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ABSTRACT

The magnetic field line Hamiltonian and the associated canonical form for the magnetic field are important concepts both for understanding toroidal plasma physics and for practical calculations. A number of important properties of the canonical or Hamiltonian representation are derived a.d their importance is explained.

I. INTRODUCTION

Significant motivation is required for a physicist to become interested in an abstract concept like the magnetic field line Hamiltonian. Therefore, before we define precisely what the canonical Hamiltonian is, let us consider why the concept is important to toroidal plasma physics.[1] First, the Hamiltonian description allows one to determine the extent to which the magnetic field lines lie on nested tori throughout a finite volume by doing only a few field line integrations.[2,3] Second, it clarifies the constraints of Faraday's law and ideal MHD on the formation of islands or other topological changes in the magnetic field structure. Third, it separates the problem of finding the shape of the field from finding its topology. By topology, we mean the presence or absence of islands, stochastic regions, etc. Fourth, magnetostatic equilibrium

$$\vec{\nabla} p = \frac{1}{c} \vec{j} \times \vec{B}$$

is satisfied if the shape of the magnetic field is chosen to extremize the sum of the magnetic and plasma energy with fixed magnetic field line Hamiltonian. This implies that the energy principle can be extended from a study of the sensitivity of plasma equilibrium to small changes in shape to also study the sensitivity to small changes in topology, which are small magnetic islands. Fifth, the magnetic field line Hamiltonian is used in the particle drift Hamiltonian to evaluate the motion of charged particles in stochastic magnetic fields.[4,5]

II. MAGNETIC HAMILTONIAN

Let us first understand the concept of a magnetic field line Hamiltonian or, more briefly, magnetic Hamiltonian.[6] Suppose a toroidal magnetic configuration is described using cylindrical coordinates, R, ϕ, z (see Fig. 1). For simplicity, assume B_ϕ does not vanish in the volume of interest. One can show[2] that such a magnetic field can always be written in a form which will be called canonical

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\theta + \vec{\nabla}\phi \times \vec{\nabla}\Psi_p(\psi,\theta,\phi) \tag{1}$$

with θ the canonical poloidal angle (see the appendix).

The canonical form for the magnetic field, Eq. (1), requires interpretation. There are two coordinate systems implied, the ordinary cylindrical coordinates R, ϕ, z and the canonical coordinates ψ, θ, ϕ. The poloidal angle θ is the canonical coordinate of the magnetic Hamiltonian. The toroidal flux function ψ is the action-like canonical momentum, and ϕ is the time-like coordinate of the Hamiltonian system. The magnetic Hamiltonian is $\Psi_p(\psi,\theta,\phi)$. Magnetic field lines are traced by making a succession of infinitesimal steps $\delta\vec{x}$ along the magnetic field. For small enough δ,

$$\delta\vec{x} = \vec{B}\delta \quad . \tag{2}$$

Using $\delta\psi = \vec{\nabla}\psi\cdot\delta\vec{x}$, etc., one can show

$$\frac{d\psi}{d\phi} = \frac{\vec{B}\cdot\vec{\nabla}\psi}{\vec{B}\cdot\vec{\nabla}\phi} \qquad \frac{d\theta}{d\phi} = \frac{\vec{B}\cdot\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\phi} \quad . \tag{3}$$

If the indicated dot products are formed with $\vec{B}$ of Eq. (1), then

$$\frac{d\psi}{d\phi} = -\frac{\partial\Psi_p}{\partial\theta} \qquad \frac{d\theta}{d\phi} = \frac{\partial\Psi_p}{\partial\psi} \quad , \tag{4}$$

which are Hamilton's equations.

The canonical form for the magnetic field, Eq. (1), therefore, consists of two parts. First, there is the magnetic Hamiltonian $\Psi_p(\psi,\theta,\phi)$. Second, there are the transformation equations $R(\psi,\theta,\phi)$ and $z(\psi,\theta,\phi)$ from the canonical coordinates to ordinary cylindrical coordinates. The basic structure or topology, like the existence of magnetic islands, is determined by Ψ_p. The shape of the magnetic configuration is determined by R and z.

One implication of Eq. (1) is that any three smooth functions, $\Psi_p(\psi,\theta,\phi)$, $R(\psi,\theta,\phi)$, and $z(\psi,\theta,\phi)$, which are periodic in θ and ϕ, define a unique magnetic field. To give this field explicitly, the Jacobian of the transformation must be defined

$$D = \frac{\partial R}{\partial\theta}\frac{\partial z}{\partial\psi} - \frac{\partial R}{\partial\psi}\frac{\partial z}{\partial\theta} \quad . \tag{5}$$

Using standard Jacobian identities, one can show

$$(\vec{\nabla}\psi \times \vec{\nabla}\theta)\cdot\vec{\nabla}\phi = \frac{1}{D} \quad . \tag{6}$$

From Eq. (1), the canonical form for $\vec{B}$, it then follows that

$$D = \frac{1}{\vec{B}\cdot\vec{\nabla}\phi} \quad . \tag{7}$$

Using $\vec{B} = \vec{B}\cdot\vec{\nabla}\vec{x}$, one finds using Eqs. (3) and (4) that

$$\vec{B} = \frac{1}{D}\left(\frac{\partial\vec{x}}{\partial\phi} + \frac{\partial\Psi_p}{\partial\psi}\frac{\partial\vec{x}}{\partial\theta} - \frac{\partial\Psi_p}{\partial\theta}\frac{\partial\vec{x}}{\partial\psi}\right) \quad . \tag{8}$$

The position vector in cylindrical coordinates is

$$\vec{x} = R(\psi,\theta,\phi)\hat{R} + z(\psi,\theta,\phi)\hat{z} \tag{9}$$

with

$$\frac{d\hat{R}}{d\phi} = \hat{\phi} \tag{10}$$

and $\hat{R}$, $\hat{\phi}$, and $\hat{z}$ the standard unit vectors.

