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By

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A KINETIC EQUATION FOR SPIN-POLARISED PLASMAS

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ABSTRACT

The usual kinetic description of a plasma is extended to include variables to describe the spin. The distribution function, over phase-space and the new spin variables, provides a sufficient description of a spin-polarised plasma. The evolution equation for the distribution function is given. The equations derived are used to calculate depolarisation due to four processes, inhomogeneous fields, collisions, collisions in inhomogeneous fields, and waves.

It is found that depolarisation by field inhomogeneity on scales large compared with the gyroradius is totally negligible. The same is true for collisional depolarisation. Collisions in inhomogeneous fields yield a depolarisation rate of order $10^{-4}s^{-1}$ for deuterons and a negligible rate for tritons in a typical fusion reactor design. This is still sufficiently small on reactor time scales. However, small amplitude magnetic fluctuations (of order one gauss) resonant with the spin precession frequency can lead to significant depolarisation (depolarises triton in ten seconds and deuteron in a hundred seconds.) In a companion paper, Ref. 2. it is shown that such fluctuations may not be present.

1. INTRODUCTION

Recent work [1] has shown that polarising the nuclear spin in a fusion plasma may be an effective way of increasing reaction rates or directing emmission of reaction products. The crucial physical question is the rate of depolarisation. In order to discuss this question with precision, it is necessary to prescribe the distribution of polarisation states among all the nuclei in the fusion plasma at a given time. For this purpose we introduce a distribution function, f, over configuration space, velocity space, and a space introduced to describe the nuclear polarisation state. Then the rate of evolution of f in time, given by a Fokker-Planck equation, gives the rate of depolarisation. In Ref. 2 the rates of depolarisation were estimated for a test ion. The kinetic approach of this paper provides a systematic way to derive accurate expressions for these rates for a distribution of particles.

In Sec. 2 we describe the necessary quantum mechanics, in Sec. 3 we construct a Fokker-Planck equation for f and in Sec. 4 we derive depolarisation rates for four processes. In the first process considered, motion in fields changing slowly in the ion frame, the polarisation vector follows the magnetic field preserving its component along the magnetic field. The second depolarisation process, collisional depolarisation, arises from small random changes in the polarisation vector during collisions. In the third process, collisions in inhomogeneous fields, the decorrelation of nearly resonant orbital and precessional motion leads to depolarisation. Depolarisation by waves, the fourth process, occurs when the perturbed perpendicular magnetic field is in phase with the precessing polarisation vector.

2. SPIN QUANTUM MECHANICS

The spin state of the entire system of particles is described by the density matrix formulation of quantum mechanics [2]. The density matrix is defined as

$$\rho = |\psi\rangle\langle\psi| \tag{2.1}$$

where $|\psi\rangle$ is the set of the whole system. By Schödinger's equation $|\psi\rangle$ evolves as

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H|\psi\rangle \tag{2.2}$$

where H is the Hamiltonian. This equation gives the evolution of the density matrix

$$i\hbar\dot{\rho} = [H,\rho] \tag{2.3}$$

We employ a quasiclassical approximation to the spinning particle treating its velocity and position classically (a good approximation in a fusion plasma) and treating its spin quantum mechanically. The quasiclassical approximation breaks down for short range collisions, but these have little effect on polarisation. (see [2]).

2.1 Spin 1/2 particles

In a plasma the dynamics of the spins of the individual particles are predominantly uncorrelated. That is, if we flip the spin of one particle, the spins of the others remain unchanged. Consider a plasma density of

10^{14} cm^{-3}. The magnetic field due to completely aligned spins is 10^{-9} G, a field 10^{-13} of a typical magnetic field confining a plasma. The spins therefore interact with the magnetic field independently. The independence of the spin dynamics enables us to treat the particle density matrices independently. This approximation breaks down for spin-spin interactions. We demonstrate how to include the small effect of two particle spin correlations in Appendix A.

The single particle density matrix for the i-th spin 1/2 particle is given by

$$\rho_i = [\underline{\underline{I}} + \underset{\sim}{p}_i \cdot \underline{\underline{\sigma}}] \tag{2.4}$$

where

$$\underset{\sim}{p}_i = \langle\psi|\underline{\underline{\sigma}}_i|\psi\rangle \, , \tag{2.5}$$

the expectation value of the Pauli spin matrices,

$$\underline{\underline{\sigma}} = \hat{\underset{\sim}{x}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \hat{\underset{\sim}{y}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \hat{\underset{\sim}{z}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.6}$$

The full Hamiltonian for the spins reduces to the sum of the Hamiltonians for each spin. The i'th particle Hamiltonian is

$$H_i = \hbar\mu_i \underset{\sim}{B}_i \cdot \underline{\underline{\sigma}}_i \tag{2.7}$$

where $\underset{\sim}{B}_i = \underset{\sim}{B}(\underset{\sim}{x}_i,t) + c^{-1}\underset{\sim}{v}_i \times \underset{\sim}{E}(\underset{\sim}{x}_i,t)$ is the B field seen by the i'th particle, with position $\underset{\sim}{x}_i$ and velocity $\underset{\sim}{v}_i$, and $\mu_i = g_i q_i/_{mc}$, g_i is the gyromagnetic ratio. Using Eqs. (2.4), (2.7), and (2.3) we obtain the evolution equation for $\underset{\sim}{p}_i$

$$\dot{\underset{\sim}{p}}_i = \frac{1}{i\hbar} \langle\psi|[\underset{\approx}{g}_i, H]|\psi\rangle = \mu_i \underset{\sim}{B}\times\underset{\sim}{p}_i . \tag{2.8}$$

This result is important. It may also be obtained from Eq. (2.5) and Schrödinger's equation (2.2) with the assumption that the Hamiltonian reduces to the sum of the single particle Hamiltonians, Eq. (2.7).

We expect the spins to remain close to pure states because the mixing term arising from spin-spin interactions is small (see Appendices A and C). For the remainder of the paper, except the appendices, we deal with pure states.

From Eq. (2.5) we can obtain $\underset{\sim}{p}_i$ from $|\psi_i\rangle$ the single particle wave function, replacing $|\psi\rangle$ with $|\psi_i\rangle$. Conversely, $|\psi_i\rangle$ can be expressed in terms of $\underset{\sim}{p}_i$. Let

$$|\psi_i\rangle = \binom{\alpha}{\beta} . \tag{2.9}$$

Then from Eq. (2.5) and the normalisation,

$$1 = |\alpha|^2 + |\beta|^2 , \tag{2.10}$$

$$p_z = |\alpha|^2 - |\beta|^2 ,$$

$$p_+ = p_x + ip_y = 2\alpha^*\beta ,$$

$$p_- = p_x - ip_y = 2\alpha\beta^* .$$

Solving Eq.(2.10), we obtain

$$\alpha = [1/2(1 + p_z)]^{1/2} \exp(i\phi) \ , \qquad (2.11)$$

$$\beta = p_+ \ (2(1 + p_z))^{-1/2} \exp(i\phi)$$

where ϕ is an undetermined phase. The probabilities of finding the particles in an up (m=1/2) or down (m=-1/2) state (where m is the eigenvalue of $\underset{=i}{\sigma} \cdot \hat{\underset{\sim}{z}}$) are

state	probability
$m = 1/2$	$1/2(1 + p_z)$
$m = -1/2$	$1/2(1 - p_z)$.

(2.12)

Note that $\underset{\sim}{p}$ is defined for both pure states, where $|\psi\rangle$ factors into $|\psi_i\rangle$ and for mixed states, Eq. (2.12) holds for both. $\underset{\sim}{p}$ transforms as a vector so it is a more convenient quantity to work with than the wave function.

2.2 Spin 1 particles

A spin 1 particle is more complicated than a spin 1/2 particle and its wave function cannot be represented completely by a single vector. However, it is possible to treat it as consisting of two spin 1/2 particles. The wave function for the spin 1 particle can be constructed from those of the spin 1/2 particles by a symmetrised product

$$|\psi\rangle = \frac{1}{N} \{|a\rangle_1 |a'\rangle_2 + |a\rangle_2 |a'\rangle_1\} \ . \qquad (2.13)$$

This product is automatically in a spin 1 state, that is, no spin zero component results from adding the two spins. N is a normalisation constant determined below. We can rewrite the Hamiltonian for the system

$$H_i = 2\mu_i \underset{\sim}{B}_i \cdot \underline{S}_i = h\mu_i \underset{\sim}{B}_i \cdot (\underline{g}_1 + \underline{g}_2) = H_1 + H_2 \tag{2.14}$$

where $\mu_i = 1/2\ g_i q_i / m_i c$ is half the magnetic moment of the spin-one particle. If the evolution of the spin 1/2 wave functions are given by

$$i\hbar \frac{\partial}{\partial t} |a\rangle_{1,2} = H_{1,2} |a\rangle_{1,2} \ , \tag{2.15}$$

$$i\hbar \frac{\partial}{\partial t} |a'\rangle_{1,2} = H_{1,2} |a'\rangle_{1,2} \ ,$$

we obtain from Eq. (2.13)

$$H|\psi\rangle = i\hbar \frac{\partial}{\partial t} |\psi\rangle \ . \tag{2.16}$$

Clearly the two spin 1/2 wave functions evolve separately and Eq. (2.13) combines then to produce the correct evolution of the spin 1 wave function. We henceforth introduce two vectors $\underset{\sim}{p}$ and $\underset{\sim}{p}'$ corresponding to $|a\rangle$ and $|a'\rangle$, respectively. They evolve according to

$$\begin{aligned} \dot{\underset{\sim}{p}} &= \mu_i \underset{\sim}{B} \times \underset{\sim}{p} \ , \\ \dot{\underset{\sim}{p}}' &= \mu_i \underset{\sim}{B} \times \underset{\sim}{p}' \ . \end{aligned} \tag{2.17}$$

Knowing $\underset{\sim}{p}$ and $\underset{\sim}{p}'$ we can obtain the spin 1 wave function from Eqs. (2.11) and (2.13). The normalisation constant N can be expressed in terms of $\underset{\sim}{p}$ and $\underset{\sim}{p}'$

$$N^2 = N^2\langle\psi|\psi\rangle = (2 + 2\ |\langle a|a'|^2) \tag{2.18}$$
$$= 3 + \underset{\sim}{p} \cdot \underset{\sim}{p}'$$

where we have used Eqs. (2.13) and (2.11).

The probabilities of the different eigenstates of S_z obtained by adding the angular momentum of the two spin 1/2 particles are

state	probability
$m = 1$	$N^{-2}(1 + p_z)\ (1 + p'_z)$
$m = -1$	$N^{-2}(1 - p_z)\ (1 - p'_z)$
$m = 0$	$N^{-2}(1 + \underset{\sim}{p} \cdot \underset{\sim}{p}' - 2p_z p_z')$.

