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ABSTRACT

We introduce a class of infinite dimensional algebras with a 'generalized loop structure' by considering the global symmetries of the four dimensional Lagrangian obtained by compactifying general relativity coupled to Yang-Nills in six dimensions down to $M^4 \times S^2$. The generalization to arbitrary dimensions is then obvious. We show by explicit construction that such algebras possess an infinite number of finite sub-algebras. Among which, for the six dimensional case, is $so(1,3)$ realized on S^2 with vanishing Casimir invariants. This $so(1,3)$ may be interpreted, in accord with a previous conjecture of Salam and Strathdee [1], as the 'ladder' symmetry for the Kaluza-Klein towers.

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Recently, Dolan and Duff[2] identified a Kac-Moody infinite parameter symmetry, namely Poincare X $C(t,t^{-1})$, as the symmetry of the four dimensional Lagrangian obtained by compactifying General Relativity in five dimensions down to $M^4 \times S^1$. They conjectured, but did not carry-out, the extension to dimensions greater than five. In this letter this extension is carried out, including the case when Yang-Mills interactions are present in the higher dimensional theory.

We proceed by first briefly recapitulating the procedure followed in ref.[2] and the difficulties attendent upon a naive extension of the method used there. We then demonstrate the circumvention of these difficulties by constructing the infinite dimensional algebra for the case of six dimensions compactified to $M^4 \times S^2$. We next consider the finite sub-algebras, of these infinite algebras, and show that both for five and six dimensions there are an infinite number of finite sub-algebras. For d=6 one finds the non-compact algebra $so(1,3)$ among these. We show that, as for $so(1,2)$ in the d=5 case, (both) the Casimir invariants vanish identically when the explicit realizations of its generators are used. We also indicate why it deserves the name of a 'ladder' or 'spectrum generating' algebra for the Kaluza Klein towers.

One begins, in the case of d=5 [2], with the relevant Hilbert action:

$$S = (2\pi K^2)^{-1} \int d^4x\, d\theta\, (-\gamma)^{1/2} R(\gamma) \tag{1}$$

which is invariant under general coordinate transformations in five dimensions:

$$Z^M \longrightarrow Z'^M = Z^M - \xi^M(z)$$
$$Z^M = (x^\mu, \theta/m) \quad . \tag{2}$$

Assuming that the ground state is given by $M^4 X S^1$ implies that $\theta \in [0, 2\pi]$. Thus the fields in the five dimensional metric tensor are periodic in θ and hence to be expanded in a Fourier series in θ:

$$\gamma_{MN} = \varphi^{-1/3} \begin{bmatrix} g_{\mu\nu} + K^2 A_\mu A_\nu & K\varphi A_\mu \\ K\varphi A_\nu & \varphi \end{bmatrix} \tag{3}$$

$$g_{\mu\nu}(x,\theta) = \sum_{n=-\infty}^{\infty} g_{\mu\nu n}(x)\, e^{in\theta}$$

etc.

The ground state symmetry (Poincaré X U(1)) is determined by the vacuum expectation values:

$$\langle g_{\mu\nu 0} \rangle = \eta_{\mu\nu} \qquad \langle A_{\mu 0} \rangle = 0 \qquad \langle \varphi_0 \rangle = 1 \tag{4}$$

To identify the infinite parameter symmetry one then makes the corresponding expansion of the infinitesimal general coordinate parameters:

$$\xi^\mu(x,\theta) = \sum_{n=-\infty}^{\infty} \xi^\mu_n(x)\, e^{in\theta}$$

$$\xi^5(x,\theta) = \sum_{n=-\infty}^{\infty} \xi^5_n(x)\, e^{in\theta} \qquad \xi^{I*}_n = \xi^I_{-n} \tag{5}$$

