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CONSTRUCTION OF LOCAL AND NON-LOCAL CONSERVATION LAWS FOR NON-LINEAR FIELD EQUATIONS *

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ABSTRACT

A method of constructing conserved currents for non-linear field equations is presented. More explicitly for non-linear equations, which can be derived from compatibility conditions of some linear system with a parameter, a procedure of obtaining explicit expressions for local and non-local currents is developed. Some examples such as the classical Heisenberg spin chain and supersymmetric Yang-Mills theory are considered.

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I. INTRODUCTION

It is well known that conservation laws are important for understanding the dynamics of non-linear equations. Usually local currents are considered, see for example, Refs.[1,2]. However, as it was shown by Lüscher and Polhmeyer [3] for the non-linear sigma model, non-local currents may also be important. In particular they give a possibility to calculate the quantum S-matrix [4]. These currents are related to an infinite-dimensional Lie-algebra of symmetry transformations acting on the space of solutions of the sigma model [5].

In this paper we shall show that non-local currents exist for a very general class of non-linear equations in multi-dynamical space-time. To this end we consider a corresponding linear equation and write down the currents for this linear system using the procedure of Ref.[6]. However, generally such currents are non-local in both time and space coordinates. On the other hand, non-trivial currents which are local in time (but non-local only in the space variables) and which could be explicitly calculated exist most probably only for integrable systems.

Let us consider two-dimensional non-linear equations, which can be derived from compatibility conditions of some linear system (i.e. zero curvature representation) with a parameter λ. It will be shown that it is always possible, up to gauge transformation, to construct an infinite number of non-local currents. These currents are obtained by expanding the solution of the linear system around a regular point in the complex λ-plane. To derive the currents for the non-linear system we simply write down the currents for the corresponding linear system. In particular we obtain the non-local currents for the classical Heisenberg spin chain. An analogous procedure is applied to the supersymmetric Yang-Mills theory.

Furthermore, we shall also propose a general procedure for the derivation of local currents through expansion around a singular point of the λ-plane.

II. NON-LOCAL CURRENTS FOR TWO-DIMENSIONAL INTEGRABLE SYSTEMS

Let us consider the following system of linear differential equations

$$\partial_1 \psi = U(x,\lambda)\psi \tag{2.1}$$

$$\partial_2 \psi = V(x,\lambda)\psi \,, \tag{2.2}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $\partial_i = \partial/\partial x_i$, $i = 1,2$; $U(x,\lambda)$, $V(x,\lambda)$ are the matrix function meromorphic on λ. The Frobenius condition

$$\partial_2 U - \partial_1 V + [U,V] = 0 \qquad (2.3)$$

is a necessity and sufficient condition for compatibility of the system in Eqs.(2). A great variety of two-dimensional non-linear differential equations was solved by means of such system [2]. From Eqs.(2.1) and (2.2) it follows that currents

$$J_1 = -V\psi \quad , \quad J_2 = U\psi \qquad (2.4)$$

satisfy the conservation law

$$\partial_1 J_1 + \partial_2 J_2 = 0 \quad . \qquad (2.5)$$

Let $\lambda = \lambda_0$ be a regular point for the functions $U(x,\lambda)$, $V(x,\lambda)$. Using the espansion of ψ and currents (2.4) in a series with respect to $(\lambda - \lambda_0)$:

$$\psi(x,\lambda) = \sum_{n=0}^{\infty} (\lambda-\lambda_0)^n \psi^{(n)}(x)$$

$$J_\mu(x,\lambda) = \sum_{n=0}^{\infty} (\lambda-\lambda_0)^n J_\mu^{(n)}(x) \quad , \mu = 1,2$$

we get

$$J_1^{(n)}(x) = -\sum_{k=0}^{n} V^{(k)}(x)\psi^{(n-k)}(x) \quad , \quad J_2^{(n)}(x) = \sum_{k=0}^{n} U^{(k)}(x)\psi^{(n-k)}(x) \quad , \qquad (2.6)$$

where $\psi^{(n)}(x)$ satisfy the equations:

$$\partial_1\psi^{(n)} = \sum_{k=0}^{n} U^{(k)}\psi^{(n-k)} \quad , \quad \partial_2\psi^{(n)} = \sum_{k=0}^{n} V^{(k)}\psi^{(n-k)} \quad . \qquad (2.7)$$