(2.19)

N^2 is a constant of the motion because $\underset{\sim}{p} \cdot \underset{\sim}{p}'$ is a constant of the motion. This follows from Eq. (2.15) which gives $\partial/\partial t\ (\underset{\sim}{p} \cdot \underset{\sim}{p}') = 0$. The introduction of two spin 1/2 particles has no physical relevance but it does simplify the formalism by replacing the treatment of the wave function with the more easily transformed vectors $\underset{\sim}{p}$ and $\underset{\sim}{p}'$. (Note that $\underset{\sim}{p}^2 = 1$ and $\underset{\sim}{p}'^2 = 1$. Here we deal with pure states, so the four degrees of freedom of $\underset{\sim}{p}$ and $\underset{\sim}{p}'$ correspond to the four degrees of freedom of the wave function of a spin 1 particle ignoring the phase factor.)

3. THE KINETIC EQUATIONS

3.1 Spin 1/2

The state of a spin 1/2 particle is described by three vectors $\underset{\sim}{x}$, $\underset{\sim}{v}$, and $\underset{\sim}{p}$ the position, velocity, and polarisation, respectively. The particle can be thought of as moving through a nine-dimensional phase space. (The normal $\underset{\sim}{x}$, $\underset{\sim}{v}$ phase space is extended to include $\underset{\sim}{p}$.) We define a distribution function f in this space by

$$\delta n(\underset{\sim}{x}, \underset{\sim}{v}, \underset{\sim}{p}, t) = f(\underset{\sim}{x}, \underset{\sim}{v}, \underset{\sim}{p}, t)\, d^3\underset{\sim}{x}\, d^3\underset{\sim}{v}\, d^3\underset{\sim}{p} \tag{3.1}$$

where $\delta n(\underset{\sim}{x},\underset{\sim}{v},\underset{\sim}{p},t)$ is the number of particles with position in $d\underset{\sim}{x}$ at $\underset{\sim}{x}$, velocity in $d\underset{\sim}{v}$ at $\underset{\sim}{v}$, and polarisation in $d\underset{\sim}{p}$ at $\underset{\sim}{p}$. The equations of motion are

$$\dot{\underset{\sim}{x}} = \underset{\sim}{v} \, , \tag{3.2}$$

$$\dot{\underset{\sim}{v}} = \frac{2}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{v}\times\underset{\sim}{B}}{c}\right) ,$$

$$\dot{\underset{\sim}{p}} = \mu_i\left(\underset{\sim}{B} + \frac{\underset{\sim}{\cdot}\times\underset{\sim}{E}}{c}\right) \times \underset{\sim}{p} \, .$$

We note that usually $\underset{\sim}{v} \times \underset{\sim}{E}/c \ll \underset{\sim}{B}$. We can write the conservation equation for particles in this extended phase space.

$$\frac{\partial f_i}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f_i}{\partial \underset{\sim}{x}} + \frac{q_i}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{v}\times\underset{\sim}{B}}{c}\right) \cdot \frac{\partial f_i}{\partial \underset{\sim}{v}} + \mu_i\left(\underset{\sim}{B} + \frac{\underset{\sim}{v}\times\underset{\sim}{E}}{\cdot}\right)\times\underset{\sim}{p} \cdot \frac{\partial f_i}{\partial \underset{\sim}{p}} \tag{3.3}$$

$$+ \left(\frac{\partial f_i}{\partial t}\right)_{\text{collisions}} = 0$$

If $(\partial f_i/\partial t)_{\text{collisions}} = 0$, f_i is constant along a phase space trajectory.

3.2 Spin 1

The state of a spin 1 particle is described by $\underset{\sim}{x},\underset{\sim}{v}$ and two $\underset{\sim}{p}$ vectors, $\underset{\sim}{p}$ and $\underset{\sim}{p}'$. The situation is analogous to the spin 1/2 case with three more dimensions in phase space. The distribution function is defined by

$$\delta n(\underset{\sim}{x},\underset{\sim}{v},\underset{\sim}{p},\underset{\sim}{p}',t) = f_i(\underset{\sim}{x},\underset{\sim}{v},\underset{\sim}{p},\underset{\sim}{p}',t)\; d^3\underset{\sim}{x}\; d^3\underset{\sim}{v}\; d^3\underset{\sim}{p}'\; d^3\underset{\sim}{p}' \; . \tag{3.4}$$

$\delta n(\underset{\sim}{x},\underset{\sim}{v},\underset{\sim}{p},\underset{\sim}{p}',t)$ is the number of particles with position in $d^3\underset{\sim}{x}$ at $\underset{\sim}{x}$, velocity in $d^3\underset{\sim}{v}$ at $\underset{\sim}{v}$, polarisation $\underset{\sim}{p}$ in $d^3\underset{\sim}{p}$ at $\underset{\sim}{p}$, and polarisation $\underset{\sim}{p}'$ in $d^3\underset{\sim}{p}'$ at $\underset{\sim}{p}'$. Using the equations of motion, we arrive at the conservation equation for spin 1 particles.

$$\frac{\partial f_i}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f_i}{\partial \underset{\sim}{x}} + \frac{q_i}{mi} (\underset{\sim}{E} + \frac{\underset{\sim}{v}\times\underset{\sim}{B}}{c}) \cdot \frac{\partial f_i}{\partial \underset{\sim}{v}} + \mu_i (\underset{\sim}{B} + \frac{\underset{\sim}{v}\times\underset{\sim}{E}}{c}) \times \underset{\sim}{p} \cdot \frac{\partial f_i}{\partial \underset{\sim}{p}} \tag{3.5}$$

$$+ \mu_i \; (\underset{\sim}{B} + \frac{\underset{\sim}{v}\times\underset{\sim}{E}}{c}) \times \underset{\sim}{p}' \cdot \frac{\partial f_i}{\partial \underset{\sim}{p}'} + \left(\frac{\partial f_i}{\partial t}\right)_{collisons} = 0$$

If $(\partial f_i/\partial t)_{collisions} = 0$, we have a Liouville equation for spin 1 particles.

3.3 Collision terms

The contribution to $(\partial f_i/\partial t)_{collisions}$ produced by changes in $\underset{\sim}{p}$ during a collision come from the possible interactions spin - orbit, spin - spin, and quadrupole interactions. The collision process is Markovian for the vector $\underset{\sim}{p}$. It produces only small $\Delta\underset{\sim}{p}$'s and these can be determined for the collision knowing the initial particle phase space coordinates. The full collision terms are

Spin 1/2

$$\left(\frac{\partial f}{\partial f}\right)_{collisions} = \frac{\partial}{\partial \underset{\sim}{v}} \cdot \langle\Delta\underset{\sim}{v}\rangle f + \frac{1}{2}\frac{\partial^2}{\partial \underset{\sim}{v}\partial \underset{\sim}{v}} : \langle\Delta\underset{\sim}{v}\Delta\underset{\sim}{v}\rangle f + \frac{\partial}{\partial \underset{\sim}{p}} \cdot \langle\Delta\underset{\sim}{p}\rangle f \tag{3.6}$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial \underset{\sim}{p}\partial \underset{\sim}{p}} : \langle\Delta\underset{\sim}{p}\Delta\underset{\sim}{p}\rangle f + \frac{1}{2}\frac{\partial^2}{\partial \underset{\sim}{v}\partial \underset{\sim}{p}} : \langle\Delta\underset{\sim}{v}\Delta\underset{\sim}{p}\rangle f$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial \underset{\sim}{p}\partial \underset{\sim}{v}} : \langle\Delta\underset{\sim}{p}\Delta\underset{\sim}{v}\rangle f$$

Spin 1

$$\left(\frac{\partial f}{\partial t}\right)_{collisions} = \frac{\partial}{\partial \underset{\sim}{v}} \cdot \langle \Delta \underset{\sim}{v} \rangle f + \frac{1}{2} \frac{\partial^2}{\partial \underset{\sim}{v} \partial \underset{\sim}{v}} : \langle \Delta \underset{\sim}{v} \Delta \underset{\sim}{v} \rangle f + \frac{\partial}{\partial \underset{\sim}{p}} \cdot \langle \Delta \underset{\sim}{p} \rangle f + \frac{\partial}{\partial \underset{\sim}{p}'} \cdot \langle \Delta \underset{\sim}{p}' \rangle f$$
$$+ \frac{1}{2} \frac{\partial^2}{\partial \underset{\sim}{p} \partial \underset{\sim}{p}} : \langle \Delta \underset{\sim}{p} \Delta \underset{\sim}{p} \rangle f + \frac{1}{2} \frac{\partial^2}{\partial \underset{\sim}{p}' \partial \underset{\sim}{p}'} : \langle \Delta \underset{\sim}{p}' \Delta \underset{\sim}{p}' \rangle f + \frac{1}{2} \frac{\partial^2}{\partial \underset{\sim}{v} \partial \underset{\sim}{p}} : \langle \Delta \underset{\sim}{v} \Delta \underset{\sim}{p} \rangle f$$
$$+ \frac{1}{2} \frac{\partial^2}{\partial \underset{\sim}{p} \partial \underset{\sim}{v}} : \langle \Delta \underset{\sim}{p} \Delta \underset{\sim}{v} \rangle f + \frac{1}{2} \frac{\partial^2}{\partial \underset{\sim}{p}' \partial \underset{\sim}{v}} : \langle \Delta p' \Delta \underset{\sim}{v} \rangle f + \frac{1}{2} \frac{\partial^2}{\partial \underset{\sim}{v} \partial \underset{\sim}{p}'} : \langle \Delta \underset{\sim}{v} \Delta \underset{\sim}{p}' \rangle f$$
$$+ \frac{1}{2} \frac{\partial^2}{\partial \underset{\sim}{p}' \partial \underset{\sim}{p}} : \langle \Delta \underset{\sim}{p}' \Delta \underset{\sim}{p} \rangle f + \frac{1}{2} \frac{\partial^2}{\partial \underset{\sim}{p} \partial \underset{\sim}{p}'} : \langle \Delta \underset{\sim}{p} \Delta \underset{\sim}{p}' \rangle f \tag{3.7}$$

These expressions, (3.6) and (3.7), are very complicated. Fortunately, all terms except the velocity terms are small. The velocity - polarisation correlations $\langle \Delta \underset{\sim}{v}\ \Delta \underset{\sim}{p} \rangle$ are lost in the many gyration and precession times between collisions. Spin orbit and spin - spin Fokker-Planck collision terms are derived in the appendix and the resulting depolarisation rates are given in Sec. 4.2.

3.4 Moment equations

The large numbers of dimensions of the phase spaces of Eqs. (3.3) and (3.5) make them very difficult to solve. Fortunately, it is possible to reduce their dimensionality to those of ordinary phase space by taking moments over the polarisation space. If we neglect the very small spin - spin collision term, each moment equation is homogeneous and thus closed. We are interested in those moments necessary to calculate the rate of nuclear reactions. For spin 1/2, only the zeroth and first moments are needed. For spin 1 the second moment is also needed.

