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Mechanism for Coherent Production of Pions from the Decay of Resonances

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ABSTRACT

Conditions and signatures for the induced emission of bosons from a microscopic device are discussed. Detailed formulae for a simple two-level model are derived. In this model the decay of N Δ-resonances residing in a microscopic volume V is discussed. It is shown that large amplification factors for unusual charge bunching of pions are obtained as a result of induced emissions.

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1. INTRODUCTION

The induced emission mechanism of photons in the LASER mechanism is not an intrinsic property of photons alone. This mechanism is shared by all bosons and we shall call it the BASER mechanism[1] (B for Boson instead of L for Light). In particular, we shall use the name PASER for pions and KASER for kaons.

The BASER mechanism relies on the preference of bosons to occupy the same quantum state. This Bose-Einstein nature of pions has in fact been detected in two-particle correlation experiments in hadron-hadron and hadron-nucleus[2,3] collisions, and the coherency of these pions is actually used to probe the volume of interaction.[4]. Because of limited statistics, we know very little about N(>2) particle correlations. These data would be particularly valuable in places where the BASER mechanism is operating.

Consider N pions produced in a common spatial volume V and a common momentum-space volume V_p. The number of pion states is given by $W=V.V_p/(2\pi)^3$. Therefore $\eta=W^{-1}$ is the probability for two pions to occupy the same quantum state. The probability for all N pions to occupy the same quantum state is expected to be enhanced by a factor $A=(N-1)!\,\eta^{(N-1)}$, where (N-1)! is the well known factorial of the PASER mechanism. The importance of the PASER mechanism is therefore measured by the magnitude of A.

The feasibility of a macroscopic PASER device will be discussed elsewhere[5]. If V is of nuclear dimensions, then we are dealing with a microscopic PASER. In this article, we will concentrate on a two-level microscopic PASER. In a subsequent article[6], we will discuss another kind of microscopic PASER which may be responsible for the Centauro events observed in cosmic ray experiments.[7].

A microscopic PASER is unlike a macroscopic PASER or LASER in many ways. Because of the nuclear size of the volume, quantum mechanics and uncertainty principle are of paramount importance. Moreover, the number of pions produced is measured in tens, perhaps hundreds, but not by the Avagadro's number. As a consequence, there is very little directional collimation because of the uncertainty principle. The coherence and intensity of the pions are negligible compared to a macroscopic device because of the small multiplicity. One might ask what then is the signature of a microscopic PASER?

A quantum state is labelled not only by its momentum but also by other quantum numbers such as its electric charge. Since the latter is not affected by the small size of the configuration volume, coherence, or bunching, in charge states would then be an observable signature.

In the rest of this paper we consider a specific model in which N Δ-resonances with negligible fermi momenta are confined to a volume N. For the sake of definiteness we shall take them to be Δ^0's. Each resonance can then decay via 'channel A' into $\pi^- P$, or 'channel B' into $\pi^0 N$. The relative probabilities for these two channels are given by isospin conservation to be 1/3 and 2/3 respectively. If these N Δ's decay independently, α of them via channel A and β of them via channel B, then the probability is governed by the binomial distribution $(\frac{1}{3})^\alpha (\frac{2}{3})^\beta N!/\alpha!\beta!$. On the other hand, if PASER mechanism is at work, then this charge distribution is expected to be augmented by an enhancement factor A similar to but more complicated than the one discussed before. This enhancement factor depends on N, α and β, and because of the PASER mechanism, is expected to favour pion states of same charges. In other words, unusual charge bunchings ($\alpha \sim N$ or $\beta \sim N$) will be greatly enhanced.

This problem has been outlined in Ref. 1 to which we refer the reader for numerical results. We provide in this paper derivations of the formulae used in Ref. 1. In Ref. 1, discrete normalizations were used. Since the final state pions and nucleons are not confined to the volume V, it is perhaps more satisfactory to use continuous normalizations for them. This we shall also do in the following.

2. Δ resonance decay model.

The interaction Hamiltonian describing the decay of Δ^0 can be written in the form

$$H = g\int d\vec{q}\, d\vec{p}\, d\vec{k}\, \delta(\vec{q}-\vec{p}-\vec{k})\left[\sqrt{\frac{1}{3}}\frac{P^{+}(\vec{p})}{(2\pi)^{3/2}}\frac{\pi_{-}^{+}(\vec{k})}{(2\pi)^{3/2}}\frac{\Delta(\vec{q})}{(2\pi)^{3/2}}\right.$$

$$\left. + \sqrt{\frac{2}{3}}\frac{N^{+}(\vec{p})}{(2\pi)^{3/2}}\frac{\pi_{0}^{\dagger}(\vec{k})}{(2\pi)^{3/2}}\frac{\Delta(\vec{q})}{(2\pi)^{3/2}}\right] + \text{h.c.}, \qquad (1)$$