The relation between the Jacobian D and $\vec{B}\cdot\vec{\nabla}\phi$, Eq. (7), is an important constraint on the canonical description of a given magnetic field $\vec{B}(\vec{x})$. In cylindrical coordinates, $\vec{B}\cdot\vec{\nabla}\phi$ is just B_ϕ/R. A given magnetic field can be written in canonical form, Eq. (1), using transformation equations $R(\psi,\theta,\phi)$ and $z(\psi,\theta,\phi)$, if and only if the transformation equations satisfy Eqs. (5) and (7), which are the Jacobian constraint. There is considerable freedom in the choice of canonical coordinates ψ, θ since canonical transformations of the magnetic Hamiltonian leave the magnetic field in canonical form, Eq. (1). See Sec. III. The rigidity of the Jacobian constraint follows from the result that a transformation of a one degree of freedom Hamiltonian is canonical if and only if the Jacobian of the transformation is unity. Since ordinary space coordinates are not canonical coordinates of the magnetic field, the Jacobian between ordinary space and canonical coordinate space, ψ, θ, ϕ is not unity. The canonical form of the magnetic field, Eq. (1), can be maintained under more general canonical transformations in which the toroidal angle ϕ, the Hamiltonian time, is also changed. In classical mechanics these are called canonical transformation in the extended phase space, that is, the $\psi, \theta, \phi, \Psi_p$ space. With extended phase-space transformations, there is considerable choice as to the functional form of the Jacobian, although the Jacobian remains $1/\vec{B}\cdot\vec{\nabla}\phi$.

Another expression of the Jacobian constraint may be useful. The transformation equations R and z at fixed ψ define a closed toroidal tube. The Jacobian constraint requires that the magnetic flux, generally called toroidal flux, which is contained in the tube, be independent of ϕ. The value of the toroidal flux is $2\pi\psi$. To prove these statements, one need only note that the toroidal flux is defined by an integration over the toroidal area element

$$d\vec{a}_t = D\vec{\nabla}\phi \; d\psi d\theta \quad . \tag{11}$$

So, the toroidal flux is

$$\int \vec{B}\cdot d\vec{a}_t = 2\pi\psi \tag{12}$$

using Eq. (7) for the Jacobian, D. The toroidal flux is determined by ψ and is therefore independent of the toroidal angle ϕ.

In summary, any magnetic field can be described by a set of transformation equations $R(\psi,\theta,\phi)$ and $z(\psi,\theta,\phi)$ and a Hamiltonian $\Psi_p(\psi,\theta,\phi)$. The only restriction on the transformation equations is that they satisfy the Jacobian constraint, Eqs. (5) and (7). A computer code,[3] written by G. Kuo-Petravic, can find transformation equations $\vec{x}(\psi,\theta,\phi)$ and magnetic Hamiltonian $\Psi_p(\psi,\theta,\phi)$ for a given field $\vec{B}(\vec{x})$. Conversely, any three smooth functions R, z, Ψ_p, which are periodic in θ and ϕ, define a magnetic field.

III. CANONICAL AND GAUGE TRANSFORMATIONS

In electromagnetic theory, we learn that an arbitrary gradient, $\vec{\nabla}G$, can be added to the vector potential without changing the magnetic field, the so-called gauge transformation. In classical mechanics, we learn that identical dynamics can be described by different phase-space coordinates through the use of canonical transformations, which are defined by a generating function S. In this section, it will be shown that the canonical transformations of the magnetic Hamiltonian are precisely the gauge transformations of the vector potential.

The obvious vector potential for a magnetic field written in canonical form, Eq. (1), is

$$\vec{A} = \phi\vec{\nabla}\theta - \psi_p\vec{\nabla}\phi \quad . \tag{13}$$

Consider the canonical transformation to $\hat{\phi}$, $\hat{\theta}$, $\hat{\psi}_p$ defined by the generating function $S(\hat{\phi},\theta,\phi)$. The well-known generating function relations are

$$\hat{\theta} = \frac{\partial S}{\partial\hat{\phi}} \tag{14}$$

$$\phi = \frac{\partial S}{\partial\theta} \quad , \quad \hat{\psi}_p = \psi_p + \frac{\partial S}{\partial\phi} \quad . \tag{15}$$

Inserting these expressions into Eq. (13) for $\vec{A}$, one finds

$$\vec{A} = \hat{\phi}\vec{\nabla}\hat{\theta} - \hat{\psi}_p\vec{\nabla}\phi + \vec{\nabla}(S - \hat{\theta}\hat{\phi}) \quad . \tag{16}$$

Equation (14) implies the generating function is not a single-valued function of position, but that $S - \hat{\theta}\hat{\phi}$ is. Defining G by

$$G = S - \hat{\theta}\hat{\psi} \quad , \tag{17}$$

demonstrates that a canonical transformation is a gauge transformation.