Spin 1/2

We take moments of the equation (3.3), the zeroth moment comes from integrating over the polarisation. Define

$$F_{1/2} \equiv \int d^3\underset{\sim}{p} f \ . \tag{3.8}$$

$F_{1/2}$ obeys the normal Fokker-Planck equation as we expect. We obtain the equation for the first moment of $\underset{\sim}{p}$ defined as

$$\underset{\sim}{P} \equiv \int d^3\underset{\sim}{p}\, \underset{\sim}{p}\, f \tag{3.9}$$

by multiplying Eq. (3.3) by $\underset{\sim}{p}$ and integrating, with the result

$$\frac{\partial \underset{\sim}{P}}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial \underset{\sim}{P}}{\partial \underset{\sim}{x}} + \frac{q}{m}\left[\underset{\sim}{E} + \frac{\underset{\sim}{v}\times\underset{\sim}{B}}{c}\right] \cdot \frac{\partial \underset{\sim}{P}}{\partial \underset{\sim}{v}} + \mu \underset{\sim}{P}\times\left(\underset{\sim}{B} + \frac{\underset{\sim}{v}\times\underset{\sim}{E}}{c}\right) \tag{3.10}$$

$$= \left(\frac{\partial \underset{\sim}{P}}{\partial t}\right)_{\text{collision}} \ .$$

The expression for $(\partial\underset{\sim}{P}/\partial t)_{\text{collisions}}$ is derived in Appendix A. Including only velocity and spin-orbit terms, the right-hand side is closed (that is, the right-hand side is a function of $\underset{\sim}{P}$ only).

Spin 1

To make the expressions simpler, we define all our moments over $\underset{\sim}{p}$ and $\underset{\sim}{p}'$ space to include the normalisation N. This is possible because N is a constant of the motion. Define

$$F_1 \equiv \int d^3\underset{\sim}{p}\, d^3\underset{\sim}{p}'\, \frac{f}{N^2} \ . \tag{3.11}$$

As with $F_{1/2}$, F_1 obeys the normal Fokker-Planck equations. There are two first moments

$$\underset{\sim}{P} \equiv \int d^3\underset{\sim}{p}\, d^3\underset{\sim}{p}\, \frac{\underset{\sim}{p} f}{N^2} \quad \text{and} \quad \underset{\sim}{P}' \equiv \int d^3\underset{\sim}{p}\, d^3\underset{\sim}{p}'\, \frac{\underset{\sim}{p}' f}{N^2} \,. \tag{3.12}$$

The equation for $\underset{\sim}{P}$ is from Eq. (3.5) [like (3.10)]

$$\frac{\partial \underset{\sim}{P}}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial \underset{\sim}{P}}{\partial \underset{\sim}{x}} + \frac{q}{m}\left[\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}\right] \cdot \frac{\partial \underset{\sim}{P}}{\partial \underset{\sim}{v}} + \mu \underset{\sim}{P} \times \left(\underset{\sim}{B} + \frac{\underset{\sim}{v} \times \underset{\sim}{E}}{c}\right) = \left(\frac{\partial \underset{\sim}{P}}{\partial t}\right)_{\text{collisions}} . \tag{3.13}$$

$\underset{\sim}{P}'$ satisfies the same equation. The symmetrised second moment

$$\underset{\approx}{Q} \equiv \int d^3\underset{\sim}{p}\, d^3\underset{\sim}{p}'\, N^{-2} (\underset{\sim}{p}\underset{\sim}{p}' + \underset{\sim}{p}'\underset{\sim}{p}) f \tag{3.14}$$

obeys the equation

$$\frac{\partial \underset{\approx}{Q}}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial \underset{\approx}{Q}}{\partial \underset{\sim}{x}} + \frac{q}{m}\left[\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}\right] \cdot \frac{\partial \underset{\approx}{Q}}{\partial \underset{\sim}{v}} + \mu\, \underset{\approx}{Q} \times \left[\underset{\sim}{B} + \frac{\underset{\sim}{v} \times \underset{\sim}{E}}{c}\right] + \mu \left[\underset{\sim}{B} + \frac{\underset{\sim}{v} \times \underset{\sim}{E}}{c}\right] \times \underset{\approx}{Q} = \left(\frac{\partial \underset{\approx}{Q}}{\partial t}\right)_{\text{collisions}} . \tag{3.15}$$

These moments suffice to specify the number of particles with velocity $\underset{\sim}{v}$ and position $\underset{\sim}{x}$ in a given m state where m is the eigenvalue of the z component of the spin (see Table 1). One can see from the calculations in Ref. 4 that these numbers are all that is needed to evaluate nuclear reaction rates and the angular distribution of reaction products.

4. VARIOUS DEPOLARISATION PHENOMENA IN A PLASMA

We may employ Eqs. (3.10) and (3.13) to study the depolarisation of a plasma for the cases considered in the preceeding paper [4]. These are a) the motion of plasma nuclei in an inhomogeneous field, b) collisional depolarisation, c) depolarisation by gyrocollisional effects, and

d) depolarisation by wave particle interactions. We consider only spin 1/2 nuclei. The results in each case are easily generalized to spin 1 nuclei.

4.1 Guiding centre approximation

If the fields seen by the nucleus change little during a precession time, we may solve Eq. (3.10) in the guiding center approximation. Upon dropping the collision terms in Eq. (3.9), we have

$$\frac{\partial \underset{\sim}{P}}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial \underset{\sim}{P}}{\partial t} + \frac{q}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{v}\times\underset{\sim}{B}}{c}\right) \cdot \frac{\partial \underset{\sim}{P}}{\partial \underset{\sim}{v}} + \mu \underset{\sim}{P} \times \underset{\sim}{B} = 0 \ . \qquad (4.1)$$

We have neglected the $\underset{\sim}{v} \times \underset{\sim}{E}$ term in the last term on the left-hand side of Eq. (3.9) because we assume $|\underset{\sim}{E}| \sim v/c\ |\underset{\sim}{B}|$ and $v \ll c$. Treating q/m as large, we can order the terms in Eq. (4.1) (this derivation is similar to the normal drift kinetic derivation in Ref. 6). To this end, let

$$\underset{\sim}{P} = \underset{\sim}{P}_o + \underset{\sim}{P}_1 \ , \qquad (4.2)$$

where $P_1 \sim 0(m/q)\ P_o$. To lowest order, Eq. (4.1) is

$$\frac{q}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{v}\times\underset{\sim}{B}}{c}\right) \cdot \frac{\partial \underset{\sim}{P}_o}{\partial \underset{\sim}{v}} + \mu\ \underset{\sim}{P}_o \times \underset{\sim}{B} = 0 \ . \qquad (4.3)$$

We take $\underset{\sim}{E} \cdot \underset{\sim}{b} \sim 0(m/q)$ (see Ref. 6) where $\underset{\sim}{b} = \underset{\sim}{B}/|B|$. To facilitate the solution of Eq. (4.3), we introduce the velocity $\underset{\sim}{v}_E = c\underset{\sim}{E} \times \underset{\sim}{B}/|B|^2$ and change the velocity space variable to $\underset{\sim}{v}'$ where $\underset{\sim}{v}' = \underset{\sim}{v} - \underset{\sim}{v}_E$. Then Eq. (4.3) becomes

$$\Omega \frac{\partial}{\partial \alpha} \underset{\sim}{P}_o + \Omega_p\ \underset{\sim}{P}_o \times \hat{\underset{\sim}{b}} = 0 \qquad (4.4)$$

where $\Omega \equiv qB/mc$ is the cyclotron frequency, $\Omega_p \equiv g\ qB/mc$ is the precession frequency, and α is the gyrophase angle of $\underset{\sim}{v}'$. The general solution of Eq. (4.4) which is periodic in α is

$$\underset{\sim}{P} = P_o \hat{\underset{\sim}{b}} \tag{4.5}$$

with

$$\frac{\partial P_o}{\partial \alpha} = 0 \; . \tag{4.5b}$$

In the next order, we determine the dependence of P_o on $W = 1/2|\underset{\sim}{v}' - \underset{\sim}{v}' \cdot \underset{\sim}{b}|^2$, $v_\parallel = \underset{\sim}{v}' \cdot \underset{\sim}{b}$, $\underset{\sim}{x}$ and t. Transforming Eq. (4.1) to these variables, we obtain

$$\left(\frac{\partial \underset{\sim}{P}_o}{\partial T} + \underset{\sim}{v} \cdot \frac{\partial \underset{\sim}{P}_o}{\partial \underset{\sim}{x}} + \frac{q}{m} E_\parallel \hat{\underset{\sim}{b}} \cdot \frac{\partial \underset{\sim}{P}_o}{\partial \underset{\sim}{v}}\right)_{\underset{\sim}{x},\, W,\, v_\parallel,\, \alpha} + \Omega \frac{\partial \underset{\sim}{P}_1}{\partial \alpha} + \Omega_p \underset{\sim}{P}_1 \times \underset{\sim}{b} \tag{4.6}$$

where the expression in the parenthesis is to be expressed in the subscript variables. The necessary and sufficient condition for the existence of a solution $\underset{\sim}{P}_1$ of Eq. (4.6) is

$$\int d\alpha \; \hat{\underset{\sim}{b}} \cdot \left[\frac{\partial \underset{\sim}{P}_o}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial \underset{\sim}{P}_o}{\partial \underset{\sim}{x}} + \frac{\varepsilon}{m} E_\parallel \hat{\underset{\sim}{b}} \cdot \frac{\partial \underset{\sim}{P}_o}{\partial \underset{\sim}{v}}\right]_{\underset{\sim}{x}, W, v_\parallel, \alpha} = 0 \; . \tag{4.7}$$

Performing this integration, we obtain the guiding centre drift equation for P_o

$$\frac{\partial P_o}{\partial t} + \dot{\underset{\sim}{r}} \cdot \frac{\partial P_o}{\partial \underset{\sim}{x}} + \dot{W} \frac{\partial P_o}{\partial W} + \dot{v}_\parallel \frac{\partial P_o}{\partial v_\parallel} = 0 \; , \tag{4.8}$$

where

$$\dot{r} = \underset{\sim}{v}_E + v_\parallel \hat{\underset{\sim}{b}} \; , \tag{4.9a}$$

$$\dot{W} = - \nabla \cdot \underset{\sim}{v}_E + v_\parallel \nabla \cdot \hat{\underset{\sim}{b}} + \hat{\underset{\sim}{b}}\hat{\underset{\sim}{b}} : \nabla \underset{\sim}{v}_E , \tag{4.9b}$$

and

$$\dot{v}_\parallel = \underset{\sim}{v}_E \underset{\sim}{v}_E : \nabla \hat{\underset{\sim}{b}} + v_\parallel \underset{\sim}{v}_E \hat{\underset{\sim}{b}} : \frac{\partial \hat{\underset{\sim}{b}}}{\partial \underset{\sim}{x}} + W \frac{\partial}{\partial \underset{\sim}{x}} \cdot \hat{\underset{\sim}{b}} + \underset{\sim}{v}_E \cdot \frac{\partial \underset{\sim}{b}}{\partial t} + \frac{q}{m} E_\parallel . \tag{4.9c}$$

If the plasma is initially totally polarised, $P_o = f_o$, then from Eq. (4.8), and the fact that f_o satisfies the same equation, we may conclude $P_o = f_o$ for all time. Thus, inhomogeneous fields will not change the polarisation state in this approximation.