where $\sqrt{\frac{1}{3}}$ and $\sqrt{\frac{2}{3}}$ are the isospin clebsch-gordan coefficients for $\Delta^0 \to \pi^- P$ (channel A) and $\Delta^0 \to \pi^0 N$ (channel B) respectively. Spins are ignored throughout these calculations. The field operators are normalized so that they create one-particle plane-wave states from the vacuum. For example

$$\Delta^{+}(\vec{q})|0\rangle = |\vec{q}\rangle$$

$$\langle \vec{q}' | \vec{q}\rangle = (2\pi)^3\, \delta(\vec{q}-\vec{q}') \, . \qquad (2)$$

The factors $(2\pi)^{-3/2}$ are put in so that the coupling constant g in (1) is the same as the coupling constant g defined in Ref. 1. Since pion and nucleon states really belong to the continuum, it is better to carry out the calculations using the continuous normalization (2) rather than the discrete normalizations adopted in Ref. 1.

The width of Δ^0 can be calculated easily from (1) and (2) to be

$$\Gamma = g^2\int d^3k\, 2\pi\, \delta(\sqrt{k^2+\mu^2} + \sqrt{k^2+m^2} - M) = 8\pi^2 g^2 k^2 \frac{dk}{dE}$$

$$= 8\, \pi^2\, g^2\, k k_1^0\, k_2^0/M \, , \qquad (3)$$

where μ, m, M are the masses of pion, nucleon and Δ, k, k_1^0, k_2^0 are the c.m. momentum and the energies of the decay products, and $E = k_1^0 + k_2^0$.

Now we want to consider the decay of N Δ^0's residing in a volume V. We shall assume the fermi momentum q of the Δ^0 to be negligible compared to its width Γ. We shall also neglect Pauli blockings of the nucleons and treat the Δ's and the nucleons as distinguishable particles. In other words, we assume the i^{th}

Δ^0 decays only to the i^{th} nucleon, plus any of the identical pions.

We shall treat this problem by lowest order perturbation theory, which is the N^{th} order. Since higher order contributions are neglected, unitarity is not automatically satisfied and must be implemented partly by hand. This is accompanied by the 'narrow width' approximation as we shall see below.

Of the N Δ's, suppose α of them decay into $P\pi^-$ (Channel A), and $\beta = N-\alpha$ of them decay into $N\pi^0$ (channel B). The matrix element describing the decay of the first α Δ's via channel A and the last β Δ's via channel B is given by Nth order perturbation theory to be

$$M = \langle \vec{p}\vec{k};\vec{p}'\vec{k}' | H \frac{1}{\Delta E} H \frac{1}{\Delta E} \cdots H \frac{1}{\Delta E} | \vec{q};\vec{q}' \rangle \; . \qquad (4)$$

Here $|\vec{q};\vec{q}'\rangle \equiv |\vec{q}_1 \ldots \vec{q}_\alpha; \vec{q}_1{}' \ldots \vec{q}_\beta{}'\rangle$ denotes the initial N-particle Δ-state, made up of products of single-particle wave functions normalized according to eq.(2). Similarly, $\langle \vec{p}\vec{k};\vec{p}'\vec{k}'| \equiv \langle \vec{p}_1 \ldots \vec{p}_\alpha \vec{k}_1 \ldots \vec{k}_\alpha; \vec{p}_1' \ldots \vec{p}_\beta{}' \, \vec{k}_1 \ldots \vec{k}_\beta|$ denotes the final nucleon-pion state wherein the channel A nucleon and pion momenta are denoted by $\vec{p}_i$ and $\vec{k}_i$ and the channel B nucleon and pion momenta are denoted by $\vec{p}_i{}'$ and $\vec{k}_i{}'$, respectively. As discussed before, we shall regard the Δ's and the nucleons to be distinguishable but of course we must treat the pions of a given charge to be identical particles in order to have the PASER effect. Consequently $\vec{q}_i \to \vec{p}_i + \vec{k}_{\pi(i)}$, $\vec{q}_i{}' \to \vec{p}_i{}' + \vec{k}_{\pi'(i)}$, are arbitrary permutations of α and β objects, respectively.

