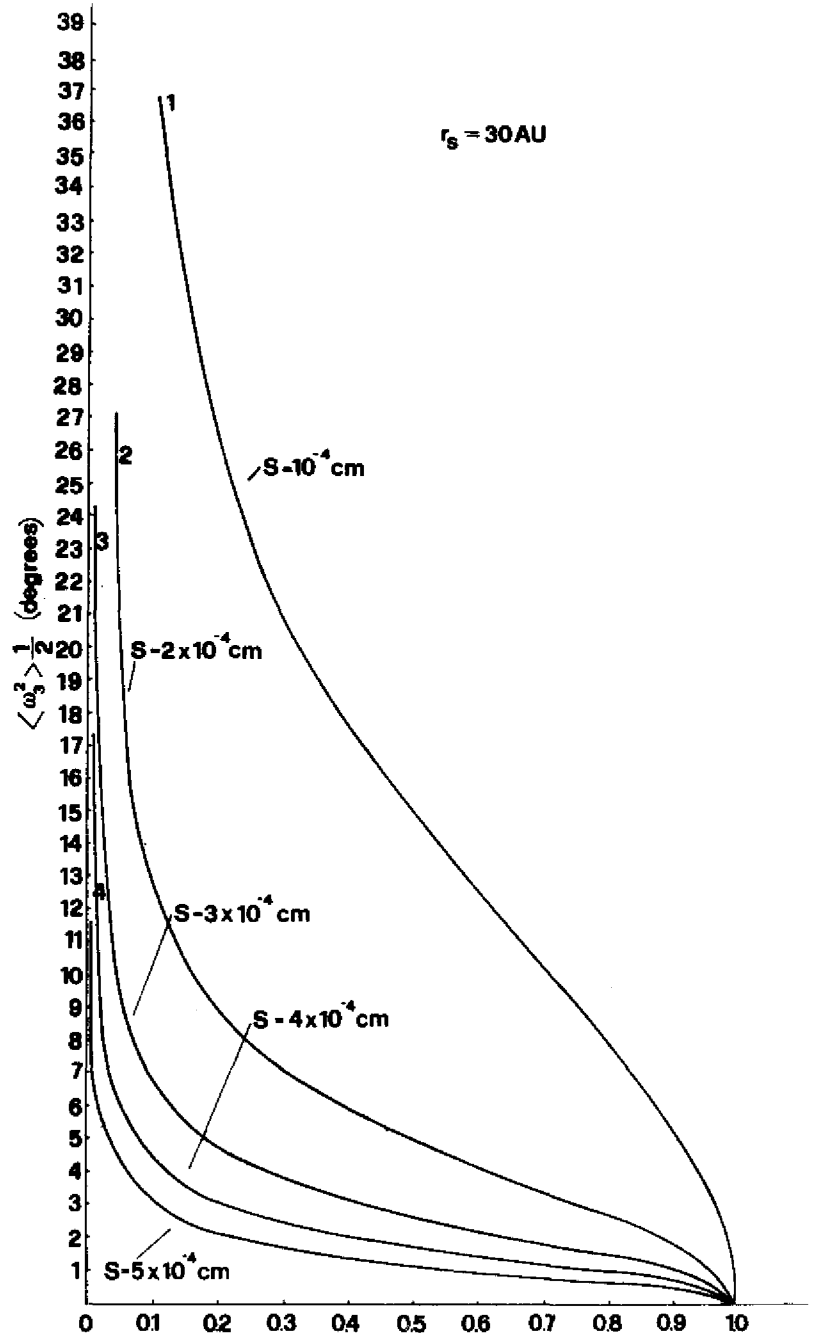
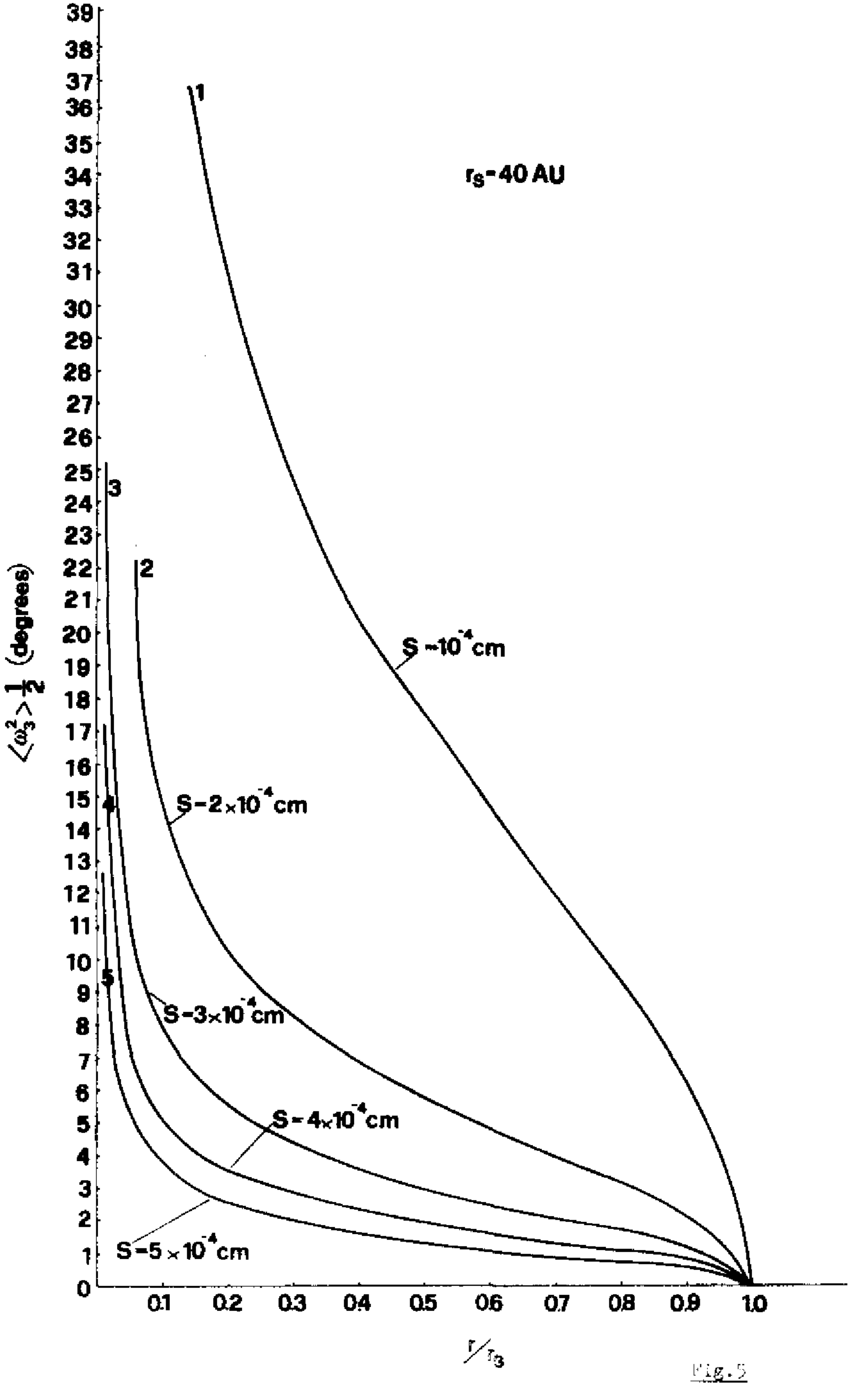

r_s = 20 AU
S = 10^{-4} cm
S = 2×10^{-4} cm
S = 3×10^{-4} cm
S = 4×10^{-4} cm
S = 5×10^{-4} cm
$\langle\omega_3^2\rangle^{\frac{1}{2}}$ (degrees)
r/r_s

Fig.3



$r_s = 30\,AU$
$\langle \omega_3^2 \rangle^{\frac{1}{2}}$ (degrees)
1
2
3
4
S-10^{-4}cm
S-2x10^{-4}cm
S-3x10^{-4}cm
S-4x10^{-4}cm
S-5x10^{-4}cm
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

Fig.4



r_s-40 AU
$\langle \omega_3^2 \rangle^{\frac{1}{2}}$ (degrees)
1
2
3
4
5
S-10^{-4}cm
S-2×10^{-4}cm
S-3×10^{-4}cm
S-4×10^{-4}cm
S-5×10^{-4}cm
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
r/r_3

Fig.5