4.2 Depolarisaton by collisions

We now employ the Fokker-Planck coefficients derived in Appendix A to calculate the depolarisation rate of spin 1/2 particles by collisions. In a homogeneous plasma, we can write Eq. (3.10) with Eq. (3.6) as

$$\frac{\partial \underset{\sim}{P}}{\partial t} + \frac{q}{mc} \underset{\sim}{v} \times \underset{\sim}{B} \cdot \frac{\partial \underset{\sim}{P}}{\partial \underset{\sim}{v}} + \mu \underset{\sim}{B} \times \underset{\sim}{P} = \left(\frac{\partial \underset{\sim}{P}}{\partial t}\right)_L + \int d^3 \underset{\sim}{p} \; \{<\Delta \underset{\sim}{p}> + \frac{\partial}{\partial \underset{\sim}{v}} <\Delta \underset{\sim}{v} \, \Delta \underset{\sim}{p}>\} f \tag{4.10}$$

where $(\partial \underset{\sim}{P}/\partial t)_L \equiv C(F,\underset{\sim}{P})$ represents the Landau collision operator (see Appendix B). In the strong field approximation (see Sec. 4.1)

$$\underset{\sim}{P}_o = P_o \underset{\sim}{b} \text{ where } \frac{\partial P_o}{\partial \alpha} = 0 . \tag{4.11}$$

Hence the solubility condition for $\underset{\sim}{P}_1$ is

$$\frac{\partial P_o}{\partial t} = \left(\frac{\partial P_o}{\partial t}\right)_L + \int d\alpha \; \underset{\sim}{b} \cdot \int d^3 p \; \{<\Delta \underset{\sim}{p}> + \frac{\partial}{\partial \underset{\sim}{v}} <\Delta \underset{\sim}{p} \Delta \underset{\sim}{v}>\} \; f_o(\underset{\sim}{x},\underset{\sim}{v},\underset{\sim}{p},t) . \tag{4.12}$$

We now make a subsidiary expansion treating the Landau operators as large. $(\partial P_o/\partial t)_L$ is the dominant term in Eq. (3.12). Writing $\underset{\sim}{P}_o = \underset{\sim}{P}^o{}_o + \underset{\sim}{P}^1{}_o$, the zeroth order equation is

$$\left(\frac{\partial P_o{}^o}{\partial t}\right)_L = C(F_{1/2}, P^o{}_o) = 0 \ . \qquad (4.13)$$

From the equation for $F_{1/2}$ in the same ordering

$$C(F_{1/2}, F_{1/2}) = 0 \ . \qquad (4.14)$$

The unique solution to Eq. (4.14) is the Maxwellian distribution. In Appendix B we show that Eq. (4.13) then implies

$$P_o^o = \beta F_{1/2} \ , \qquad (4.15)$$

where β is a function of time only. Writing Eq. (4.12) to next order, we obtain

$$\left(\frac{\partial P_o^1}{\partial t}\right)_L = \frac{\partial P_o{}^o}{\partial t} - \int d\alpha \ \underset{\sim}{b} \cdot \int d^3p \ \{\langle\Delta \underset{\sim}{p}\rangle + \frac{\partial}{\partial \underset{\sim}{v}} \langle\Delta \underset{\sim}{p} \Delta \underset{\sim}{v}\rangle\} f_o^o \ (\underset{\sim}{x}, \underset{\sim}{v}, \underset{\sim}{p}, t) \ . \qquad (4.16)$$

The condition for $\underset{\sim}{P}_o{}^1$ to exist is that the integral over velocity of the right-hand side of Eq. (4.16) must vanish (see Appendix B). This condition determines β. Integrating Eq. (4.16) over velocity annihilates the term containing $\langle\Delta \underset{\sim}{p} \ \Delta v\rangle$, which represents the mixing in velocity space. We assume that $\underset{\sim}{P}_o$ is being depolarised by Maxwellian electrons. Using $v_e \gg v_i$ and assuming the electrons are unpolarised in evaluating $\langle\Delta \underset{\sim}{p}\rangle$ for spin-orbit and spin-spin (see Appendix A for the detailed calculation of $\Delta \underset{\sim}{p}$), we obtain

$$\frac{\partial\beta}{\partial t} = -\left[\left(\frac{\mu e}{c}\right)^2 N_e v_{the} \ell n\Lambda + 2(2\pi)^{3/2} \,{}^2 N_e b_{min}^{-2} \; v_{the}^{-1} \right]\beta \tag{4.17}$$

where v^2_{the} = 2Te/me and the other symbols are defined in Appendix A. It is convenient for comparison to present this in terms of the number of particles in the down state, $|c(\downarrow)|^2 n_o$

$$\frac{\partial |c(\downarrow)|^2}{\partial t} = \frac{N_e v_{the}}{3} \left[\left(\frac{\mu e}{2c}\right)^2 \pi^{-1/2} \; \ell n\Lambda + 2 \; \alpha^2 b_{min}^{-2} v_{the}^{-2} \right] (1-2|c(\downarrow)|^2) \; . \tag{4.18}$$

4.3. Collisions and inhomogeneous field

In this section we calculate the effect of collisions on a polarised distribution in an inhomogeneous magnetic field. The physical mechanism for depolarisation can be attributed to the decorrelation of the orbital motion and the spin precession. As a particle moves in a gyro-orbit it sees a changing magnetic field. Superimposed on the precessional motion is a small periodic oscillation of the $\underset{\sim}{p}$ vector. When a collision changes the center of gyromotion, the correlation between gyromotion and oscillation is lost leading to a small nonperiodic change in $\underset{\sim}{p}$. The result of many collisions is a diffusion in $\underset{\sim}{p}$ space.

Consider a sheared magnetic field of constant field strength

$$\begin{aligned} \underset{\sim}{B} &= B \, \underset{\sim}{b} \, , \\ \underset{\sim}{b} &= b_z(x) \, \hat{\underset{\sim}{z}} + b_y(x) \hat{\underset{\sim}{y}} \; . \end{aligned} \tag{4.19}$$

(The case of varying field strength has been treated with a model collision operator and shows no qualitative changes.) We first calculate the distribution function F.

Write the Fokker-Planck equation for F in the variables $\underset{\sim}{x}$, c, $v_\parallel$ and α,

$$\Omega \frac{\partial F}{\partial \alpha} + v_\perp \sin\alpha \frac{\partial F}{\partial x} + \frac{v_\perp^2 \sin\alpha\cos\alpha}{L} \frac{\partial F}{\partial v_\parallel} + \frac{v_\parallel^2 \sin\alpha}{L} \frac{\partial F}{\partial \alpha} = C_L(F,F) \qquad (4.20)$$

where $C_L(F,F)$ is the Landau collision operator and $v_\parallel = \underset{\sim}{v} \cdot \underset{\sim}{b}$, $v^2 = \varepsilon$, $\underset{\sim}{v} \cdot \underset{\sim}{x} = v_\perp \sin\alpha$, and $v_\perp = |\underset{\sim}{v} - v_\parallel \underset{\sim}{b}|$. L is the shear length $L^{-1} = b_z^{-1}\, \partial b_y/\partial x = b_y^{-1}\, \partial b_z/\partial_x$, and we assume it is constant. Make the strong magnetic field approximation and write.

$$F = F_o + F_1 \cdots\cdot \qquad (4.21)$$

where $F_1 = O(\frac{v}{\Omega L})\ F_o$.

Solving Eq. (4.20) order by order, we get for the zeroth order distribution

$$\Omega \frac{\partial F_o}{\partial \alpha} = 0 \ . \qquad (4.22)$$

To first order, Eq. (4.20) becomes

$$\Omega \frac{\partial F_1}{\partial \alpha} = -v_\perp \sin\alpha \frac{\partial F_o}{\partial x} - \frac{v_\perp^2 \sin\alpha \cos\alpha}{L} \frac{\partial F_o}{\partial v_\parallel} + C(F_o,F_o) \ . \qquad (4.23)$$

The solubility condition for F_1 is obtained by integrating Eq. (4.23) over α

$$C(F_o,F_o) = 0 \ . \qquad (4.24)$$

Therefore F_o is a Maxwellian, in general dependent on space, but we assume $\partial F_o/\partial x = 0$ for this calculation. With this result Eq. (4.23) becomes

$$\Omega \frac{\partial F_1}{\partial \alpha} = 0 \; . \tag{4.25}$$

The solubility condition to next order is

$$C(F_o,F_1) + C(F_1,F_o) = 0 \; . \tag{4.26}$$

$F_1 + F_o$ is again a Maxwellian and by induction we are entitled to set $F = F_{Maxwellian}$ to all orders. The procedure for finding the evolution of $\underset{\sim}{P}$ is similar, however the result is less trivial. Making the strong field approximation, Eq. (3.10) becomes, upon neglecting all but the normal Landau collision operator,

$$(\ell_o+\ell_1)\, \underset{\sim}{P} = 0 \tag{4.27}$$

where $\ell_o = \Omega \frac{\partial}{\partial \alpha} - \Omega_p \underset{\sim}{bx}$

and

$$\ell_1 = v_\perp \sin\alpha \frac{\partial}{\partial x} + v_\perp^2 \frac{\sin\alpha \cos\alpha}{L} \frac{\partial}{\partial v_\parallel} + v_\parallel \frac{\sin^2\alpha}{L} \frac{\partial}{\partial \alpha} \; . \tag{4.28}$$

Formally, $\ell_1 = O(v/\Omega L)\, \ell_o$ and $\partial \underset{\sim}{P}/\partial t$ is withheld to third order where it is needed to find a consistent solution. Expand $\underset{\sim}{P}$

$$\underset{\sim}{P} = \underset{\sim}{P}_o + \underset{\sim}{P}_1 + \ldots\ldots \tag{4.29}$$

The zeroth order equation gives

$$\ell_o \underset{\sim}{P}_o = 0 \tag{4.30}$$

The solution of Eq. (4.30) is (see Sec. 4.1)

$$\frac{\partial P_o}{\partial \alpha} = 0 \tag{4.31}$$

$$\underset{\sim}{P}_o = P_o \underset{\sim}{b} \ .$$

The solubility condition for $\underset{\sim}{P}_{n+1}$ is

$$\int \partial\alpha \, \underset{\sim}{b} \cdot \ell_1 \, \underset{\sim}{P}_n = 0 \ . \tag{4.32}$$

For $\underset{\sim}{P}_1$ to exist P_o must satisfy

$$C(F,P_o) = 0 \ , \tag{4.33}$$

which by a modification of the H theorem contained in Appendix B is shown to imply

$$P_o = \Pi F \tag{4.34}$$

where Π is independent of $\underset{\sim}{v}$. Additionally, we show that if Π is independent of $\underset{\sim}{x}$ at any time, it remains; therefore we make $d\Pi/d\underset{\sim}{x} = 0$. $\underset{\sim}{P}_1$ is now determined by

$$\ell_1 \, \underset{\sim}{P}_o = \ell_o \, \underset{\sim}{P}_1 = v_\perp \sin\alpha \, P_o \frac{\partial \underset{\sim}{b}}{\partial x} \ . \tag{4.35}$$

It is convenient to introduce basis vectors $\underset{\sim}{x}$, $\underset{\sim}{b}$, and $\underset{\sim}{i}$ where $\underset{\sim}{x} \cdot \underset{\sim}{i} = 0$, and $\underset{\sim}{b} \cdot \underset{\sim}{i} = 0$, and $\underset{\sim}{i} = b_z(x)\underset{\sim}{y} - b_y(x)\underset{\sim}{z}$. $\underset{\sim}{b}$ and $\underset{\sim}{i}$ form a useful pair as $d\underset{\sim}{b}/dx = \underset{\sim}{i}/L$ and $d\underset{\sim}{i}/dx = -\underset{\sim}{b}/L$.