At each vertex H of (4), one of the Δ's decays into a nucleon and a pion. Let it be the i^{th}. Let

$$X_i \equiv \sqrt{k_i^2+\mu^2} + \sqrt{p_i^2+m^2} - \sqrt{q_o^2+M^2} + \frac{i\Gamma}{2} \equiv \Delta E_i + \frac{i\Gamma}{2} \, ,$$

$$X_i{}' \equiv \sqrt{k_i'^2+\mu^2} + \sqrt{p_i'^2+m^2} - \sqrt{q_i'^2+M^2} + \frac{i\Gamma}{2} \equiv \Delta E'_i + \frac{i\Gamma}{2}$$

be the (complex) energy difference corresponding to this decay into channel A and channel B respectively. For the following discussions, it is convenient to define $X_{i+\alpha} = X'_i$ so that X_j denotes the energy difference of a channel A decay if $1 \leq j \leq \alpha$, and that of a channel B decay if $\alpha+1 \leq j \leq N$.

The decay of the N Δ's in (4) can occur in any time sequence. Let $\sigma \epsilon S_N$ be any permutation of N objects, then different time sequences can be represented by different σ. The matrix element of the first (reading from right to left) energy denominator operator $\hat{\Delta E}$ in (4) is $X_{\sigma(1)}$, that of the second is $X_{\sigma(1)} + X_{\sigma(2)}$, etc. For a given σ, the product of the matrix elements of the energy denominators in (4) is given by

$$V_\sigma \equiv \prod_{i=1}^{N-1} \frac{1}{\sum_{j=1}^{i} X_{\sigma(j)}} \qquad (5)$$

On the other hand, the matrix element of the i^{th} vertex H (again reading from right to left) is given by (1) and (2) to be

$$(2\pi)^{9/2} g(CG)\ \delta_V\left(\vec{q}_{\sigma(i)} - \vec{p}_{\sigma(i)} - \tilde{\vec{k}}_{\sigma(i)}\right) \equiv \langle H \rangle_i \qquad (6)$$

Here (CG) is the Clebsch-Gordan coefficient which is $\sqrt{\frac{1}{3}}$ and $\sqrt{\frac{2}{3}}$ respectively for channels A and B.

In eq.(6), we encounter the spread-out momentum conservation δ-function δ_V rather than the exact-momentum-conservation Dirac δ-function $\delta = \delta_\infty$ because the Δ's are assumed to be confined to a volume V. If no such confinement existed, then the momentum conservation δ-function comes from the overlap of the plane-wave wave-functions of the Δ, the nucleon, and the pion:

$$\delta(\vec{q}-\vec{p}-\vec{k}) = \frac{1}{(2\pi)^3} \int e^{i(\vec{q}-\vec{p}-\vec{k})\cdot\vec{X}}\, d^3X.$$

Since the Δ's are confined to a volume V, this overlapped integral would be changed to

$$\delta_V(\vec{q}-\vec{p}-\vec{k}) \equiv \frac{1}{(2\pi)^3} \int_V e^{i(\vec{q}-\vec{p}-\vec{k})\cdot\vec{X}} d^3X \qquad (7)$$

where the integration is confined to the volume V wherein the Δ's reside. This is the origin of the spread-out δ-function.

As in the case of X_i, we have extended the definition of $\vec{q}_i$, $\vec{p}_i$, $\vec{k}_i$ in eq.(6) by $\vec{q}_{i+\alpha} = \vec{q}_i'$, etc. The tilde above the momenta $\tilde{\vec{k}}_j$ ($j = \sigma(i)$) reminds us of the fact that the pions are not distinguishable, thac the j^{th} Δ decays into the j^{th} nucleon plus any pion. In the notation explained somewhere between eqs.(4) and (5), the tilde momenta are defined by $\tilde{\vec{k}}_j = \vec{k}_{\pi(j)}$ if $1 \leq j \leq \alpha$, and $\tilde{\vec{k}}_j = \vec{k}_{\pi'(j-\alpha)}$ if $\alpha+1 \leq j \leq N$.

It is the product of all the $\langle H\rangle_i$ that appears in (4). This product is independent of σ but depends on $\pi \in S_\alpha$ and $\pi' \in S_\beta$. It is

$$\prod_{i=1}^{N} \langle H\rangle_i = \left(\sqrt{\tfrac{1}{3}}\right)^{\alpha} \left(\sqrt{\tfrac{2}{3}}\right)^{\beta} g^N \prod_{i=1}^{\alpha} (2\pi)^{9/2} \delta_V(\vec{q}_i - \vec{p}_i - \vec{k}_{\pi(i)}) \prod_{j=1}^{\beta} (2\pi)^{9/2} \delta_V(\vec{q}_j{}' - \vec{p}_j{}' - \vec{k}_{\pi'(j)}) \tag{8}$$

Finally, the matrix element in (4) is given by

$$M = \sum_{\pi \in S_\alpha} \sum_{\pi' \in S_\beta} \left(\prod_{i=1}^{N} \langle H\rangle_i \right) \sum_{\sigma \in S_N} V_\sigma \tag{9}$$