Restricting oneself to transformations global in x :

$$\xi^\mu_n(x) = a^\mu_n + \omega^\mu_{\nu n} x^\nu$$

$$\xi^5_n(x) = c_n \qquad \omega_{\mu\nu} = -\omega_{\nu\mu} \tag{6}$$

where $c_n, a^\mu_n, \omega^n_{\mu\nu}$ are independent of x, allows one to identify the generators of the transformations associated with this infinite family of parameters:

$$P^n_\mu = e^{in\theta}\partial^\mu$$

$$M^{\mu\nu}_n = -e^{in\theta}(x^\mu\partial^\nu - x^\nu\partial^\mu) \tag{7}$$

$$Q_n = i e^{in\theta}\partial_\theta \qquad n = 0, \pm 1, \pm 2, \ldots$$

Using the explicit forms given above, it is easy to show that these generators form a Lie algebra under commutation, namely the Kac-Moody loop algebra Poincare $\times C(t,t^{-1})$. For instance one has:

$$[Q_n, Q_m] = (n-m)\, Q_{n+m}$$

$$[Q_n, P^m_\mu] = -m\, P^{m+n}_\mu \tag{8}$$

$$[M^n_{\mu\nu}, P^m_\sigma] = \eta_{\mu\sigma} P^{n+m}_\nu - \eta_{\nu\sigma} P^{n+m}_\mu$$

Note the characteristic 'loop property', namely that the commutator of two generators of 'floor number' n and m results in generators of 'floor number' n+m. This follows trivially from the exponent adding property of the product of two exponentials. We consider the finite sub-algebras further on. Since the Hilbert action is invariant under arbitrary reparametrizations one may think of the five dimensional general coordinate invariance as having been 'Fourier analyzed' into a discrete infinity of local four dimensional transformations associated with the discrete infinity of generators in (7). Finally we note that the breaking of this infinite set of symmetries leads[2] via the Higgs mechanism to a spectrum consisting of a single tower of spin two particle masses $|n|M$, charges nKM. Here M is the inverse radius of S^1.

Let us now turn to the six dimensional case with an assumed compactification to $M^4 \times S^2$, for instance via the vacuum expectation values(VEVs) of gauge fields coupled to the original six dimensional gravity theory as in ref.[3]. Let θ, φ be the angular coordinates on S^2. One now has six infinitesimal parameters ξ^M for the coordinate transformations that leave the action invariant together with the parameters Λ^a of the six dimensional Yang-Mills gauge transformations.

Thus one has under coordinate transformations:

$$x^\mu \longrightarrow x^\mu + \xi^\mu(x,\theta,\varphi)$$

$$\theta \longrightarrow \theta + \xi^\theta(x,\theta,\varphi) \tag{9}$$

$$\varphi \longrightarrow \varphi + \xi^\varphi(x,\theta,\varphi)$$

Since ξ^μ are scalars on S^2 they may be expanded in the usual spherical harmonics Y_{lm}:

$$\xi^\mu(x,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \xi^\mu_{lm}(x)\, Y_{lm}(\theta,\varphi) \tag{10}$$

The same holds for the gauge transformation parameters Λ^a. Just as in the five dimensional case one can specialize to transformations global in x :

$$\xi^\mu_{lm}(x) = a^\mu_{lm} + \omega^\mu_{\nu lm} x^\nu$$

$$\Lambda^a_{lm}(x) = \Lambda^a_{lm}. \tag{11}$$

This leads one to identify (Q^a are the generators of the gauge group)

$$\left.\begin{array}{l} P_{\mu lm} = Y_{lm}\partial_\mu \\ M^{\mu\nu}_{lm} = -Y_{lm}(x^\mu\partial^\nu - x^\nu\partial^\mu) \\ Q^a_{lm} = Y_{lm} Q^a \end{array}\right\} X_{lm} \tag{12}$$

as candidate generators for the infinite algebra in this case.

The case of the parameters on the two dimensional manifold is more subtle. Here $\xi^I = (\xi^\theta, \xi^\varphi)$ form a vector field on S^2 and hence can be expanded in vector spherical harmonics $Y^\pm_{lm}$ [4]:

$$\xi^I(\theta,\varphi) = \sum_{lm} \xi^+_{lm} Y^{+I}_{lm} + \xi^-_{lm} Y^{-I}_{lm} \tag{13}$$