Inverting ℓ_o in Eq. (4.35) we obtain

$$P_{1x} = -\frac{\Omega_p P_o v_\perp \sin\alpha}{\Omega^2 - \Omega_p^2}\frac{1}{L} , \qquad (4.36a)$$

$$P_{1i} = \frac{\Omega P_o v_\perp \cos\alpha}{\Omega^2 - \Omega_p^2}\frac{1}{L} , \qquad (4.36b)$$

$$P_{1b} = P_{1b} \quad , \frac{\partial P_{1b}}{\partial \alpha} = 0 \qquad (4.36c)$$

where $\underset{\sim}{P} = P_x \underset{\sim}{x} + P_i \underset{\sim}{i} + P_b \underset{\sim}{b}$. P_{1b} is determined by the solubility condition for $\underset{\sim}{P}_2$. We write the next order equations

$$\ell_1 \underset{\sim}{P}_1 = \ell_o \underset{\sim}{P}_2 \qquad (4.37)$$

as

$$-C(F,P_{1x}) = \Omega \frac{\partial P_{2x}}{\partial \alpha} + \Omega_p P_{2i} , \qquad (4.37a)$$

$$\frac{\Omega v_\perp v_\parallel \sin\alpha}{L^2(\Omega^2-\Omega_p^2)} - \frac{\Omega v_\perp^2 \sin\alpha\cos\alpha}{L^2(\Omega^2-\Omega_p^2)} \qquad (4.37b)$$

$$-\frac{v_\perp \sin\alpha}{L} P_{1b} - C(F,P_{1i}) = \Omega\frac{\partial P_{2i}}{\partial\alpha} - \Omega_p P_{2x} ,$$

$$-\frac{\Omega P_o v_\perp^2 \cos\alpha\sin\alpha}{L^2(\Omega^2-\Omega_p^2)} + v_\perp \sin\alpha \frac{\partial P_{1b}}{\partial x} + \frac{v_\perp^2 \sin\alpha\cos\alpha}{L^2}\frac{\partial P_{1b}}{\partial v_\parallel} - C(F,P_{1b}) = \Omega\frac{\partial P_{2b}}{\partial\alpha}$$

(4.37c)

where Eqs. (4.37a), (4.37b), and (4.37c) are the $\underset{\sim}{x}$, $\underset{\sim}{i}$, and $\underset{\sim}{b}$ components, respectively, of Eq. (4.37). The solubility condition for $\underset{\sim}{P}_2$ is

$$C(F,P_{1b}) = 0 . \qquad (4.38)$$

P_{1b} is therefore proportional to F (see Appendix B) so we can include it in P_o and set $P_{1b} = 0$. We write the third order solubility condition

$$\int d\alpha \left[\frac{\partial P_o}{\partial t} + v_\perp \sin\alpha \frac{\partial P_{2b}}{\partial x} + \frac{v_\perp \sin\alpha}{L} P_{2i} + \frac{v_\perp^2 \sin\alpha \cos\alpha}{L} \frac{\partial P_{2b}}{\partial v_\parallel} + \frac{v_\parallel \sin\alpha^2}{L} \frac{\partial P_{2b}}{\partial \alpha} + C(F,P_{2b})\right] = 0 . \tag{4.39}$$

Rather than solve Eq. (4.37) we extract the terms with the α dependence that lead to nonvanishing contributions to Eq. (4.39). From Eq. (4.37c) it is clear that the only contribution from P_{2b} is in the collision term $C(F,P_{2b})$, in which only the α independent part of P_{2b} contributes. It is useful to write

$$\int \partial\alpha\, v_\perp \frac{\sin\alpha}{L} P_{2i} = - \int \partial\alpha \frac{v_\perp \sin\alpha}{L (\Omega^2 - \Omega_p^2)} \left(\Omega^2 \frac{\partial^2 P_{2i}}{\partial\alpha^2} + \Omega_p^2 P_{2i}\right) . \tag{4.40}$$

Eliminating $_{2x}$ from Eqs. (4.37a) and (4.37b) and extracting the term proportional to $\sin\alpha$ gives

$$\int d\alpha \left\{\frac{\partial P_o}{\partial t} - \frac{1}{L^2 (\Omega^2 - \Omega_p^2)^2} \left[\Omega_p^2 v_x C(F, v_x P_o) + \Omega^2 v_y C(F, v_y P_o)\right] + C(F,P_{2b})\right\} = 0 . \tag{4.41}$$

Equation (4.41) is an equation relating $\partial\pi/\partial t$ and P_{2b}. The condition that a solution for P_{2b} exist is that the integral over velocity space of the first two terms disappear (see Appendix B). Integrating Eq. (4.41) over velocity we obtain the evolution of Π.

$$n_o \frac{d\Pi_o}{dt} + \{\frac{2}{3} \frac{\Gamma}{L^2} \frac{(\Omega_p^2+\Omega^2)}{(\Omega^2-\Omega_p^2)^2} \int \frac{d^3\underset{\sim}{v} d^3\underset{\sim}{v}' \; F(v) \; F(v')}{|\underset{\sim}{v}-\underset{\sim}{v}'|}\} \, \Pi_o = 0 \tag{4.42}$$

where $\Gamma = 8\pi \; q^4 \; \ln \Lambda \; n_o/m^2$. In deriving Eq. 4.32 we make use of the property of the Landau operator

$$\int d^3v \; d^3\underset{\sim}{v}' \; \hat{\underset{\sim}{x}} \cdot \underset{\approx}{G} \cdot \hat{\underset{\sim}{x}} \, F(v) \; F(v') \equiv \frac{2}{3} \int d^3v \; d^3v' \; \frac{F(v) \; F(v')}{|\underset{\sim}{v}-\underset{\sim}{v}'|} \quad , \tag{4.43}$$

$$\underset{\approx}{G} = \frac{|\underset{\sim}{v}-\underset{\sim}{v}'|^2 \; \underset{\approx}{I} - (\underset{\sim}{v}-\underset{\sim}{v}') \; (\underset{\sim}{v}-\underset{\sim}{v}')}{|\underset{\sim}{v}-\underset{\sim}{v}'|^3} \quad . \tag{4.44}$$

The integral in Eq. (4.42) is evaluated easily because F is a Maxwellian. Equation (4.42) shows that $\partial\Pi_o/\partial x$ remains zero if zero initially. The final result is

$$\Pi_o = \Pi_{oo} \exp(-\gamma t) \tag{4.45}$$

where

$$\gamma = (\frac{2}{\pi})^{1/2} \; \frac{2}{3} \frac{\rho^2}{L^2} \frac{1 + \Omega_p^2/\Omega^2}{(1-\Omega_p^2/\Omega^2)^2} \; \nu \; , \tag{4.46}$$

and $\nu = 2\pi(\ln\Lambda) q^4 m^{-1/2} T^{-3/2} \; n_o$ and $\rho = T^{1/2} m^{1/2} c q^{-1} B^{-1}$, the collision frequency and gyroradius, respectively. The calculation shows how depolarisation arises. The inhomogeneous field perturbs the distribution to give $\underset{\sim}{P}_1$. Then the collisions acting on $\underset{\sim}{P}_1$ drive a $\underset{\sim}{P}_2$ which in the subsequent motion gives depolarisation. This depolarisation mechanism is strong for electrons because $\Omega-\Omega_p = 0.001\Omega_p$ (almost a resonance) and a typical electron depolarisation time in a 10 keV, 50 kilogauss $n = 10^{14}$ cm^{-3} plasma with $L = 10^3$ is 6 sec. For deuterons $\Omega-\Omega_p = 0.4\Omega$ and for the same conditions the depolarisation time is approximately, 2000 sec.

4.4 Quasilinear analysis

We investigate the effect of magnetic field fluctuations on polarisation using Eq. (3.10). Consider an infinite plasma with a uniform magnetic field in the $\underset{\sim}{z}$ direction

$$\underset{\sim}{B}_o = B_o \hat{\underset{\sim}{z}} . \tag{4.47}$$

The fluctuation level is assumed to be small so that $\delta\underset{\sim}{B}$, $\delta\underset{\sim}{E} << \underset{\sim}{B}_o$. Fourier transforming $\delta\underset{\sim}{B}$ and $\delta\underset{\sim}{E}$ we have for $\delta\underset{\sim}{B}$

$$\delta\underset{\sim}{B}(r,t) = \int d\underset{\sim}{k} d\omega \; \delta B_{\underset{\sim}{k},\omega} \exp(-i\omega t+i\underset{\sim}{k}\cdot\underset{\sim}{r}) . \tag{4.48}$$

By the random phase approximation on $\delta\underset{\sim}{B}$ and $\delta\underset{\sim}{E}$ we have

$$<\delta B^*_{\underset{\sim}{k},\omega},\; \delta B_{\underset{\sim}{k},\omega'}> = <\delta B_{-\underset{\sim}{k}',-\omega'} \delta B_{\underset{\sim}{k}\omega}> = \sum_j \underset{\approx}{I}^j_k \; \delta(\omega'-\omega_k) \; \delta(\underset{\sim}{k}'-\underset{\sim}{k}) \; \delta(\omega-\omega^j_k) \tag{4.49}$$

where $\underset{\approx}{I}^j_k$ represents the intensity of the magnetic field in the j'th plasma mode.

We drop collisions in Eq. (3.10) and order the operators in the perturbation strength. Equation (3.10) becomes

$$(L_o+L_1) \cdot (\underset{\sim}{P}_o+\underset{\sim}{P}_1+\underset{\sim}{P}_2 \ldots) = 0 , \tag{4.50}$$

where

$$L_o = \frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} + \Omega_o \frac{\partial}{\partial \alpha} + \mu \underset{\sim}{B}_o \times \tag{4.51}$$

$$L_1 = \frac{q}{m}\{\delta \underset{\sim}{E} + \frac{\underset{\sim}{v}\times\delta\underset{\sim}{B}}{c}\}\cdot\frac{\partial}{\partial\underset{\sim}{v}} + \mu\delta\underset{\sim}{B}\times$$

and α is gyrophase angle. We have dropped the $\underset{\sim}{v} \times \delta\underset{\sim}{E}/c$ term as being small (its effect is similar in form to $\delta\underset{\sim}{B}$). The zeroth order solution is obtained from

$$L_o\underset{\sim}{P}_o = 0 \tag{4.52}$$

which gives, as before,

$$P_o = \Gamma_o\hat{\underset{\sim}{b}}\ ,\ \frac{\partial P_o}{\partial\alpha} = 0 \tag{4.53}$$

where $\underset{\sim}{b} = \underset{\sim}{B}_o\ |\underset{\sim}{B}|^{-1}$.