To prove that any gauge transformation is a canonical transformation, we need only prove the theorem for infinitesimal transformations since a finite gauge transformation can be formed from a succession of infinitesimal ones. Let the transformed vector potential have the form

$$\vec{A} = (\psi + \delta\psi)\vec{\nabla}(\theta + \delta\theta) - (\Psi_p + \delta\Psi_p)\vec{\nabla}\phi + \vec{\nabla}\delta G(\psi, \theta, \phi) \quad . \tag{18}$$

Let

$$\delta S = \delta G - \psi\delta\theta \quad . \tag{19}$$

If linear differential terms are equated, one finds

$$\delta\psi = \frac{\partial\delta S}{\partial\theta} \quad , \quad \delta\theta = -\frac{\partial\delta S}{\partial\psi} \quad , \quad \delta\Psi_p = \frac{\partial\delta S}{\partial\phi} \quad . \tag{20}$$

These are the well-known equations for an infinitesimal canonical transformation and prove the assertion.

There is a broader class of canonical transformations than just the ψ, θ transformations. These are the so-called extended phase-space transformations of classical mechanics, which are transformations in ψ, θ, ϕ, Ψ_p space. A typical extended phase-space generating function $S_e(\hat{\psi}, \theta, \phi, \hat{\Psi}_p)$ obeys the following relations

$$\psi = \frac{\partial S_e}{\partial \theta} \qquad \hat{\theta} = \frac{\partial S_e}{\partial \hat{\psi}}$$

$$\Psi_p = -\frac{\partial S_e}{\partial \phi} \qquad \hat{\phi} = -\frac{\partial S_e}{\partial \hat{\Psi}_p} \quad . \tag{21}$$

Under an extended phase-space transformation the vector potential of Eq. (16) is transformed to

$$\vec{A} = \hat{\psi}\vec{\nabla}\hat{\theta} - \hat{\Psi}_p\vec{\nabla}\hat{\phi} + \vec{\nabla}(S_e - \hat{\theta}\hat{\psi} + \hat{\phi}\hat{\Psi}_p) \quad . \tag{22}$$

The ordinary ψ, θ canonical transformations correspond to

$$S_e = S(\hat{\psi}, \theta, \phi) - \phi\hat{\Psi}_p \quad . \tag{23}$$

There are only three independent variables in the magnetic Hamiltonian description of the field topology. The existence of a four variable generating function corresponds to the considerable freedom in the choice of independent variables and in the specification of an auxiliary condition. For example, the toroidal angle ϕ can be chosen to be the cylindrical angle. The freedom to choose independent variables is illustrated by the generating function

$$S_e = \hat{\Psi}_p\theta - \hat{\psi}\phi \quad , \tag{24}$$

which switches the roles of the toroidal and the poloidal variables.

IV. FARADAY'S LAW

Consider a magnetic field which changes with time. In the Hamiltonian picture, both the transformation equations $\vec{x}(\psi,\theta,\phi)$ and the magnetic Hamiltonian $\Psi_p(\psi,\theta,\phi)$ may become functions of time. Time plays the role of a parameter, changing one Hamiltonian system into another. To the extent the time evolution in Ψ_p is a canonical transformation, the topology of the magnetic field is conserved. Consequently, when Ψ_p is changed by at most a canonical transformation, we say the magnetic Hamiltonian is conserved. Of course, the shape of the magnetic field may change, even when the magnetic Hamiltonian is conserved, due to changes in the transformation equations.

Faraday's law is just

$$\left(\frac{\partial B}{\partial t}\right)_{\vec{x}} = -c\vec{\nabla} \times \vec{E} \quad . \tag{25}$$

The subscript $\vec{x}$ has been added to the time derivative since we wish to transform Faraday's law into canonical coordinates. Since the transformation equations depend on time, the transformation of Faraday's law into canonical coordinates is actually a Lorentz transformation. Using Eq. (1) for $\vec{B}$, tensor calculus implies[7]

$$\left(\frac{\partial \vec{B}}{\partial t}\right)_{\vec{x}} = \vec{\nabla} \times \left[\vec{v}_B \times \vec{B} - \left(\frac{\partial \Psi_p}{\partial t}\right)_{\psi,\theta,\phi} \vec{\nabla}\phi\right] \tag{26}$$

with the velocity of the magnetic field defined by

$$\vec{v}_B \equiv \left(\frac{\partial \hat{x}}{\partial t}\right)_{\psi,\theta,\phi} \quad . \tag{27}$$

Faraday's law is, therefore, equivalent to

$$\vec{E} + \frac{\vec{v}_B}{c} \times \vec{B} = \frac{1}{c} \frac{\partial \Psi_p}{\partial t} \vec{\nabla}\phi - \vec{\nabla}\left(\frac{\partial G}{\partial t}\right) \quad . \tag{28}$$

The single-valued function of position and time $\partial G/\partial t$ represents the freedom to change gauge as time evolves. The subscripts ψ, θ, ϕ have been dropped from the time derivatives of Ψ_p and G with the convention that partial derivatives are to be taken in canonical coordinates unless otherwise stated. There is clearly only one component of the electric field which is relevant to the time development of the magnetic Hamiltonian, the parallel component $E_\parallel$. One has

$$\frac{\partial \Psi_p}{\partial t} = \frac{c}{\vec{B}\cdot\vec{\nabla}\phi} \left(\vec{B}\cdot\vec{E} + \vec{B}\cdot\vec{\nabla} \frac{\partial G}{\partial t}\right) \quad . \tag{29}$$

First, we will prove that if $E_\parallel$ is derivable from a single-valued potential, then the magnetic Hamiltonian is changed by at most a canonical transformation. Of course, the parallel component of the electric field is always locally derivable from a potential V with

$$\vec{B}\cdot\vec{E} = -\vec{B}\cdot\vec{\nabla}V \quad . \tag{30}$$

The issue is whether V exists globally as a single-valued function of position. It should be noted that the condition that V be a single-valued function neither implies nor is implied by the vector $\vec{E}$ being locally curl-free. Of course, if $\vec{E}$ is curl-free throughout space, then a single-valued V exists. The proof of the assertion, that a single-valued potential for $E_\parallel$ implies Ψ_p is conserved, is trivial. We need only absorb V into the gauge change $\partial G/\partial t$ and remember a gauge transformation is a canonical transformation.