Using the formula (A.1) of Appendix A, and noting that

$T = \sum_{j=1}^{N} X_{\sigma(j)} = \sum_{j=1}^{N} X_j$ is independent of σ, we obtain from (5) that

$$\sum_{\sigma \in S_N} V_\sigma = T \sum_{\sigma \in S_N} V_\sigma \frac{1}{T} = T \prod_{i=1}^{N} \frac{1}{X_i} \tag{10}$$

Substituting (8) and (10) into (9), we obtain

$$M = g^N \left(\sqrt{\tfrac{1}{3}}\right)^{\alpha} \left(\sqrt{\tfrac{2}{3}}\right)^{\beta} T \sum_{\pi \in S_\alpha} \sum_{\pi' \in S_\beta} \left(\prod_{i=1}^{\alpha} \frac{(2\pi)^{9/2}}{X_i} \delta_V(\vec{q}_i - \vec{p}_i - \vec{k}_{\pi(i)}) \right) \left(\prod_{j=1}^{\beta} \frac{(2\pi)^{9/2}}{X_j'} \delta_V(\vec{q}_j' - \vec{p}_j' - \vec{k}_{\pi'(j)}) \right) \tag{11}$$

The transition rate for the decay of N Δ's, α of which through channel A and β of which through channel B, is given by the golden rule to be

$$R = \frac{N!}{\alpha!\,\beta!} \cdot \frac{1}{V^N} \cdot \int \frac{1}{\alpha!}\left(\prod_{i=1}^{\alpha} \frac{d^3k_i}{(2\pi)^3}\, \frac{d^3p_i}{(2\pi)^3} \right) \frac{1}{\beta!}\left(\prod_{j=1}^{\beta} \frac{d^3k_j'}{(2\pi)^3}\, \frac{d^3p_j'}{(2\pi)^3} \right)$$

$$\cdot\; 2\pi\; \delta(E_f - E_o)\; |M|^2, \qquad (12)$$

where

$$E_f - E_o = \sum_{i=1}^{\alpha} \Delta E_i + \sum_{j=1}^{\beta} \Delta E_j' \qquad (13)$$

The combinatorial factor $N!/(\alpha!\,\beta!)$ in (12) comes about because M describes only the particular decay mode in which the first α Δ's decay through channel A and the last β Δ's decay through channel B. In reality any α of the N Δ's can decay through channel A and the remaining Δ's can decay through channel B, hence the combinatorial factor $N!/\alpha!\,\beta!$. The flux factor V^{-N} occurs because the single-particle-state normalization (2) for Δ normalizes to V in the volume considered rather than unity.

So far the calculations have been exact. To proceed further, we must begin to contemplate approximations.

If Γ is the true total width of the Δ^0, and if the N Δ's are widely separated, viz, if $V \to \infty$, then no induced transition will take place and R is given by the spontaneous decay rate R_S. This is the decay rate of N independent Δ's,

$$R_S = \frac{N!}{\alpha!\,\beta!} \left(\frac{1}{3}\right)^{\alpha} \left(\frac{2}{3}\right)^{\beta} N\Gamma \qquad (14)$$

because the total width is $N\Gamma$ and, from combinatorial considerations, the distributions among channels A and B must be binomial. Formally, one may say that (14) is the result of unitarity. Now if no approximation is made in (11) and (12), we will not recover (14) when $V \to \infty$.

As discussed before, this is because the present calculation is carried out using the lowest order perturbation formula where unitarity is not respected. To make physical sense we must restore unitarity by hand. It turns out that this can be achieved by the 'narrow width' approximation described below.

To motivate that, first note that a factor $|X_i|^{-2}$ for every i occurs in (12). Since $X_i = \Delta E_i + i\Gamma/2$,

$$\frac{1}{|X_i|^2} = \frac{1}{(\Delta E_i)^2+\Gamma^2/4} \equiv \frac{2\pi}{\Gamma}\, \delta_\Gamma(\Delta E_i), \tag{15}$$

where $\delta_\Gamma(\Delta E_i)$ is a spread-out δ-function with width Γ and normalized to unity. [Note: δ_V and δ_Γ are not the same spread-out δ-functions].

Now when $V \to \infty$ and all the Δ's decay independently, energy must be conserved in each decay. Comparing (11), (12) and (15), clearly this conservation is obtained only when we make the narrow width approximation and use $\delta_\Gamma(\Delta E_i)$, $\delta(\Delta E_i)$ interchangably:

$$\delta_\Gamma(\Delta E_i) \simeq \delta(\Delta E_i) = \delta_0(\Delta E_i) \tag{16}$$

That this approximation is necessary is because (11) and (12) are computed to the lowest perturbative order. Thus the coupling constant g must be assumed to be small and the width Γ must be assumed narrow. In such a case (16) is eminantly reasonable. In reality, width Γ is not small but we will then simply regard (16) as a way of implementing unitarity and other physical requirements (e.g., energy conservation) by hand.