We now solve Eq. (4.50) to first order to determine $\underset{\sim}{P}_1$. The equation for $\underset{\sim}{P}_1$ is

$$L_o\underset{\sim}{P}_1 = -L_1\underset{\sim}{P}_o\ . \tag{4.54}$$

It is convenient to introduce then a change of variable to solve this equation. Let

$$\underset{\sim}{X} = \underset{\sim}{x} + \frac{\underset{\sim}{v}\times\hat{\underset{\sim}{b}}}{\Omega}\ . \tag{4.55}$$

In terms of $\underset{\sim}{X}$

$$L_o = \frac{\partial}{\partial t} + v_\parallel\frac{\partial}{\partial x_\parallel} + \Omega_o\frac{\partial}{\partial\alpha} + \underset{\sim}{B}_o\times\ . \tag{4.56}$$

Decompose $\underset{\sim}{P}_1$ into a Fourier series in α

$$\underset{\sim}{P}_1 = \sum_n \underset{\sim}{P}_n e^{in\alpha} . \tag{4.57}$$

We can invert L_o in Eq. (4.54) by Fourier transforming $\underset{\sim}{P}_n$ in the variables $(\underline{X}, t)$

$$\underset{\sim}{P}_n(\underset{\sim}{X}, t) = \int d^3\underset{\sim}{k} d\omega \; \underset{\sim}{\tilde{P}}_n(\underset{\sim}{k}, \omega) \exp\{i\underset{\sim}{k}\cdot\underset{\sim}{X} - i\omega t\} . \tag{4.58}$$

Transform $\underset{\sim}{P}_1$ in $\underset{\sim}{x}, t$ space

$$\underset{\sim}{P}_1(\underset{\sim}{x}, t) = \int d^3\underset{\sim}{k} \; d\omega \; \underset{\sim}{P}(\underset{\sim}{k}, \omega) \exp\{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t\} \tag{4.59}$$

and note that $\underset{\sim}{P}(\underset{\sim}{k}, \omega)$ and $\underset{\sim}{\tilde{P}}_n$ are related by

$$\underset{\sim}{\tilde{P}}(\underset{\sim}{k}, \omega) = \sum_n e^{in\alpha} \underset{\sim}{\tilde{P}}_n \exp(-i\underset{\sim}{k} \cdot \frac{\underset{\sim}{v}\times\hat{\underset{\sim}{b}}}{\Omega}) \tag{4.60}$$

$$= \sum_{nn'}\sum e^{i(n-n')\alpha} J_n(\lambda)\underset{\sim}{\tilde{P}}_n$$

where $\lambda = \frac{k_\perp v_\perp}{\Omega}$. After some algebra Eq. (4.54) can be transformed into an equation for the $\underset{\sim}{P}_n$'s,

$$(\omega - v_\parallel k_\parallel - n\Omega)\underset{\sim}{\tilde{P}}_n + \Omega_p \hat{\underset{\sim}{b}} \times \underset{\sim}{\tilde{P}}_n = \frac{q}{m} \delta\underset{\sim}{E}(\underset{\sim}{k}, \omega) \cdot \Big[(v_\parallel \underset{\sim}{b} J_n + v_\perp \frac{\underset{\sim}{b} \times \underset{\sim}{k}_\perp}{k_\perp} J_n') \frac{\partial P_o}{\partial v^2} \tag{4.61}$$

$$+ (\underset{\sim}{b} J_n + \frac{\underset{\sim}{b}\times\underset{\sim}{k}}{k_\perp} \frac{v_\perp k_\parallel}{\omega} J_n') \frac{\partial P_o}{\partial v_\parallel} \Big] \underset{\sim}{b} + \mu J_n (\delta\underset{\sim}{B}(\underset{\sim}{k}, \omega) \times \hat{\underset{\sim}{b}}) P_o .$$

The evolution of P_o can be calculated from Eq. (4.50) to second order

$$\frac{\partial \underset{\sim}{P}_o}{\partial t} + L_o \underset{\sim}{P}_2 = L_1 \underset{\sim}{P}_1 \qquad (4.62)$$

where $\underset{\sim}{P}_1$ is obtained from Eqs. (4.57)-(4.61). In order to find $\partial P_o/\partial t$ we take the $\underset{\sim}{b}$ component of Eq. (4.62) and average over the ensemble and α. We obtain

$$\frac{\partial \underset{\sim}{P}_o}{\partial t} = \int d\alpha \{ \frac{q}{m} \langle (\delta \underset{\sim}{E} + \frac{(\delta \underset{\sim}{E} \times \underset{\sim}{k}) \times \underset{\sim}{v}}{\omega} \cdot \frac{\partial \underset{\sim}{P}_1 \cdot \underset{\sim}{b}}{\partial \underset{\sim}{v}} \rangle + \mu \langle (\delta \underset{\sim}{B} \times \underset{\sim}{P}_1) \cdot \underset{\sim}{b} \rangle \ . \qquad (4.63)$$

The brackets < > denote the ensemble average. The first term (after $\underset{\sim}{P}_1 \cdot \underset{\sim}{b}$ is expressed in terms of P_o) is just the normal quasilinear expression (see for instance Ref. 4) for wave-particle interactions and we do not quote it here. (Its term represents velocity space diffusion and does not contribute to the bulk depolarisation.) We will write the second term explicitly. Let

$$\delta B_{\pm} = (\hat{\underset{\sim}{x}} \pm i\hat{\underset{\sim}{y}}) \cdot \delta \underset{\sim}{B} \ ,$$

$$P_{\pm} = (\hat{\underset{\sim}{x}} \pm i\hat{\underset{\sim}{y}}) \cdot \underset{\sim}{P}_1 \qquad (4.64)$$

so that

$$(\delta \underset{\sim}{B} \times \underset{\sim}{P}_1) \cdot \underset{\sim}{b} = \frac{-i}{2}(\delta B_- P_{1+} - \delta B_+ P_{1-}) \ . \qquad (4.65)$$

From Eqs. (4.60) and (4.61)

$$P_{\pm} = \mp \mu P_o \int d^3\underset{\sim}{k} d\omega \delta B_{\pm}(\underset{\sim}{k},\omega) \sum_{nn'}\sum \frac{e^{i(n-n')^2} J_n(\lambda) J_{n'}(\lambda)}{(\omega - v_{\parallel} k_{\parallel} - n\Omega \pm \Omega_p)} e^{i\underset{\sim}{k} \cdot \underset{\sim}{x} - i\omega t} \ . \qquad (4.66)$$

From Eqs. (4.66), (4.65), and (4.63), omitting the normal quasilinear terms, we find

$$\frac{\partial P_o}{\partial t} = \frac{-i\mu^2 P_o}{2} \sum_n \int d^3\underset{\sim}{k} d^3\underset{\sim}{k}' d\omega d\omega' \; \frac{\langle \delta \underset{\sim}{B}(\underset{\sim}{k},\omega)\,\delta B_{+}(\underset{\sim}{k}',\omega')\rangle}{(\omega - v_{\parallel} k_{\parallel} - n\Omega \pm \Omega_p)}$$

$$+ \frac{\langle \delta B_{+}(\underset{\sim}{k},\omega)\,\delta B_{-}(\underset{\sim}{k}',\omega)\rangle}{(\omega - v_{\parallel} k_{\parallel} - n\Omega \pm \Omega_p)} \; J_n^{\,2}(\lambda) e^{i(\underset{\sim}{k}+\underset{\sim}{k}')\underset{\sim}{x} - i(\omega+\omega')t} \; . \qquad (4.67)$$

Using Eq. (4.49), [with $(-\underset{\sim}{k}, -\omega')$ replacing $(\underset{\sim}{k}', \omega')$] to evaluate the ensemble average we get

$$\frac{\partial P_o}{\partial t} = -\mu^2 P_o \int d^3k \sum_{nj}\sum J_n^{\,2}(\lambda) \;\; \delta(\omega_k^{\,j} - k_{\parallel} v_{\parallel} - n\Omega - \Omega_p) \; (\hat{\underset{\sim}{x}} - i\hat{\underset{\sim}{y}}) \cdot \underset{\approx}{I}_k^{\,j} \cdot (\hat{\underset{\sim}{x}} + i\hat{\underset{\sim}{y}}) \; . \qquad (4.68)$$

In the standard way we have replaced the imaginary part of the resonant denominator of Eq. (4.67) by a delta function. Equation (4.68) expresses the evolution of P_o in terms of the wave spectrum.

5. CONCLUSION

We have generalised the usual kinetic description of a plasma to include variables specifying the spin state of each particle in addition to its position and velocity. We have obtained a new kinetic equation for the evolution of the generalised distribution function. This equation provides the basis for accurately calculating the depolarisation rate in an arbitarily polarised plasma. In this investigation we found the vector $\underset{\sim}{p}$ to be useful for describing spin.

The depolarisation rates we have calculated are sufficiently low, provided significant fluctuations at the precession frequency are absent, to make polarised fusion reactors an attractive concept (see discussion in Ref. 2). Equations (4.8) and (4.9) show that in the collisionless guiding centre limit the plasma stays polarised. Collisional depolarisation is negligible, see Eq. (4.17), where the depolarisation cross section is of order 10^{-29} cm^2. Depolarisation by collisions in inhomogeneous magnetic fields, given in Eqs. (4.45) and (4.46), gives a rate of order 10^{-4} S^{-1} in a typical tokamak plasma. Equation (4.68) gives the rate of depolarisation by waves. For a fluctuating field of one gauss at the precession frequency, Eq. (4.68) predicts depolarisation of tritons in about ten seconds. The results are more accurate versions of those given in Refs. 1 and 2. More generally, the use of these kinetic equations, Eqs. (3.3) and (3.5), and the corresponding moment equations, Eqs. (3.10), (3.13), and (3.15), is valid whenever spin-spin forces are small and the particle orbits are classical.