In the so-called ideal MHD limit, it is assumed that the plasma conductivity along the field lines holds $E_\parallel$ to zero. Consequently, the ideal MHD limit conserves the magnetic Hamiltonian Ψ_p.

If $E_\parallel$ is not derivable from a single-valued potential, then there is either a slippage of the two magnetic fluxes relative to each other or a topological change in the configuration. To understand the concept of flux slippage, suppose the magnetic Hamiltonian were integrable. By definition of integrable, a canonical transformation exists such that the magnetic Hamiltonian becomes $\psi_p(\psi)$, a function of ψ alone. The concept of a coordinate system ψ,θ,ϕ in which ψ_p is a function of ψ alone, so-called magnetic coordinates, dates to early papers of Kruskal and Kulsrud,[8] Hamada,[9] and Greene and Johnson.[10] One can easily show that $2\pi\psi_p$ is the poloidal magnetic flux and the derivation of ψ_p is the rotational transform ι,

$$\iota = \frac{d\psi_p}{d\psi} \quad . \tag{31}$$

If Eq. (29) is integrated over the volume inside a ψ surface, one obtains

$$\frac{\partial\psi_p}{\partial t} = \frac{\partial}{\partial\psi}\left(\frac{c}{(2\pi)^2}\int \vec{E}\cdot\vec{B}d^3x\right) \quad , \tag{32}$$

for the element of volume is

$$d^3x = \frac{d\psi d\theta d\phi}{\vec{B}\cdot\vec{\nabla}\phi} \quad . \tag{33}$$

Near the magnetic axis or 0 point of the poloidal flux, Eq. (32) implies

$$\frac{\partial}{\partial t}\left(2\pi\psi_p(0)\right) = c \oint \vec{E}\cdot d\vec{\ell} \quad . \tag{34}$$

The line integral is along the magnetic axis. This is the well-known result that the rate of change of the flux $2\pi\psi_p(0)$ penetrating a closed curve, the magnetic axis, is given by the line integral of the electric field along that curve. When the poloidal flux function ψ_p is a function of ψ and t alone, one can think of the poloidal flux as slipping relative to the toroidal flux. In the simplest case, ψ_p is the sum of a function of time and a function of ψ in some region of space. In this case, the magnetic field is locally independent of time, but the magnetic field must change somewhere in the surface bounded by the magnetic axis for ψ_p to change with time.

The absence of a single-valued potential for $E_\parallel$ can also indicate a topology change. Again we suppose that Ψ_p is locally integrable and is given by the form $\psi_p(\psi,t)$. The parallel electric field can always be Fourier decomposed in canonical coordinates

$$\frac{\vec{B}\cdot\vec{E}}{\vec{B}\cdot\vec{\nabla}\phi} = \sum_{n,m} E_{nm} \exp[i(n\phi-m\theta)] \tag{35}$$

as can $\partial G/\partial t$,

$$\frac{\partial G}{\partial t} = \sum_{n,m} g_{nm} \exp[i(n\phi-m\theta)] \quad . \tag{36}$$

When the magnetic Hamiltonian is integrable,

$$\vec{B}\cdot\vec{\nabla}\frac{\partial G}{\partial t} = \vec{B}\cdot\vec{\nabla}\phi \sum_{n,m} (n-\iota m)\, g_{nm} \exp[i(n\phi-m\theta)] \quad . \tag{37}$$

with the use of Eqs. (3) and (4). The rate of change of Ψ_p as given by Eq. (29) is

$$\frac{\partial \Psi_p}{\partial t} = \sum_{n,m} \left[E_{nm} + (n-\chi m) g_{nm}\right] \exp[i(n\phi - m\theta)] \quad . \tag{38}$$

If $n - \chi m$ is finite, then the g_{nm} can be chosen to cancel the E_{nm}. There are two ways $n - \chi m$ can vanish or force a change in the magnetic Hamiltonian. First, n and m can both be zero. In this case, the magnetic Hamiltonian can remain a function of ψ alone but is time dependent, i.e., of the form $\psi_p(\psi,t)$. This is the flux slippage case. Second, on the so-called rational surfaces, on which the transform χ equals the ratio of two integers, the factor $n - \chi m$ will vanish for certain values of n and m. This situation always corresponds to a topology change. The case with χ zero is, however, somewhat special since the magnetic Hamiltonian can remain in the form $\psi_p(\psi,t)$, but an island will nonetheless evolve unless all the E_{om} Fourier terms vanish. On the rational surfaces, with χ finite, all the Fourier terms E_{nm}, which satisfy $n = \chi m$, must vanish if the magnetic Hamiltonian is to remain in the form $\psi_p(\psi,t)$. Fourier terms, which satisfy $n = \chi m$, are called resonant. An equivalent statement is that $\oint E_\parallel d\ell$ must be identical on each field line of every rational surface if the magnetic Hamiltonian is to remain integrable, that is, canonically equivalent, using ordinary canonical transforms, to the form $\psi_p(\psi,t)$. It should be noted that a change in the form of the magnetic Hamiltonian in a rational surface also implies a change over a broader region since a Hamiltonian must be a smooth function of the canonical coordinates.