We still have to show that approximation (16) enables us to obtain (14) from (11) and (12) when $V \to \infty$. To do that note that as $V \to \infty$, the Δ's, and the corresponding nucleons and pions, are very far apart so that the pions may be regarded as distinguishable. In other words, the different pions cannot interfere so that we only have to retain the diagonal terms when we square the sums over $\pi \varepsilon S_\alpha$ and $\pi' \varepsilon S_\beta$ in (11).

Now note from (13) that

$$E_f - E_o = \sum_{i=1}^{\alpha} \Delta E_i + \sum_{j=1}^{\beta} \Delta E'_j = \mathrm{Re}\left(\sum_{i=1}^{\alpha} X_i + \sum_{j=1}^{\beta} X_j' \right)$$

and note that the corresponding imaginary part of these sums of X_i and X_j' is $N\Gamma/2$. Thus in the spirit of (16) we may replace the overall energy conservation δ function in (12) as follows:

$$|T|^2 2\pi\delta(E_f-E_o) = 2\pi\left(\frac{N\Gamma}{2}\right)^2 \delta(E_f-E_o) \simeq 2\pi\left(\frac{N\Gamma}{2}\right)^2 \delta_{N\Gamma}(E_f-E_o)$$

$$= 2\pi\left(\frac{N\Gamma}{2}\right)^2 \delta_{N\Gamma}(o) = 2\pi\left(\frac{N\Gamma}{2}\right)^2 \frac{2}{\pi N\Gamma} = N\Gamma \qquad (17)$$

With these approximations, we can carry out the $\vec{p}_i$ and $\vec{p}_i{}'$ integrations in (12). We get

$$R = R_S \prod_{i=1}^{\alpha} \left((2\pi)^3 g^2 \int \frac{2\pi}{\Gamma} \delta_\Gamma(\Delta E_i) \lim_{V\to\infty} \frac{\delta_V(\vec{o})}{V} d^3k_i \right)$$

$$\prod_{j=1}^{\beta} \left((2\pi)^3 g^2 \int \frac{2\pi}{\Gamma} \delta_\Gamma(\Delta E_j{}') \lim_{V\to\infty} \frac{\delta_V(\vec{o})}{V} d^3k_j{}' \right) \qquad (18)$$

where R_S is given by eq.(14). Now from (7), $(2\pi)^3 \delta_V(o)/V=1$; from (3), the remaining factors inside each square bracket integrates to 1 if (16) is used. Hence $R = R_S$ as demanded by physical considerations.

Now we return to a finite volume V when interference and PASER effect may take place. We shall assume that the volume V is sufficiently large to allow the fermi momenta q_i of the Δ to be small compared to Γ (i.e., after we suitably convert the momentum scale into energy scale). This allows all the pions to occupy the same momentum-space region to enable them to PASE effectively. When the volume is large, not only energy but also momentum are very well conserved. We will then be justified to approximate the integral

$$\int d^3p_i \ \delta_V(\vec{q}_i-\vec{p}_i-\vec{k}_{\pi_1(i)}) \frac{1}{|x_i|^2} \delta_V(\vec{q}_i-\vec{p}_i-\vec{k}_{\pi_2(i)}) \qquad (19)$$

by

$$\frac{2\pi}{\Gamma} \delta_\Gamma(\Delta E_i) \ \delta_V(\vec{k}_{\pi_1(i)}-\vec{k}_{\pi_2(i)}) \qquad (20)$$

With ΔE_i given from now on to be

$$\Delta E_i = \sqrt{k_i^2+\mu^2} + \sqrt{k_i^2+m^2} - M. \qquad (21)$$

In this way eq.(12) yields the result

$$R = R_S \sum_{\pi_1 \varepsilon S_\alpha} \sum_{\pi_2 \varepsilon S_\alpha} \frac{1}{\alpha!} \prod_{i=1}^{\alpha} \left(\int d^3k_i \frac{(2\pi)^4 g^2}{V\Gamma} \delta_\Gamma(\Delta E_i)\delta_V(\vec{k}_{\pi_1(i)}-\vec{k}_{\pi_2(i)})\right) \cdot$$

$$\sum_{\pi_1' \varepsilon S_\beta} \sum_{\pi_2' \varepsilon S_\beta} \frac{1}{\beta!} \prod_{j=1}^{\beta} \left(\int d^3k_j' \frac{(2\pi)^4 g^2}{V\Gamma} \delta_\Gamma(\Delta E_j')\delta_V(\vec{k}_{\pi_1'(j)}-\vec{k}_{\pi_2'(j)})\right)$$

$$= R_S \sum_{\pi \varepsilon S_\alpha} \prod_{i=1}^{\alpha} \left(\int d^3k_i \frac{(2\pi)^4 g^2}{V\Gamma} \delta_\Gamma(\Delta E_i)\delta_V(\vec{k}_i-\vec{k}_{\pi(i)})\right)$$

$$\sum_{\pi' \varepsilon S_\beta} \prod_{j=1}^{\beta} \left(\int d^3k_j' \frac{(2\pi)^4 g^2}{V\Gamma} \delta_\Gamma(\Delta E_j')\delta_V(\vec{k}_j'-\vec{k}_{\pi'(j)})\right) \qquad (22)$$