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APPENDIX A: FOKKER-PLANCK COLLISION TERMS FOR SPIN

We derive some Fokker-Planck collision terms for the spin-orbit and spin-spin interactions during collisions. The spin 1/2 case is treated here; no qualitative changes are expected with spin 1. The collisional orbit is treated as known and the change in the spin is calculated for this orbit. (There is a change in the orbit due to the spin interactions but it is small for orbital angular momentum greater than h, so within the normal classical approximation to the collisions this treatment is sufficient.) An exhaustive treatment of collisions is not included here because their effect on polarisation is insignificant.

a) <u>Spin-Orbit interaction.</u>

A particle moving nonrelativistically with velocity $\underset{\sim}{v} = d\underset{\sim}{r}/dt$ through the electric field

$$\underset{\sim}{E} = \frac{e\underset{\sim}{r}}{r^3} \tag{A.1}$$

of another particle experiences a magnetic field

$$\underset{\sim}{B} = \frac{\underset{\sim}{v} \times \underset{\sim}{E}}{c} \quad . \tag{A.2}$$

The equation of motion of the polarisation vector is

$$\dot{\underset{\sim}{p}} = \mu\underset{\sim}{B}\times\underset{\sim}{p} = \frac{\mu e}{c}\frac{1}{r^3}\,\underset{\sim}{\ell}\times\underset{\sim}{p} \tag{A.3}$$

where $\underset{\sim}{\ell} = \underset{\sim}{v} \times \underset{\sim}{r}$ is the orbital angular momentum. We iterate this equation to find the first and second order changes in $\underset{\sim}{p}$ due to the collision. For simplicity we take straight line orbits as shown in Fig. A.1. To first order

$$\Delta \underset{\sim}{p}^{(1)}(\theta) = \frac{\mu e}{cb}(\hat{\underset{\sim}{\ell}}\times \underset{\sim}{p})(\sin\theta + 1) \tag{A.4}$$

where $\hat{\underset{\sim}{\ell}} = \underset{\sim}{\ell}/\ell$, b is the impact parameter and $\underset{\sim}{v} \cdot \underset{\sim}{r} = |\underset{\sim}{v}|\ |\underset{\sim}{r}|\ \sin\theta$. $\Delta\underset{\sim}{p}^{(1)}(\pi/2)$ averages to zero when $\underset{\sim}{\ell}$ is average over the directions perpendicular to $\underset{\sim}{v}$. The drag term, $\Delta\underset{\sim}{p}^{(2)}$ is obtained from the second order iteration of Eq. (A.3), the diffusion term $\Delta\underset{\sim}{p}\Delta\underset{\sim}{p}$ is calculated from Eq. (A.4). We obtain from Eq. (A.4) and Eq. (A.3)

$$\Delta \underset{\sim}{p}^{(2)}\left(\frac{\pi}{2}\right) = -2\left(\frac{\mu e}{cb}\right)^2 (\hat{\underset{\sim}{\ell}} \times(\hat{\underset{\sim}{\ell}}\times \underset{\sim}{p})) , \tag{A.5}$$

and from Eq. A.4,

$$\Delta \underset{\sim}{p}^{(1)}\left(\frac{\pi}{2}\right)\Delta \underset{\sim}{p}^{(1)}\left(\frac{\pi}{2}\right) = -4\left(\frac{\mu e}{cb}\right)^2 (\underset{\sim}{p}\times\hat{\underset{\sim}{\ell}})(\hat{\underset{\sim}{\ell}}\times\underset{\sim}{p}) . \tag{A.6}$$

We average Eq. (A.5) and Eq. (A.6) over random orientations of $\underset{\sim}{\ell}$, integrate over all impact parameters greater than b_{min}, and average over the relative velocity. (The appropriate choice of b_{min} is given in Ref. 2.) The results are

$$\langle \Delta \underset{\sim}{p}_i \rangle = -4\pi\left(\frac{\mu e}{c}\right)^2 n \ell n\ \Lambda\left[\underset{\sim}{p}_i \times\left[\int v(\underset{=}{I} - \hat{\underset{\sim}{v}}\ \hat{\underset{\sim}{v}})F_2 d^3v\right]\times \underset{\sim}{p}_i\right] , \tag{A.7}$$

$$\langle \Delta \underset{\sim}{p}_i \rangle = -2\pi\left(\frac{\mu e}{c}\right)^2 n\ \ell n\Lambda\left[\underset{\sim}{p}_i \cdot \int v(\underset{=}{I} + \hat{\underset{\sim}{v}}\hat{\underset{\sim}{v}})\ F_2 d^3v\right] , \tag{A.8}$$

where $\hat{\underset{\sim}{v}} = v^{-1}(\underset{\sim}{v}_2 - \underset{\sim}{v}_1)$ and $v = |\underset{\sim}{v}_2 - \underset{\sim}{v}_1|$ and $\ell n\ \Lambda$ is the cut-off logarithm. The spin-velocity correlation term is similarly found from Eq. (A.4) and $\Delta\underset{\sim}{v} = 2\ e^2/mvb\ \underset{\sim}{b}$.

$$\langle \Delta \underset{\sim}{p}_i \Delta \underset{\sim}{v} \rangle = 4\pi \frac{e}{m} \left(\frac{\mu e}{c}\right) n \ \ell n \ \Lambda \left[\underset{\sim}{p} \times \underline{\underline{I}} \times \int \hat{\underset{\sim}{v}} F_2 d^3 / \underset{\sim}{v} \right] . \tag{A.8b}$$

Equations (A.7), (A.8), and (A.9) can be used in Eq. (3.6) to calculate the spin-orbit depolarisation rate (see Sec. 4.2).

b) Spin-Spin interaction.

The purpose of showing the derivation of these Fokker-Planck terms is to demonstrate the effect of spin-spin correlations and the consequent mixing of spin states on this formalism. During the collision the spin-spin forces mix the spin states of the two particles. We can still define $\underset{\sim}{p}$ for each particle as $\underset{\sim}{p}_i = \langle \psi | \underset{\sim}{s}_i | \psi \rangle$ where $|\psi\rangle$ is a mixed state wave function. However, we cannot easily determine $\langle \psi | \underset{\sim}{s}_1 \underset{\sim}{s}_2 | \psi \rangle$, for instance. We describe a formal perturbative method for obtaining the change in $\underset{\sim}{p}$.

Consider two interacting particles. Let us calculate the change $\Delta \underset{\sim}{p}_1$ of the $\underset{\sim}{p}$ of particle 1

$$i\hbar \frac{\partial}{\partial t} \langle \psi | \underset{\sim}{s}_1 | \psi \rangle = \langle \psi | [\underset{\sim}{s}_1, H] | \psi \rangle \tag{A.9}$$

The interaction Hamiltonian for spin-spin interaction dropping the contact potential (Ref.2) is

$$H = h^2 \mu_1 \mu_2 \frac{\underset{\sim}{s}_1}{r^3} (\underline{\underline{I}} - 3\hat{\underset{\sim}{r}}\hat{\underset{\sim}{r}}) \cdot \underset{\sim}{s}_2 . \tag{A.10}$$

Let $\alpha = h \mu_1 \mu_2$. Combining Eq. (A.9) and (A.10), we obtain

$$\dot{\underset{\sim}{p}}_1 = \alpha \langle \psi | \underset{\sim}{s}_1 \times \frac{(\underline{\underline{I}} - 3\hat{\underset{\sim}{r}}\hat{\underset{\sim}{r}})}{r^3} \cdot \underset{\sim}{s}_2 | \psi \rangle . \tag{A.11}$$

$\underset{\sim}{p}$ and $|\psi\rangle$ change little during the interaction so we expand the wave function in powers of α

$$|\psi\rangle = |\psi_o\rangle + \alpha|\delta\psi^1(t)\rangle + \alpha^2|\delta\psi^{(2)}(t)\rangle \tag{A.12}$$

where $|\psi_o\rangle$ is the wave function for the spins of the two interacting particles prior to the interaction. We solve Schrödinger's equation to next order by obtaining

$$|\delta\psi^{(1)}(t)\rangle = i\underset{\sim}{s}_1 \cdot \int_{-\infty}^{t} dt\left[\frac{\underline{\underline{I}}-3\hat{\underset{\sim}{r}}\hat{\underset{\sim}{r}}}{r^3}\right] \cdot \underset{\sim}{s}_2|\psi_o\rangle \,. \tag{A.13}$$

We now write $\dot{\underset{\sim}{p}}$, accurate to second order in α

$$\dot{\underset{\sim}{p}}_1 = \alpha\langle\psi_o|\underset{\sim}{s}_1\times\left[\frac{\underline{\underline{I}}-3\hat{\underset{\sim}{r}}\hat{\underset{\sim}{r}}}{r^3}\right]\cdot\underset{\sim}{s}_2|\psi_o\rangle + \mathrm{Re}\{2i\alpha^2\langle\psi_o|s_1\times\left[\frac{\underline{\underline{I}}-3\hat{\underset{\sim}{r}}\hat{\underset{\sim}{r}}}{r^3}\right]\cdot\underset{\sim}{s}_2\underset{\sim}{s}_1\cdot$$
$$\int_{-\infty}^{t} dt\left[\frac{\underline{\underline{I}}-3\hat{\underset{\sim}{r}}\hat{\underset{\sim}{r}}}{r^3}\right]\cdot\underset{\sim}{s}_2|\psi_o\rangle\}. \tag{A.14}$$

We assume that the particles are initially in pure states, that is, they can be in mixed states with third particles but not with each other. This assumption is discussed in Appendix C but the point is somewhat academic because for times of interest, the particles are approximately in pure state. Proceeding, we write $|\psi_o\rangle$ as the product of the individual particle wave functions. With this assumption we can work out $\Delta\underset{\sim}{p}_1\Delta\underset{\sim}{p}_1$ and $\Delta\underset{\sim}{p}_1$ by integrating Eq. (A.14) over the particle orbit. A useful identity is

$$\underset{\sim}{s}_i\underset{\sim}{s}_i = \underline{\underline{I}} + i\underline{\underline{I}}\times\underset{\sim}{s}_i. \tag{A.15}$$

Notice also $\langle\psi_o|\underset{\sim}{s}_1|\psi_o\rangle = \underset{\sim}{p}_1\ (t=0)$. We omit the details of the algebra involved in performing the averages over the initial conditions of the collisional orbit (orientation, impact parameters, relative velocity, and spin of the scatterer). The final results for the Fokker-Planck coefficients are

$$\langle\Delta\underset{\sim}{p}_1\rangle = -2\pi\alpha^2 n \int\left[2\underset{\sim}{p}_1 \cdot (\underline{\underline{I}}+\hat{\underset{\sim}{v}}\hat{\underset{\sim}{v}}) + 4\underset{\sim}{p}_2 \cdot \hat{\underset{\sim}{v}}\hat{\underset{\sim}{v}}\right] \frac{f_2}{vb^2_{min}}\, d^3\underset{\sim}{v}_2 d^3\underset{\sim}{p}_2$$

$$\langle\Delta\underset{\sim}{p}_1\Delta\underset{\sim}{p}_1\rangle = -4\pi\alpha^2 n\int\left[\underset{\sim}{p}_1\times(\underline{\underline{I}} - \hat{\underset{\sim}{v}}\hat{\underset{\sim}{v}})\times\underset{\sim}{p}_1\right] \left[\underset{\sim}{p}_2^2-(\underset{\sim}{p}_2\cdot\underset{\sim}{v})^2\right] \frac{f_2}{vb^2_{min}}\, d^3\underset{\sim}{v}_2 d^3\underset{\sim}{p}_2$$

$$\langle\Delta\underset{\sim}{p}_1\Delta\underset{\sim}{v}_1\rangle = 0\ . \qquad (A.16)$$

Once again, we refer to Ref. 2 for a discussion of b_{min}.