Resonant Fourier terms in the parallel electric field create resonant terms in the magnetic Hamiltonian and therefore open a magnetic island. The structure of this island can be most simply studied in helical canonical coordinates with

$$\phi_* = \frac{\phi}{M} \quad , \quad \theta_* = \frac{M\theta - N\phi}{M} \tag{39}$$

and M and N chosen so that N/M is the transform on the rational surface under study. By substitution in Eq. (1), one finds

$$\vec{B} = \vec{\nabla}\phi \times \vec{\nabla}\theta_* + \vec{\nabla}\phi_* \times \vec{\nabla}\Psi_* \tag{40}$$

with

$$\Psi_* = M\Psi_p - N\phi \quad . \tag{41}$$

The transformation to helical coordinates is a canonical transformation in extended phase space. This transformation possesses the important feature that functions which are periodic in θ and ϕ and are therefore single valued, are also periodic in θ_* and ϕ_*. Since the derivation of Eq. (32) was of a formal nature, this equation can be used to study the increase in helical flux Ψ_* associated with the magnetic surfaces in the interior of an island.[7]

V. ENERGY INTEGRAL

The energy associated with a stationary plasma in a magnetic field is

$$W = \int \left(\frac{B^2}{8\pi} + \frac{p}{\gamma - 1}\right) d^3x \tag{42}$$

with γ the adiabatic index and p the pressure. Kruskal and Kulsrud[8] showed that equilibrium,

$$\vec{\nabla}p = \frac{1}{c}\,\vec{J} \times \vec{B} \quad , \tag{43}$$

corresponds to stationary points of the energy integral when the shape of the plasma is varied under certain constraints. These constraints are conservation laws for the toroidal and the poloidal fluxes, the entropy, and the number of particles. In addition, no energy should cross the surface bounding the volume of interest. The proof of Kruskal and Kulsrud was based on the existence of perfect magnetic surfaces throughout the volume of interest. This proof can be generalized using the concept of a magnetic Hamiltonian.

The variational property of the energy has been used to calculate plasma equilibrium and to test for ideal stability.[11,12] A plasma is ideal-stable if and only if the energy is not only stationary but also minimal under properly constrained, small changes in plasma shape.

More general consequences of the energy integral can be considered. Suppose a small change is made in the physics. Such changes can be of two types. First, the energy integral itself may be changed slightly. For example, the stress tensor may be closely approximated by, but not equal, the pressure. Second, a constraint used in the energy variation may be weakened. For example, the magnetic fluxes may be extremely well but not perfectly conserved over the time scale of interest. In either case, ideal-stable equilibria should be accurately approximated by the standard energy variational procedure. The singular cases, in which this is not true, require special attention. In the first case, a small change in the energy integral implies a large change in the plasma shape. This is possible if a properly constrained, small change in the plasma shape gives a particularly small or zero change in the energy. This first singular case can be studied with existing ideal-stability codes. The second case, in which the energy can be reduced by a large amount by a small change in a constraint, leads to the presumption that small dissipative effects will cause the plasma to evolve rapidly to a lower energy configuration. Resistive instabilities[13] are of this type. The study of the energy integral can yield more information than just equilibrium and ideal-stability. The magnetic Hamiltonian would appear especially useful for studying the sensitivity of plasma equilibria to the flux conservation constraint. The equivalence of canonical and gauge transformations implies that any canonically equivalent form for the Hamiltonian gives the same equilibrium, provided the entropy and density are given in the appropriate canonical coordinates.

Let us prove the variational property of the energy integral with fixed but arbitrary magnetic Hamiltonian. First, consider the magnetic energy,

$$W_m = \frac{1}{8\pi} \int B^2 d^3x \quad . \tag{44}$$

In Sec. II it was demonstrated that the transformation equations $\vec{x}(\psi,\theta,\phi)$ plus a magnetic Hamiltonian $\Psi_p(\psi,\theta,\phi)$ define a unique magnetic field. Inserting this representation, Eq. (8), into the magnetic energy integral, one finds

$$W_m = \frac{1}{8\pi} \int \left(\frac{\partial \vec{x}}{\partial \phi} + \frac{\partial \Psi_p}{\partial \psi}\frac{\partial \vec{x}}{\partial \theta} - \frac{\partial \Psi_p}{\partial \theta}\frac{\partial \vec{x}}{\partial \psi}\right)^2 \frac{1}{D}\, d\psi d\theta d\phi \quad . \tag{45}$$

Suppose the transformation equations are changed by a small amount $\delta\vec{x}$, then

$$\delta W_M = \frac{1}{4\pi} \int \vec{B} \cdot \left(\frac{\partial \delta\vec{x}}{\partial \phi} + \frac{\partial \Psi_p}{\partial \psi}\frac{\partial \delta\vec{x}}{\partial \theta} - \frac{\partial \Psi_p}{\partial \theta}\frac{\partial \delta\vec{x}}{\partial \psi}\right) d\psi d\theta d\phi - \frac{1}{8\pi} \int B^2 (\delta D)\, d\psi d\theta d\phi \quad . \tag{46}$$