Here I runs over θ, φ . The components $Y^{\pm I}_{lm}$ are [ ]:

$$Y^{+\theta}_{lm} = \frac{\partial Y_{lm}}{\partial\theta} = Y_{lm,\theta} \qquad Y^{+\varphi}_{lm} = \frac{im}{\sin\theta} Y_{lm}$$

$$Y^{-\theta}_{lm} = \frac{im}{\sin\theta} Y_{lm} \qquad Y^{-\varphi}_{lm} = \frac{-1}{\sin\theta} Y_{lm,\theta} \tag{14}$$

In analogy with the d=5 case one may ask whether the generators $J^\pm_{lm}$ defined by:

$$J^\pm_{lm} = Y^{\pm\theta}_{lm}\partial_\theta + Y^{\pm\varphi}_{lm}\partial_\varphi \tag{15}$$

form a closed consistent algebra together with the X_{lm} previously defined in (12). However on taking commutators $[J^+_{lm}, J_{lm}]$ etc, one finds that one does not in general obtain an expression with any obvious 'loop' property. In other words, although we have not proven this, the right-hand side of the commutator appears to include an infinite number of of harmonics rather than a finite subset of them and the $Y^{\pm}_{lm}$ are not an appropriate basis for discovering an algebra with a 'loop' property. We next show that an appropriate basis for this purpose is, in fact, close at hand.

Let $J_{\pm,0}$ be the usual Killing generators for S^2:

$$J_{\pm} = \pm e^{\pm i\varphi}\left(\partial_\theta \pm i \cot\theta\, \partial_\varphi\right) \qquad J_0 = -i\,\partial_\varphi \tag{16}$$

Then at any point on S^2 the three J_i form an overcomplete basis for vectors in the tangent space at that point. Thus if one writes a general vector field on S^2 as:

$$\xi^I(\theta,\varphi) = \xi^i(\theta,\varphi)\, J_i^I \qquad I = \theta, \varphi \tag{17}$$

where ξ^i are scalar functions on S^2, then one can determine the harmonic components $\xi^i_{\ell m}$ of two of the ξ^i in terms of the parameters $\xi^{\pm}_{\ell m}$ of the complete basis $Y^{\pm}_{lm}$, but the third set must be fixed by convention; as is the case whenever on uses an overcomplete basis. Nevertheless we are led to consider the set of generators:

$$J^{\ell m}_{\pm,0} = Y^{\ell m}(\theta,\varphi)\, J_{\pm,0} \tag{18}$$

as candidate generators for the loop algebra. We emphasize that for purposes of enumerating the number of independent parameters a complete set must be used. These generators along with the X_{lm} defined in (12) form a closed infinite algebra with the sought for 'loop property'. To see this it is only necessary to use the following well known properties of the J_i, Y_{lm} and the Clebsch–Gordon coefficents $\langle j, m+m' | l, l', m, m' \rangle$:

$$\begin{aligned} J_\pm Y_{\ell m} &= \sqrt{(\ell\mp m)(\ell\pm m+1)}\; Y_{\ell\, m\pm 1} \\ J_0 Y_{\ell m} &= m\, Y_{\ell m} \\ Y_{\ell m} Y_{\ell' m'} &= \sum_{j=|\ell-\ell'|}^{\ell+\ell'} \langle j\; m+m' | \ell, \ell', m, m' \rangle\, Y_{j, m+m'} \end{aligned} \tag{19}$$

We give some of the commutation relations to illustrate their form but, since the derivation of the rest is trivial, and for considerations of space, omit the rest. The Jacobi identities are satisfied.

$$[X_{\ell m}, J^{\pm}_{\ell' m'}] = -\sqrt{(\ell\mp m)(\ell\pm m+1)} \sum_{j=|\ell-\ell'|}^{\ell+\ell'} \langle j\; m+m'\pm 1 | m\pm 1, m' \rangle\, X_{j, m+m'\pm 1} \tag{20a}$$

$$[J^{\pm}_{\ell m}, J^{0}_{\ell' m'}] = \sum_{j=|\ell-\ell'|}^{\ell+\ell'} \Big\{ \sqrt{(\ell'\mp m')(\ell'\pm m'+1)}\, \langle j, m+m'\pm 1 | m, m'\pm 1\rangle\, J^0_{j, m+m'\pm 1} - (m\pm 1)\langle j, m+m' | m, m'\rangle\, J^{\pm}_{j\, m+m'} \Big\} \tag{20b}$$