APPENDIX B: PROPERTIES OF THE LANDAU COLLISION OPERATOR

The Landau collision operator for distribution P scattering on F is

$$C(F,P) = \Gamma\frac{\partial}{\partial \underset{\sim}{v}} \cdot \int d^3\underset{\sim}{v}' \; \underset{=}{G} \cdot \left[\frac{P(\underset{\sim}{v})}{M_F}\frac{\partial F}{\partial \underset{\sim}{v}'} - \frac{F(\underset{\sim}{v}')}{M_p}\frac{\partial P}{\partial \underset{\sim}{v}}\right] , \qquad (B.1)$$

where M_p and M_P are the masses of the particles described by F and P, respectively, and $\Gamma = 8\pi\, q^4 z_p^2 z_f^2 \ln\Lambda\, N_F M_p^{-2}$. $\underset{=}{G}$ is given by

$$\underset{=}{G} = \{|\underset{\sim}{v}-\underset{\sim}{v}'|^2\underset{=}{I} - (\underset{\sim}{v}-\underset{\sim}{v}')(\underset{\sim}{v}-\underset{\sim}{v}')\}|\underset{\sim}{v}-\underset{\sim}{v}|^{-3} . \qquad (B.2)$$

We prove the following theorem

<u>Theorem:</u> If F is a Maxwellian and $C(F,P) = 0$, then P is a Maxwellian with the same temperature and average velocity. <u>Proof:</u> We rewrite P as the product of a Maxwellian F' of the same temperature and velocity as F and an arbitrary function ϕ.

$$P = \phi F' \qquad (B.3)$$

In terms of ϕ and F', $C(F,P) = 0$ is

$$C(F,P) = I(\phi) = \Gamma\frac{\partial}{\partial \underset{\sim}{v}} \cdot \int d^3\underset{\sim}{v}' \; \underset{=}{G} \cdot \frac{\partial\phi}{\partial \underset{\sim}{v}} F(\underset{\sim}{v}')\; F'(\underset{\sim}{v}) . \qquad (B.4)$$

Multiplying Eq. (B.4) by ϕ and integrating over $\underset{\sim}{v}$ we get

$$\int d^3\underset{\sim}{v}\, d^3\underset{\sim}{v}' \; \frac{\partial\phi}{\partial \underset{\sim}{v}} \cdot \underset{=}{G}\,\frac{\partial\phi}{\partial \underset{\sim}{v}} F(\underset{\sim}{v}')\; F'(\underset{\sim}{v}) = 0 , \qquad (B.5)$$

The integrand is positive definite so

$$\frac{\partial \phi}{\partial \underset{\sim}{v}} \cdot \underset{=}{G} \cdot \frac{\partial \phi}{\partial \underset{\sim}{v}} = 0 \ , \tag{B.6}$$

for all $\underset{\sim}{v}'$ and $\underset{\sim}{v}$. The only solution to Eq. (B.6) is

$$\frac{\partial \phi}{\partial \underset{\sim}{v}} = 0 \ . \tag{B.7}$$

This completes the proof because ϕ is just a function of $\underset{\sim}{x}$ and t.

Equation (4.41) can be written as an example of equations of the type

$$I(\phi) = A(\underset{\sim}{v}, \underset{\sim}{x}, t) \tag{B.8}$$

where, for instance, in Eq. (4.41) $P_{zb} = \phi F$. The condition that ϕ exist is that A is orthogonal to the homogeneous solution ϕ_o, where

$$I(\phi_o) = 0 \ . \tag{B.9}$$

From the above theorem ϕ_o is independent of $\underset{\sim}{v}$ so the condition that ϕ exists is

$$\int d^3\underset{\sim}{v}\ A(\underset{\sim}{v}, \underset{\sim}{x}, t) = 0 \tag{B.10}$$

APPENDIX C: MIXED STATES

A pure state is one in which the wave function for the system can be represented as product of the wave functions of the individual parts (in our case the product of the individual particle wave function). The wave function for a mixed state is not capable of this factorisation. In the text we have treated the plasma as though it is in a pure state. All of the results are generalised to mixed states except the spin-spin Fokker-Planck coefficient in Appendix A (where what we give is only an upper bound). As the spin-spin depolarisation is weak compared to other processes, the difficulties associated with it do not affect the main thrust of this paper. We describe here the effect of mixing on an ensemble of spin 1/2 particles.

The definition of the $\underset{\sim}{p}$ vector of the i' th particle is

$$\underset{\sim}{p}_i = \langle\psi|\underset{\sim}{\sigma}_i|\psi\rangle \quad \text{(C.1)}$$

The probabilities that the particle is in the up state (relative to the magnetic field) or down state are sufficient to determine the enhancement of the reaction rate (Ref. 2). These probabilities are given for pure and mixed states by

$$\text{up:} \qquad \frac{1}{2}(1 + \underset{\sim}{p}_i \cdot \underset{\sim}{b}) \quad \text{(C.2)}$$

$$\text{down:} \qquad \frac{1}{2}(1 - \underset{\sim}{p}_i \cdot b) .$$

Clearly, if we know $\underset{\sim}{p}$ at any time, we know these probabilities. The complication arises in advancing $\underset{\sim}{p}$ in time; the equation of motion for $\underset{\sim}{p}_i$ is

$$\dot{\underset{\sim}{p}}_i = \langle\psi|[\underset{\approx}{g}_i, H]|\psi\rangle \quad . \qquad (C.3)$$

The Hamiltonian H can be written as

$$H = \sum_{ij} (H_i + H_{ij}) \qquad (C.4)$$

where H_i depends only on the spin of the i'th particle and H_{ij} on the spins of the i'th and j'th particles. The terms in Eq. (C.3) involving H_i can be written in terms of $\underset{\sim}{p}_i$ using Eq. (A.15). The terms involving H_{ij} introduce another quantity

$$\underset{\approx}{t}_{ij} = \langle\psi|\underset{\approx}{g}_i \underset{\approx}{g}_j|\psi\rangle \quad . \qquad (C.5)$$

It is the terms due to H_{ij} (i.e., the spin-spin term) which mix states. We can find an equation of motion for $\underset{\approx}{t}_{ij}$ which involves another new quantity $\underset{\approx}{t}_{ijk} = \langle\psi|g_i g_j g_k|\psi\rangle$. The occurrence of $\underset{\approx}{t}_{ijk}$ represents the mixing of the i'th and j'th particles due to each particle interacting (at different times) with the k'th particle. Note that for a pure state, $\underset{\approx}{t}_{ij} = \underset{\sim}{p}_i \underset{\sim}{p}_j$.

If we drop the spin-spin interaction, we have a complete description of spins irrespective of whether we are in a pure or mixed state. To include spin-spin interaction we have to make some assumption about $\underset{\approx}{t}_{ij}$. For a well polarised plasma $\underset{\approx}{t}_{ij} = \underset{\sim}{p}_i \underset{\sim}{p}_j$ is adequate. (Note that a completely polarised state is always a pure state.)

Particles become mixed together in two ways, either by interacting directly through the spin-spin interaction, or by mutually interacting with a third particle. Consider a plasma starting in a pure state. Let the time taken for a particle's $\underset{\sim}{p}$ vector to shrink from magnitude one to magnitude e^{-1} because of spin-spin interactions be ν^{-1} (see Sec. 4.2). When two particles collide a time t after the system was in a pure state, the probability that they have met before and 'mixed' their wave function is approximately $\nu t/N$ where N is the number of particles in the system. The probability that the same two particles have both met a third and mixed is $(\nu t/N)^2$. Summation over all N possible third particles makes this probability $(\nu t)^2/N$. Hence in the time it takes a particle to depolarise by spin-spin interaction only 1/N of the collisions are between particles in mixed states. Therefore for systems where N is large, the effect of mixing is small in one depolarisation time.

There are essentially two ways in which depolarisation takes place. All interactions except spin-spin rotate $\underset{\sim}{p}$ over a shell in $\underset{\sim}{p}$ space with p^2=constant (for pure state p^2=1). Depolarisation is caused by spreading over this shell. The spin-spin interaction not only rotates the $\underset{\sim}{p}$ vector but also makes p^2 decrease.

In principle a better approximation to the collision-term can be constructed by assuming $\underset{\sim}{t}_{ijk} = \underset{\sim}{p}_i\ \underset{\sim}{p}_j\ \underset{\sim}{p}_k$ and developing a two particle distribution function in analogy to the work in normal kinetic theory (Ref. 7). However, we do not carry this out here because our calculation of Appendix A shows that the zeroth order term in the 1/N expansion is small.

Table 1

	m state	No. of particles
spin 1/2	1/2	$1/2(F_{1/2}+P_z)$
	-1/2	$1/2(F_{1/2}-P_z)$
spin 1	1	$F_1 + P_z + P'_z + Q_{zz}$
	0	$F_1 + Q_{xx} + Q_{yy} - Q_{zz}$
	1	$F_1 - P_z - P'_z + Q_{zz}$

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FIGURE 1. Orbit geometry for collision terms.

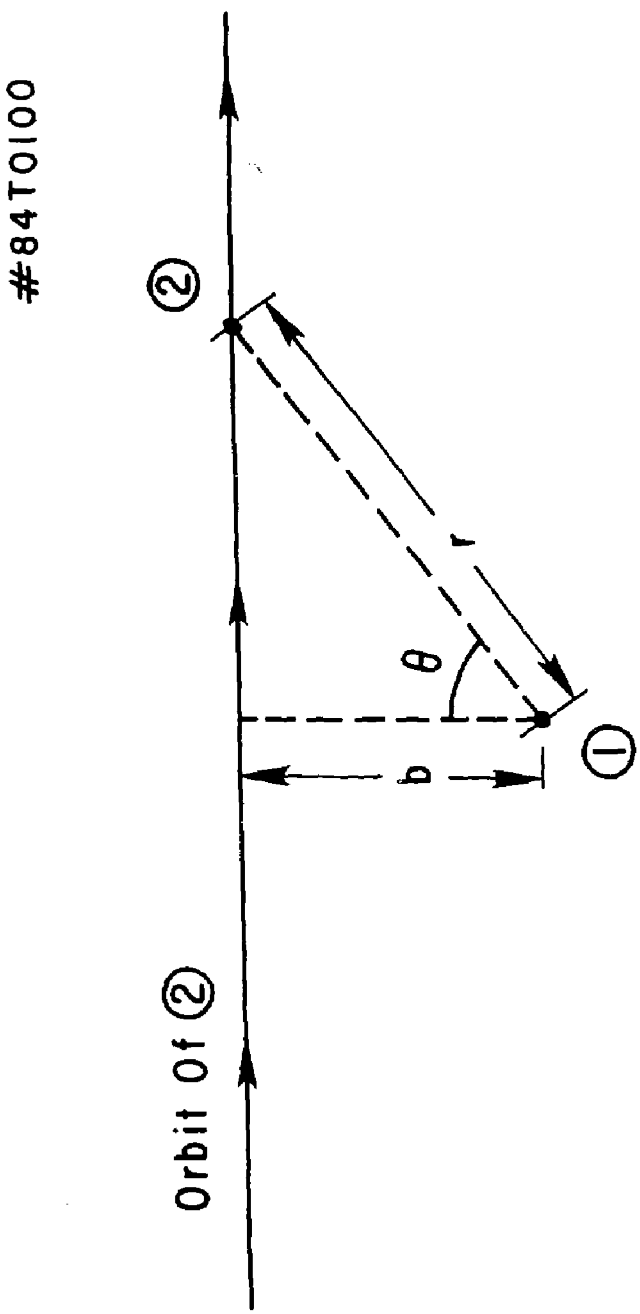

Fig. 1

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