The first integral can be put in simpler form by the use of

$$\vec{B}\cdot\vec{\nabla} = \frac{1}{D}\left(\frac{\partial}{\partial\phi} + \frac{\partial\Psi_p}{\partial\psi}\frac{\partial}{\partial\theta} - \frac{\partial\Psi_p}{\partial\theta}\frac{\partial}{\partial\psi}\right) \quad . \tag{47}$$

Simplification of the second integral requires the relation

$$\frac{\delta D}{D} = \vec{\nabla}\cdot\delta\vec{x} \quad . \tag{48}$$

This relation can be demonstrated by explicit calculation. More elegantly, consider the change in volume δV of an arbitrary region defined by the canonical coordinates. One expression for δV is clearly

$$\delta V = \int (\delta D)\, d\psi d\theta d\phi \quad . \tag{49}$$

Another expression is

$$\delta V = \int \delta\vec{x} \cdot d\vec{a} \quad , \tag{50}$$

an area integral over the surface bounding the volume. The use of Gauss' theorem and writing d^3x as $D\ d\psi d\theta d\phi$ yields the desired relation. The simplified form for the change in magnetic energy is

$$\delta W_m = \frac{1}{4\pi}\int \vec{B}\cdot(\vec{B}\cdot\nabla\ \delta\vec{x} - \frac{1}{2}\vec{B}\vec{\nabla}\cdot\delta\vec{x})\, d^3x \quad . \tag{51}$$

Vector identities can be used to show

$$\vec{\nabla}\cdot[(\delta\vec{x} \times \vec{B}) \times \vec{B} + \delta\vec{x}\cdot\vec{B}\ \vec{B}] = 2\vec{B}\cdot(\vec{B}\cdot\vec{\nabla}\delta\vec{x}) - 2\delta\vec{x}\cdot(\vec{B} \times \vec{\nabla} \times \vec{B}) - B^2\vec{\nabla}\cdot\delta\vec{x} \ , \tag{52}$$

which implies

$$\delta W_m = \int \{\vec{\nabla}\cdot[(2\delta\vec{x}_\parallel - \delta\vec{x})\frac{B^2}{8\pi}] - \delta\vec{x} \cdot (\frac{\vec{J}}{c} \times \vec{B})\}d^3x \tag{53}$$

with $\delta\vec{x}_\parallel$ the component along the magnetic field.

Now consider the energy contained in the plasma

$$W_p = \int \frac{p}{\gamma-1} \, D d\psi d\theta d\phi \quad . \tag{54}$$

The variation in the pressure can be related to that of the density by the adiabatic assumption. That is, the assumption that the entropy per particle at a given point in canonical coordinate space is fixed, or

$$\delta p = \gamma \frac{p}{\rho} \delta\rho \quad . \tag{55}$$

The change in density $\delta\rho$ is determined by the condition that the number of particles in each canonical coordinate volume be conserved, or

$$\delta(\rho D) = 0 \quad . \tag{56}$$

Using Eq. (48) for δD, one obtains the well-known Lagrangian displacement relation

$$\delta\rho + \rho\vec{\nabla} \cdot \delta\vec{x} = 0 \quad . \tag{57}$$

These expressions imply

$$\delta W_p = -\int p \, \vec{\nabla} \cdot \delta\vec{x} \, d^3x \quad . \tag{58}$$

The total change in energy is then

$$\delta W = \int \vec{\nabla} \cdot \left[\frac{B^2}{4\pi}\,\delta\vec{x}_{\parallel} - \left(p + \frac{B^2}{8\pi}\right)\delta\vec{x}\right] d^3x + \int \delta\vec{x} \cdot \left(\vec{\nabla}p - \frac{\vec{j}}{c} \times \vec{B}\right) d^3x \quad . \tag{59}$$

The first integral in δW, which contains the divergence, just gives the energy crossing the surface which bounds the volume of interest. The energy integral is stationary at equilibrium provided the energy crossing the bounding surface vanishes.

In the Hamiltonian representation of the magnetic field, variations are naturally given in the Lagrangian form. Although the Lagrangian formulation was used by Bernstein et al. in their well-known energy principle paper,[12] the Eulerian form is essentially universal in textbooks. To avoid confusion, consider the relation between the variation of any quantity F in canonical coordinates, no subscript, and the variation in ordinary space, subscript $\vec{x}$. Now

$$\delta F \equiv F_n\,(\vec{x} + \delta\vec{x}) - F_o(\vec{x}) \tag{60}$$

with the subscripts n and o standing for new and old, and

$$(\delta F)_{\vec{x}} \equiv F_n\,(\vec{x}) - F_o\,(\vec{x}) \quad . \tag{61}$$

Therefore, for small $\delta\vec{x}$,

$$\delta F = (\delta F)_{\vec{x}} + \delta\vec{x} \cdot \vec{\nabla}F \quad . \tag{62}$$

For example, the variation of the magnetic field, following the argument given earlier in the section, is

$$\delta\vec{B} = \vec{B} \cdot \vec{\nabla}\,\delta\vec{x} - \vec{B}\,(\vec{\nabla} \cdot \delta\vec{x}) \qquad . \tag{63}$$

The use of a vector identity will then give the well-known result

$$(\delta\vec{B})_{\vec{x}} = \vec{\nabla}\times\,(\delta\vec{x} \times \vec{B}) \qquad . \tag{64}$$

There is a subtlety when dealing with densities, like the energy density w. The energy variation is

$$\delta W = \delta(\int w\, D\, d\psi d\theta d\phi). \tag{65}$$

The variation in D from Eq. (48) coupled with the variation in w, using Eq. (62), implies the energy variation is

$$\delta W = \int[(\delta w)_{\vec{x}} + \vec{\nabla} \cdot (w\delta\vec{x})]\, d^3x \qquad . \tag{66}$$

These simple relations allow one to calculate variations in either canonical coordinate space or in ordinary space, and transform to the other representation.