Again if we restrict π and π' to the identical permutations, we will get $R=R_S$ as in (18). Induced transitions correspond to those terms for which π and/or π' differ from the identity.

To analyse these terms systematically, we shall express the permutations π and π' in their cycle forms.

Because of the factor $\delta_V(\vec{k}_i-\vec{k}_{\pi(i)})$ $\delta_V(\vec{k}_j-\vec{k}_{\pi'(j)})$, all the momenta whose indices appear in the same cycle are equal. These momentum conservation δ-functions then allow us to carry out momentum integrations for all but one of the momenta inside a cycle. For example, if $\alpha=5$, the permutation $\pi=(124)(35)$ describes the physical process in which $\vec{k}_1=\vec{k}_2=\vec{k}_4$ and $\vec{k}_3=\vec{k}_5$. The momentum integrations, carried out in the spirit of 'narrow width' and large volume approximations, are

$$\prod_{i=1}^{5} \int d^3k_i \;\delta_\Gamma(\Delta E_i)\; \delta_V(\vec{k}_i-\vec{k}_{\pi(i)}) \simeq \int d^3k_1 (\delta_\Gamma(\Delta E_1))^3 \;\delta_V(\vec{o}) \cdot$$

$$\int d^3k_3 [\delta_\Gamma(\Delta E_3)]^2 \;\delta_V(\vec{o}) \qquad (23)$$

$$\simeq [\delta_\Gamma(o)]^{2+1} \int d^3k_1 \;\delta_\Gamma(\Delta E_1)\; \delta_V(\vec{o}) \int d^3k_3 \;\delta_\Gamma(\Delta E_3)\; \delta_V(\vec{o})$$

$$\simeq [\delta_\Gamma(o)]^{2+1} \left[\frac{V}{(2\pi)^4} \frac{\Gamma}{g^2} \right]^2$$

in which eqs.(3),(7) and (16) have been used.

It is clear from this example that a cycle of length ℓ would contribute a factor $[\delta_\Gamma(o)]^{\ell-1} \frac{V}{(2\pi)^4} \frac{\Gamma}{g^2}$ to integrations of the type (23). Thus if π has ν_ℓ cycles of length ℓ (ℓ=1,2,3,..), then

$$\prod_{i=1}^{\alpha} \int d^3k_i \frac{(2\pi)^4 g^2}{V\Gamma} \delta_\Gamma(\Delta E_i) \delta_V(\vec{k}_i - \vec{k}_{\pi(i)})$$

$$= \left(\frac{(2\pi)^4 g^2}{V\Gamma} \delta_\Gamma(o) \right)^{\nu_\ell(\ell-1)} \quad (24)$$

Note that since $\pi \epsilon S_\alpha$, the non-negative integers ν_ℓ is subject to the constraint

$$\sum_{\ell=1}^{\infty} \ell \nu_\ell = \alpha \quad (25)$$

We define the amplification factor by

$$A = R/R_S \quad (26)$$

and if we let

$$\eta = \frac{(2\pi)^4 g^2 \delta_\Gamma(o)}{\Gamma V}, \quad (27)$$

we can express (22) as

$$A = \sum_{\{\nu_\ell\}} K_\nu \, \eta^{\nu_\ell(\ell-1)} \sum_{\{\nu_\ell'\}} K'_{\nu'} \, \eta^{\nu'_\ell(\ell-1)}, \quad (28)$$

where the sums in (28) are taken over all configurations $\{\nu_\ell\}$ of $\pi \epsilon S_\alpha$ and all configurations $\{\nu_\ell'\}$ of $\pi' \epsilon S_\beta$. The first must obey the constraint (25) and the second must obey a similar constraint.