$$[Q^a_{\ell m}, Q^b_{\ell' m'}] = \sum_j \sum_c f^{abc} \langle j, m+m' | \ell, \ell', m, m' \rangle\, Q^c_{j\, m+m'} \tag{20c}$$

$$[P^\mu_{\ell m}, P^\nu_{\ell' m'}] = 0 \tag{20d}$$

$$[M^{\mu\nu}_{\ell m}, P^\sigma_{\ell' m'}] = \sum_j \langle j, m+m' | m, m' \rangle \left(\eta^{\mu\sigma} P^\nu_{j\, m+m'} - \eta^{\nu\sigma} P^\mu_{j\, m+m'}\right) \tag{20e}$$

etc.

Furthermore, note that if one drops either of the sets $\{J^+_{lm}\}$ or $\{J^-_{lm}\}$ the resultant truncated algebra still closes. This indicates that $\{J^-_{lm}, J^0_{lm}\}$ or $\{J^+_{lm}, J^0_{lm}\}$ together with the $\{X_{lm}\}$ are two alternative complete bases for a discussion of the infinite set of symmetries resulting from the harmonic analysis of the higher dimensional coordinate and gauge invariances.

We now turn to the finite sub-algebras. First consider d=5. The three generators $Q_{\pm,0}$ form a non-compact algebra, with vanishing Casimir C_1, namely $so(1,2)$:

$$\begin{aligned} [Q_{+n}, Q_{-n}] &= 2n\, Q_0 \qquad n = 1, 2, \ldots \\ [Q_{\pm n}, Q_0] &= \pm n\, Q_{\pm n} \\ C_n = \tfrac{1}{2}\{Q_{+n}, Q_{-n}\} - Q_0^2 &\equiv 0 \end{aligned} \tag{21}$$

In fact note further that $Q_{\pm n,0}$ also form an so(1,2) algebra, with zero Casimir, C_n, for arbitary n. Thus there are an infinite number of so(1,2) subalgebras!!

It is easy to see [2] that the generators $Q_{\pm n}$ rotate given fields on the m'th floor $\phi_m(x)$ into fields with floor numbers differing by $\mp n$. Thus $Q_\pm$ are precisely the 'ladder' operators that generate motion through the weight space of U(1). Prior to the set of vacuum expectation values that define the symmetry of the effective theory in four dimensions, the towers of fields with floor numbers n form irreducible representations of Poincare X so(1,2)[1]. In the true ground state the only remaining symmetry is PoincareX U(1). The Goldstone bosons resulting from the breaking of the other symmetries [2] furnish the longitudinal modes for a tower of massive gravitons. The so(1,2) generators provide a 'ladder' ($Q_\pm$) or 'non-stop elevators' ($Q_{\pm n}$) to rotate fields on one floor into those on another, but no longer commute with the mass operator. This corresponds to the mass splittings between different floors of the final tower.

We now turn to the finite sub-algebras in the six dimensional case. A realization of the non-compact generators $K_{\pm,0}$ of so(1,3) on S^2 has been derived by Randjbar-Daemi[5] :

$$K_0 = -i\sin\theta\,\partial_\theta \qquad (22)$$
$$K_\pm = ie^{\pm i\varphi}\left(\cos\theta\,\partial_\theta \pm \frac{i}{\sin\theta}\partial_\varphi\right)$$

Together with $J_{\pm,0}$ given in (16) ,which generate the compact sub-sub-algebra of isometries , these six generators form an so(1,3) algebra :

$$[J_\pm, K_\mp] = \pm 2K_0 \qquad [K_0, K_\pm] = \mp J_\mp$$
$$[K_+, K_-] = -2J_0 = -[J_+, J_-]$$
$$[J_0, K_0] = [J_\pm, K_\pm] = 0 \qquad (23)$$
$$[J_0, K_\pm] = \pm K_\pm$$
$$[J_\pm, K_0] = \mp K_\pm$$
$$[J_\pm, J_0] = \mp J_\pm$$

It is easy to show that one can write $K_\pm$ as :

$$K_\pm = i\sqrt{\frac{8\pi}{3}}\left(J_0^{1,\pm 1} \pm \frac{J_\pm^{1,0}}{\sqrt{2}}\right) \qquad (24)$$

While. K_0 can be written in any of three ways :

$$K_0 = -i\sqrt{\frac{2\pi}{3}}\left(J_-^{1,1} + J_+^{1,-1}\right)$$
or $$(25)$$
$$K_0 = -i\sqrt{\frac{8\pi}{3}}\left(J_\pm^{1,\mp 1} \pm \frac{J_0^{1,0}}{\sqrt{2}}\right)$$

Thus we have proven that so(1,3) is a sub-algebra of (20).