From Eqs.(2.6) it is clear that if we have the gauge conditions:

$$U(x,\lambda_0) = U^{(0)}(x) = 0 \quad , \quad V(x,\lambda_0) = V^{(0)}(x) = 0 \qquad (2.8)$$

then $\psi^{(n)}(x)$ is represented as a finite sum of integrals from $U^{(k)}(x)$ and

$V^{(k)}(x)$. In particular:

$$J_1^{(1)}(x) = -V^{(1)}(x) \quad , \quad J_2^{(1)}(x) = U^{(1)}(x) \qquad (2.9)$$

$$J_1^{(2)}(x) = -V^{(1)}(x)\psi^{(1)}(x) - V^{(2)}(x)$$
$$J_2^{(2)}(x) = U^{(1)}(x)\psi^{(1)}(x) + U^{(2)}(x) \quad , \qquad (2.10)$$

where
$$\psi^{(1)}(x) = \int_{x^0}^{x} [U^{(1)}(x')dx_1' + V^{(1)}(x')dx_2' \quad ,$$

but we can always choose the gauge conditions (2.7) by means of the gauge transformation

$$\psi \to \tilde{\psi} = \Omega^{-1}\psi \quad , \quad U \to \tilde{U} = \Omega^{-1}U\Omega - \Omega^{-1}\partial_1\Omega$$
$$V \to \tilde{V} = \Omega^{-1}V\Omega - \Omega^{-1}\partial_1\Omega \quad ,$$

where $\Omega = \psi(x,\lambda_0)$.

So we have the following theorem.

Theorem For every non-linear differential equation which can be derived from compatibility conditions of Eqs.(2.1) and (2.2) there exists an infinite number of non-local currents which up to a gauge transformation are given by the Eqs.(2.6).

The example of the non-linear sigma model is well known. The non-local currents [3-5] for this model can be expressed in the form (2.6). Let us consider the classical Heisenberg model, for which the explicit form for non-local currents may be new.

The Heisenberg model involves a field

$$S(x,t) = \sum_{a=1}^{3} S_a(x,t)\sigma_a \quad , \quad S^2 = I \quad ,$$

where σ_a , a = 1,2,3 denote the Pauli matrices. The equation of motion

$$\partial_t S = \frac{1}{2i}\partial_x [S, S_x] \tag{2.11}$$

is the integrability condition of the Lax pair [7]

$$\partial_x \psi = U\psi = i\lambda S\psi$$
$$\partial_t \psi = V\psi = (\lambda S S_x + 2i\lambda^2 S)\psi \quad .$$

For $\lambda_0 = 0$ we have $U(x,0) = V(x,0) = 0$, i.e. gauge conditions (2.8). Therefore according to (2.9) and (2.10), the first non-trivial currents are

$$J_1^{(1)} = -SS_x \quad , \quad J_2^{(1)} = iS \tag{2.12}$$

$$J_1^{(2)} = -SS_x\psi^{(1)} - 2iS \quad , \quad J_2^{(2)} = iS\psi^{(1)} \quad , \tag{2.13}$$

where

$$\partial_x \psi^{(1)} = iS \quad , \quad \partial_t \psi^{(1)} = SS_x \quad .$$

Let us consider the solutions of Eq.(2.11), which satisfy the following boundary conditions: $S(x,t) - S_0$, $S_x(x,t)$, $S_{xx}(x,t)$ going to 0, when $|x| \to \infty$ (here S_0-some constant matrix, $S_0^2 = I$). Then on the space of such solutions and with the help of the currents (2.12) and (2.13), we can obtain the following integrals of motion:

$$\begin{aligned} Q^{(1)} &= i\int_{-\infty}^{\infty} \bar{S}(x,t)\,dx \\ Q^{(2)} &= \iint_{-\infty}^{\infty} \bar{S}(x,t)\,\theta(x-x')\,\bar{S}(x',t)\,dx\,dx' \\ &\quad + \int_{-\infty}^{\infty} x\,[\bar{S}(x,t), S_0]\,dx \quad , \end{aligned} \tag{2.14}$$

where $\bar{S}(x,t) = S(x,t) - S_0$; $\theta(x) = 1$, $x \geq 0$, $\theta(x) = 0$, $x < 0$.