The factor K_ν is the number of permutations $\pi \epsilon S_\alpha$ with identical cycle structures $\{\nu_\ell\}$. It is

$$K_\nu = \frac{\alpha!}{\prod_{\ell=1}^{\infty} \nu_\ell! \, \ell^{\nu_\ell}} \quad (29)$$

Similarly,

$$K'_{\nu'} = \frac{\beta!}{\prod_{\ell=1}^{\infty} \nu'_{\ell}! \; \ell^{\nu'_{\ell}}} \qquad (30)$$

It turns out that these sums can be carried out exactly. This is shown in Appendix B. Using eq.(B.2), we obtain the final result for the amplification factor $A=R/R_S$ to be

$$A = \left(\prod_{i=1}^{\alpha-1} (1+i\eta) \right) \left(\prod_{j=1}^{\beta-1} (1+j\eta) \right) \qquad (31)$$

For approximate expressions of this result in different ranges of η, see Ref. 1.

The physical meaning of (28) and (31) is very simple. K_ν and $K'_{\nu'}$ give the combinatorial factors corresponding to particular pion momenta alignments. Consider for example channel A pions. If none of the pions are aligned, then $\nu_1=\alpha$ and all other $\nu_i=0$. From (29), $K_\nu=1$ and there are no PASER enhancements. On the other hand, if all the pions are aligned, then $\nu_\alpha=1$ and all other $\nu_i=0$. From (29), $K_\nu=(\alpha-1)!$, which gives the well known factorial enhancement factor of a PASER.

The factor of η in (27) and (28) must then describe the probability for two pions of the same charge to be aligned. Let us derive this directly.

Since the pions originate from the volume V, the number of quantum states W for pions of a given charge is given by the phase space volume divided by $(2\pi)^3$. This is

$$W = \frac{V}{(2\pi)^3} 4\pi k^2 \Delta k = \frac{V}{(2\pi)^3} 4\pi k^2 \frac{dk}{dE} \Delta E \qquad (32)$$

where k is the momentum of the pion in the rest frame of the Δ, and the momentum spread Δk is a reflection of the finite width Γ of the Δ. Now if we put

$$\Delta E = \frac{\pi}{2} \Gamma \qquad (33)$$

then

$$W = \eta^{-1} \qquad (34)$$

and therefore η is the probability for two pions to occupy the same quantum state. To show (34), note that η is given by (15) and (27) to be

$$\eta = \frac{(2\pi)^4 g^2}{\Gamma^2 V} \frac{2}{\pi} = \frac{32\ \pi^3 g^2}{V\Gamma^2}$$

Using (3) to replace g^2, we get

$$\eta = \frac{4\pi}{V\Gamma k^2 \frac{dk}{dE}} = W^{-1} \qquad (35)$$

if (33) is used.

The fact that (28) and (31) admit a simple physical explanation further substantiates the correctness of the approximations adopted here to deal with lowest order perturbation theory.

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APPENDIX A

Let σ be a member of the permutation group S_N of N objects. We shall show in this Appendix the following mathematical formula used in the text:

$$\sum_{\sigma \in S_N} \prod_{i=1}^{N} \frac{1}{\sum_{j=1}^{i} X_{\sigma(j)}} = \prod_{i=1}^{N} \frac{1}{X_i} \qquad (A.1)$$

Proof: (A.1) is trivially true for N=1.

Assume it to be true for N-1. The left hand side of (A.1) can be written as

$$\sum_{\sigma(N)=1}^{N} \sum_{\sigma'} \prod_{i=1}^{N} \frac{1}{\sum_{j=1}^{i} X_{\sigma(j)}} = \frac{1}{T} \sum_{\sigma(N)=1}^{N} \sum_{\sigma'} \prod_{i=1}^{N-1} \frac{1}{\sum_{j=1}^{i} X_{\sigma(j)}}, \qquad (A.2)$$

where $T = \sum_{j=1}^{N} X_{\sigma(j)} = \sum_{j=1}^{N} X_j$ is independent of σ, and where σ' denotes any permutation of the N-1 numbers $\sigma(1)$, $\sigma(2)$, ... $\sigma(N-1)$, always with $\sigma(N)$ kept fixed. By the induction hypothesis,

$$\sum_{\sigma'} \prod_{i=1}^{N-1} \frac{1}{\sum_{j=1}^{i} X_{\sigma(j)}} = \prod_{i=1}^{N-1} \frac{1}{X_{\sigma(i)}}, \qquad (A.3)$$

thus (A.2) is equal to

$$\frac{1}{T} \sum_{\sigma(N)=1}^{N} \prod_{i=1}^{N-1} \frac{1}{X_{\sigma(i)}} = \frac{1}{T} \sum_{\sigma(N)=1}^{N} \frac{X_{\sigma(N)}}{\prod_{i=1}^{N} X_{\sigma(i)}}$$

$$= \prod_{i=1}^{N} \frac{1}{X_i} \frac{1}{T} \sum_{\sigma(N)=1}^{N} X_{\sigma(N)} = \prod_{i=1}^{N} \frac{1}{X_i}, \qquad (A.4)$$

which is the same as the right hand side of (A.4). Formula (A.1) is therefore true for any N.