The usual energy principle expression for determining ideal stability follows if $\vec{\nabla}p$, $\vec{J}$, and $\vec{B}$ are varied holding the magnetic Hamiltonian fixed. Variations in quantities involving the gradient operator are much easier to perform in ordinary space, in which the operator is fixed, than in canonical coordinates. For example

$$\frac{4\pi}{c}\,(\delta\vec{j})_{\vec{x}} = \vec{\nabla}\times\,(\delta\vec{B})_{\vec{x}} \quad . \tag{67}$$

But $\delta\vec{j}$ has a complex relation to $\delta\vec{B}$ since the curl operator must itself be varied. Using the rules established in the previous paragraph

$$\delta\vec{j} = \delta\vec{x}\cdot\nabla\vec{j} + \vec{\nabla}\times\,[\vec{\nabla}\times(\delta\vec{x}\times\vec{B})] \quad . \tag{68}$$

Since the derivation of the energy principle is well-known, we will not continue the discussion other than to note that the displacement $\vec{\xi}$ of stability theory is the variation in the transformation equations $\delta\vec{x}$.

The magnetic energy can be changed not only by variations in the transformation equations but also by variations in the magnetic Hamiltonian. The Hamiltonian variation is denoted by a subscript H. Using Eq. (1),

$$(\delta\vec{B})_H = \vec{\nabla}\phi\times\vec{\nabla}\,\delta\Psi_p \quad . \tag{69}$$

The variation in the magnetic energy is just the integral of $\vec{B}\cdot\delta\vec{B}/4\pi$ over the volume, or

$$(\delta W_m)_H = -\frac{1}{c}\int\delta\Psi_p\,\vec{j}\cdot\vec{\nabla}\phi\,d^3x - \frac{1}{4\pi}\int(\delta\Psi_p\,\vec{\nabla}\phi\times\vec{B})\cdot d\vec{a} \quad . \tag{70}$$

The area integral is over the surface of the volume. Consequently, the energy variation is stationary if the toroidal current j_ϕ vanishes. A special case of this result was derived by Kruskal and Kulsrud[8] by varying the transform.

The magnetic Hamiltonian variation can be applied to the tearing mode.[13] Consider the tearing mode in a cylinder r, θ, z = ϕ/R with a strong axial field and zero pressure. Let r_o be the resonant surface with $\chi(r_o)$ =

N/M. The perturbed current density in a tearing mode is very large at the resonant surface and to a good approximation can be thought to have a component which is a delta function in r and varies on the surface as sin(Nϕ - Mϕ). Ampere's law for the perturbed field is

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} \quad . \tag{71}$$

The z component of this equation is

$$j_z = \frac{c}{4\pi}\left(\frac{\partial b_\theta}{\partial r} - \frac{\partial b_r}{\partial \theta}\right) \quad . \tag{72}$$

In the vicinity of the resonant surface, the delta function part of j_z implies the term $\partial b_r/\partial\theta$ can be ignored. The solenoidal condition on $\vec{b}$ implies

$$\frac{\partial b_r}{\partial r} + \frac{1}{r}\frac{\partial b_\theta}{\partial \theta} = 0 \quad . \tag{73}$$

Letting

$$b_r = \bar{b}_r \cos(N\phi - M\theta) \quad , \tag{74}$$

one finds the delta function part of j_z is

$$j_z = \frac{c}{4\pi}\frac{r_o}{M}\left[\frac{d\bar{b}_r}{dr}\right] \delta(r - r_o) \sin(N\phi - M\theta) \tag{75}$$

with $[d\bar{b}_r/dr]$ the jump in the radial derivation of $\bar{b}_r$ across the resonant surface. Now consider $(\delta W_m)_H$, Eq. (70). The surface integral can be ignored by assuming $\delta\Psi_p$ is zero on the distant surface. The resonant part of $\delta\Psi_p$

gives the change in the energy and creates an island around the resonant surface. At the resonant surface, the resonant part of b_r must be given by $\delta\Psi_p$, since a resonant b_r can be easily shown to produce a topology change; so

$$\delta b_r = [\frac{1}{R}\hat{z} \times \vec{\nabla}(\delta\Psi_p)] \cdot \hat{r} \quad , \tag{76}$$

consequently,

$$\delta\Psi_p = \frac{Rr}{M}\,\delta\bar{b}_r \sin(N\phi - M\theta) \quad . \tag{77}$$

Carrying out the energy integral, one finds

$$(\delta W_m)_H = -\frac{r_o}{4\pi} V \left[\frac{d\bar{b}_r}{dr}\right] \delta\bar{b}_r \tag{78}$$

with V the volume inside the resonant surface. Since $[d\bar{b}_r/dr]$ is proportional to $\bar{b}_r$ in the simple theory outlined here, the total change in the magnetic energy ΔW_m is

$$\Delta W_m = -V\, r_o . \Delta' \frac{\bar{b}_r^2}{8\pi} \tag{79}$$

with

$$\Delta' = \frac{[b_r']}{b_r} \quad . \tag{80}$$

The expression for the magnetic energy released by a tearing mode was orginally given by Furth.[14] Instability corresponds to Δ' positive, that is, the release of magnetic energy.