The non-local charge $Q^{(2)}$ is analogous to the famous non-local charges [3-5] for the sigma model. It is interesting to investigate the possibility of calculating the S-matrix of magnons by means of $Q^{(2)}$ in analogy with Lüscher's work on the calculation of the S-matrix for the non-linear sigma model. It is interesting also to understand the connection between $Q^{(2)}$ (2.14) and the hidden symmetry of Kac-Moody type for this model [8,9].

III. LOCAL CURRENTS FOR TWO-DIMENSIONAL INTEGRABLE SYSTEMS.

It is well known that for many integrable systems there exist a method of obtaining the integrals of motion with local densities by means of the trace identities, (see for example Refs.[2,10]). In this section it will be shown that it is possible to get not only the integrals of motion but both components of currents in a form of a simple general formula. To this end we use an asymptotic expansion of Eqs.(2.1) and (2.2) in a singular point λ_0 of the complex λ-plane. Let us first reduce the problem of obtaining solutions of the system: Eqs.(2.1) and (2.2), to the problem of finding solutions for Eq.(2.1) and Eq.(2.2) separately. This procedure is motivated by the following lemma.

<u>Lemma</u> Let us consider the Eqs.(2.1) and (2.2) in a certain vicinity of a point $X \neq 0$ and let $V(0,x_2,\lambda)$ be an invertible matrix. Let $\chi(x,\lambda)$ be the solution of Eq.(2.1) with the condition $\chi(0,x_2,\lambda) = I$. Then $\psi(x,\lambda)$ satisfy the following equation:

$$(\partial_2 - V)\chi = -\chi V_0 \quad , \tag{3.1}$$

where $V_0 = V(0,x_2,\lambda)$.

<u>Proof</u> Let us denote

$$\phi = (\partial_2 - V)\chi \quad . \tag{3.2}$$

Then

$$(\partial_1 - U)\phi = 0 \quad , \quad \phi|_{x_1=0} = -V_0 \quad .$$

Therefore $-\phi V_0^{-1}$ satisfy the Eq.(2.1) and $-(\phi V_0^{-1})|_{x_1=0} = I$, i.e. $-\phi V_0^{-1} = \chi$. Now from (3.2) it follows:

$$(\partial_2 - V)\chi = -\chi V_0 \quad .$$

Q.E.D.

By using this lemma we can construct the solution of the systems (2.1) and (2.2). In fact, if $\Omega = \Omega(x_2,\lambda)$ is an arbitrary solution of equation

$$\partial_2 \Omega - V(0,x_2,\lambda)\,\Omega = 0 \tag{3.3}$$

then

$$\psi = \chi\,\Omega \tag{3.4}$$

is a solution of Eqs.(2.1) and (2.2).

Let us now consider Eqs.(2.1) and (2.2) and let λ_0 be a pole for $U(x,\lambda)$ or $V(x,\lambda)$. After denoting $\lambda - \lambda_0 = \varepsilon$ we have the systems (2.1) and (2.2) in the form

$$\partial_1 \psi = \frac{1}{\varepsilon^p} A(x,\varepsilon)\,\psi \tag{3.5a}$$

$$\partial_2 \psi = \frac{1}{\varepsilon^q} B(x,\varepsilon)\,\psi \quad , \tag{3.5b}$$

where p and q are some integer number (e.g. $p > 0$),

$$A(x,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n A_n(x) \quad , \quad B(x,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n B_n(x)$$

and consider the asymptotic behaviour of the solutions of Eqs.(3.5) in the limit $\varepsilon \to 0$.