APPENDIX B

Let π be a permutation of α objects. When expressed in cycle form, suppose π has ν_ℓ cycles of length ℓ (ℓ=1,2,3,...). Since $\pi \epsilon S_\alpha$, the following constraint on ν_ℓ must be obeyed:

$$\sum_{\ell=1}^{\infty} \ell \, \nu_\ell = \alpha \, . \qquad (B.1)$$

Now the total number of perturbations in S_α with the aforementioned cycle structure is

$$K_\nu \equiv \alpha! / \prod_{\ell=1}^{\infty} (\ell^{\nu_\ell} \nu_\ell !) \, .$$

The purpose of this Appendix is to show that

$$\sum_{\{\nu_\ell\}} K_\nu \, \eta^{\sum_{\ell=1}^{\infty} \nu_\ell(\ell-1)} = \sum_{\nu_i} \frac{\alpha!}{\prod_{\ell=1}^{\infty} (\ell^{\nu_\ell} \nu_\ell !)} \eta^{\sum_{\ell=1}^{\infty} (\nu_\ell(\ell-1)} = \prod_{i=1}^{\alpha-1} (1+i\eta)$$

(B.2)

where the sum on the left hand side is carried out over all configurations $\{\nu_\ell\}$ subject to the constraint (B.1).

<u>Proof</u>: Equation (B.2) is trivially true for α=1, where the only cycle is (1). Thus ν_1=1 and ν_ℓ=0 for ℓ>1, and both sides of (B.2) are equal to 1.

Now we prove (B.2) for a general positive integer α by induction. First note that there are two and only two ways of obtaining a permutation of α objects from a permutation of α-1 objects expressed in the cycle form. Either we start with a permutation of α-1 objects and add to it an additional 1-cycle (α) consisting solely of the α^{th} fobject, or, we could insert the α^{th} object into an ℓ-cycle (ℓ arbitrary) of $\pi' \epsilon S_{\alpha-1}$ to make this an (ℓ+1)-cycle and thus to convert π' into an element of S_α. If we should do it by this latter way, we can create different members of S_α by inserting this α^{th} object behind any of the existing objects in the ℓ-cycle. In other words, for a given $\pi \epsilon S_{\alpha-1}$, we can thus create $\alpha-1$ members of S_α.

For example, if $\alpha=3$, then $\pi' \epsilon S_{\alpha-1}$ is given in cycle form by (1)(2) or (12). The first method creates (1)(2)(3) and (12)(3) from these two S_2 elements, and the second method creates (13)(2), (1)(23); (132), (123) from the same S_2 elements. In this way we obtain all 3!=6 members of S_3.

Now let us turn to eq.(B.2). Since K_ν represents the number of distinct permutations with definite cycle lengths, we can write (B.2) as

$$\sum_{\pi \epsilon S_\alpha} \eta^{\sum_{\ell=1}^{\infty} \nu_\ell (\ell-1)}$$

In other words, if we associate a factor $\eta^{\ell-1}$ with every ℓ-cycle in π and if we sum over all $\pi \epsilon S_\alpha$, the result is the left hand side of (B.2).

To prove (B.2) by induction, let us assume it to be true for $\alpha-1$,

$$\sum_{\pi' \epsilon S_{\alpha-1}} \eta^{\sum_{\ell=1}^{\infty} \nu'_\ell (\ell-1)} = \prod_{i=1}^{\alpha-2} (1+i\eta) \equiv F_{\alpha-1} , \qquad (B.3)$$

where ν'_ℓ is the number of ℓ-cycles in π'. Now obtain all $\pi \epsilon S_\alpha$ from $\pi' \epsilon S_{\alpha-1}$ by the two ways mentioned above. If π is obtained by the first method then the additional cycle (α) gives rise to an additional factor $\eta^0=1$. The contribution of this first way to (B.2), by (B.3), is therefore $F_{\alpha-1}$. If $\pi \epsilon S_\alpha$ is obtained from $\pi' \epsilon S_{\alpha-1}$ in (B.3) by the second method by inserting the α^{th} object into an ℓ-cycle of π', then the factor $\eta^{\ell-1}$ in (B.3) becomes η^ℓ. In other words, there is an additional factor of η wherever the insertion takes place. But as noted before, there are $\alpha-1$ possible insertions for every $\pi' \epsilon S_{\alpha-1}$. Thus the contribution of the second method to (B.2), by (B.3), is $F_{\alpha-1} \cdot \eta \cdot (\alpha-1)$. The total result of (B.2) is therefore

$F_{\alpha-1}[1+(\alpha-1)\eta] = \prod_{i=1}^{\alpha-1}[1+i\eta]$, which is equal to the right hand side of (B.2).