APPENDIX

EXISTENCE OF CANONICAL FORM

Although the existence of the canonical form of the magnetic field has been demonstrated elsewhere,[2] the proof is sufficiently short to be reproduced. Let ρ, θ, ϕ be any nonsingular toroidal coordinates. That is ρ, θ, and ϕ are functions of ordinary Cartesian space coordinates $\vec{x}$ with $(\vec{\nabla}\rho \times \vec{\nabla}\theta) \cdot \vec{\nabla}\phi$ finite in the region of interest. The level surfaces of ρ are tori, θ is a poloidal, and ϕ is a toroidal angle. Since $\vec{\nabla}\rho$, $\vec{\nabla}\theta$, $\vec{\nabla}\phi$ span ordinary three-dimensional space, any vector $\vec{B}$ can be written in the form

$$\vec{B} = \frac{\partial\psi}{\partial\rho}\,\vec{\nabla}\rho \times \vec{\nabla}\theta + \frac{\partial\Psi_p}{\partial\rho}\,\vec{\nabla}\phi \times \vec{\nabla}\rho + C\,\vec{\nabla}\theta \times \vec{\nabla}\phi \quad . \tag{A1}$$

The zero divergence of crossed gradients implies

$$\vec{\nabla} \cdot \vec{B} = (\vec{\nabla}\rho \times \vec{\nabla}\theta) \cdot \vec{\nabla}\phi\,\frac{\partial}{\partial\rho}\left(\frac{\partial\psi}{\partial\phi} + \frac{\partial\Psi_p}{\partial\theta} + C\right) \quad . \tag{A2}$$

The divergence-free nature of the magnetic field implies ψ and Ψ_p can be chosen so that

$$C = C_o - \left(\frac{\partial\psi}{\partial\phi} + \frac{\partial\Psi_p}{\partial\theta}\right) \tag{A3}$$

with C_o a constant. Actually C_o must be zero by noting that it is the net flux of magnetic field crossing a constant ρ surface. The insertion of Eq. (A3), with $C_o = 0$, into Eq. (A1) yields the canonical form

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\theta + \vec{\nabla}\phi \times \vec{\nabla}\Psi_p \quad . \tag{A4}$$

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FIGURE CAPTION

FIG. 1. Coordinate System. The method of describing a topology toroidal magnetic configuration using cylindrical coordinates is illustrated.

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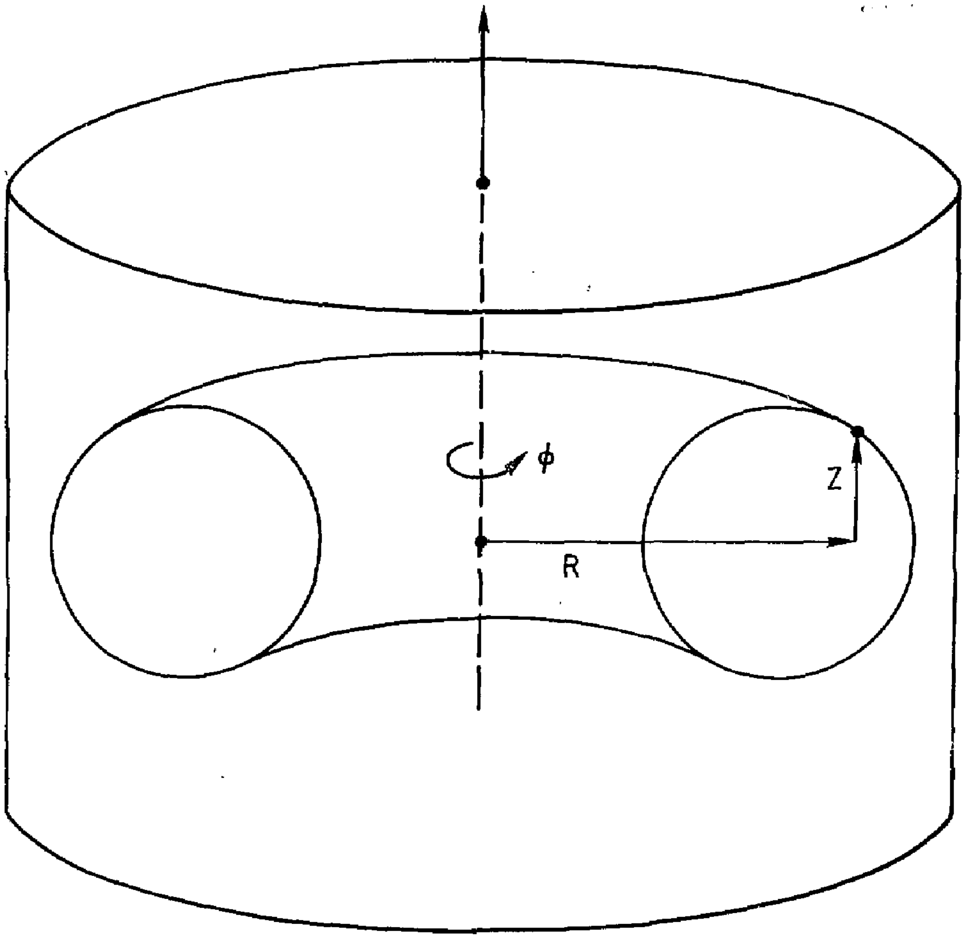

Fig. 1

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