One can also see that there are,infact, an infinite number of other finite sub-algebras of (20) . For instance we have an infinite family of algebras generated by K^l, J_+, J_0 ,where K^l is defined by:

$$K_+^{\ell} = J_+^{\ell,\ell-1} + \sqrt{2\ell}\; J_0^{\ell,\ell} \qquad (26)$$

The algebra is :

$$[K_+^{\ell}, J_+] = 0 \qquad [K_+^{\ell}, J_0] = -\ell K_+^{\ell} \qquad (27)$$
$$[J_+, J_0] = -J_+$$

The Casimir invariants F,G of so(1,3) defined by :

$$F = \vec{J}^2 - \vec{K}^2 = \frac{J_+J_- + J_-J_+}{2} + J_0^2 - \frac{K_+K_- + K_-K_+}{2} - K_0^2$$
$$(28)$$
$$G = i\vec{J}\cdot\vec{K} = i\frac{J_+K_- + J_-K_+}{2} + iJ_0K_0$$

are seen to be identically zero upon substituting the explicit realizations of the so(1,3) generators given in (12) and(22). One can determine how these generators act on the four dimensional fields by considering the behaviour of the relevant harmonics*. One finds that the non-compact generators rotate fields of a given 'floor' (labelled by l) and'room'(labelled by m) into a mixture of fields on different floors and rooms (for fields occurring in the expansion of a scalar on S^2, $\Delta l = \pm 1, 0$). A detailed analysis of how the symmetries of the Lagrangian organize the spectrum of the final theory,in which most of the intial symmetries are broken ,in particular the case of d=6 gravity coupled to gauge fields[3],into Kaluza-Klein towers with 'ladders' and 'elevators',will be given in a forthcoming publication[6].

It should be obvious that generalization to arbitary dimensions compactified to $M^4 \times G/H$, where G/H is a compact coset space is trivial in principle . One need only replace the spherical harmonics on $S^2 = SO(3)/SO(2)$ by scalar harmonics on G/H and the generators $J_{\pm,0}$ by the usual Cartan-Weyl generators $E_{\pm\alpha}, H_i$ of G.

In conclusion we have introduced a new (to the best of our knowledge) type of infinite dimensional algebra with a 'loop structure' that could be called a generalization of the well known Kac-Moody algebras $C(t,t^{-1}) \times G$. Such algebras are useful for describing the symmetries of Kaluza-Klein theories and will doubtless prove valuable in other contexts[7] where Kac-Moody symmetries have been shown to be operative.The moral of our analysis is that: 'if Nature builds towers , She does not omit to provide them with ladders'!

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* One simply makes a harmonic analysis of the variation of a six dimensional field under general coordinate transformations of θ,φ generated by the non-compact generators (vide eqns.17,24,25).The result explicitly confirms whatone expects from group theory:the K_i generate motion through the weight space (l,m) of so(1,3).

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REFERENCES

[1] Abdus Salam and J. Strathdee, Ann. Phys. 141, 316 (1982).

[2] L. Dolan and M.J. Duff, Phys. Rev. Lett. 52, 14 (1984).

[3] S. Randjbar-Daemi, A. Salam and J. Strathdee, Nucl. Phys. B214, 491 (1983).

[4] B. Shutz, Geometrical Methods in Mathematical Physics, (Cambridge Univ. Press, 1980).

[5] S. Randjbar-Daemi, unpublished, private communication.

[6] C.S. Aulakh, work in progress.

[7] L. Dolan, Phys. Rev. Lett. 47, 1371 (1981); Phys. Lett. 113B, 387 (1982).