If we have only one equation (3.5a), then the asymptotic behaviour of the solution is well known. Let us suppose that the matrix $A(x,0)$ is diagonalizable

$$T(x)\,A(x,0)\,T^{-1}(x) = \Lambda_0(x) = \operatorname{diag}[\lambda_1(x), \ldots, \lambda_k(x)] \tag{3.6}$$

and $\lambda_i(x) \neq \lambda_j(x)$, $i \neq j$. Then Eq.(3.5a) has the asymptotic solution [11]

$$\psi = \phi\, e^{\frac{1}{\varepsilon^p} S} \quad , \tag{3.7}$$

where S is a diagonal matrix, $\partial_1 S = \Lambda$ and ϕ and S are asymptotic series with respect to ε. We can always choose S such that

$$S(x,\varepsilon)\big|_{x_1=0} = 0 \quad . \tag{3.8}$$

The coefficients in the series for S and ϕ can be expressed in terms of $A(x,\varepsilon)$ by means of a recursion procedure.

We shall now use the Lemma. Let

$$\chi = \phi\, e^{\frac{1}{\varepsilon^p} S} \phi_0^{-1} \quad , \tag{3.9}$$

where $\phi_0 = \phi(0,x_2,\varepsilon)$, is the solution of Eqs.(3.5), which satisfy the initial condition [1]

$$\chi\big|_{x_1=0} = I \quad .$$

Then by Lemma we have

$$\partial_2 \chi = \frac{1}{\varepsilon^q} B\chi - \frac{1}{\varepsilon^q}\chi B_0 \quad , \quad B_0 = B\big|_{x_1=0}$$

i.e.

$$\partial_2 S \equiv M = \varepsilon^{p-q}\phi^{-1}B\phi - \varepsilon^p\phi^{-1}\partial_2\phi - e^{\frac{1}{\varepsilon^p}S}\left(\varepsilon^{p-q}\phi_0^{-1}B_0\phi_0 - \varepsilon^p\phi_0^{-1}\partial_2\phi_0\right)e^{-\frac{1}{\varepsilon^p}S} =$$

$$= T - e^{\frac{1}{\varepsilon^p}S}\,T_0\,e^{-\frac{1}{\varepsilon^p}S} \quad , \quad T = \varepsilon^{p-q}\phi^{-1}B\phi - \varepsilon^p\phi^{-1}\partial_2\phi \quad , \quad T_0 = T\big|_{x_1=0} \tag{3.10}$$

or

$$M + T_0 = e^{-\frac{1}{\varepsilon^p}S}\,T\,e^{\frac{1}{\varepsilon^p}S} \quad .$$

As $\partial_1 T_0 = 0$ we have

$$\partial_1 M = e^{-\frac{1}{\varepsilon^p}S}\left(\partial_1 T - \frac{1}{\varepsilon^p}[\Lambda, T]\right)e^{\frac{1}{\varepsilon^p}S} \tag{3.11}$$

and, because $\operatorname{diag}[\Lambda,T] = 0$, we get from (3.11)

$$\partial_1 M = \operatorname{diag}\partial_1 T \quad . \tag{3.12}$$

The solution of Eq.(3.12), with the initial condition $M|_{x_1=0} = 0$, (see Eq.(3.10)) has the form

$$M = \operatorname{diag}(T - T_0) \quad . \tag{3.13}$$

[1] Expressions such as (3.9) exist in Faddeev's paper [10].

Now, as $\Lambda = \partial_1 S$ and $M = \partial_2 S$ we have $\partial_2 \Lambda - \partial_1 M = 0$

i.e.

$$\partial_2 \,\mathrm{diag}(\phi^{-1} A \phi - \varepsilon^p \phi^{-1} \partial_1 \phi) - \partial_1 \,\mathrm{diag}(T \cdot T_0) = 0 \quad .$$

And finally, because $\partial_1 T_0 = 0$, we get

$$\partial_2 \,\mathrm{diag}(\phi^{-1} A \phi - \varepsilon^p \phi^{-1} \partial_1 \phi) - \partial_1 \,\mathrm{diag}(\varepsilon^{p-q} \phi^{-1} B \phi) - \varepsilon^p \phi^{-1} \partial_2 \phi) = 0 \quad . \tag{3.14}$$

So we have the following theorem:

Theorem For every system of non-linear equations, which can be derived from compatibility conditions of linear system (3.5) with conditions (3.6), there exists infinite number of local currents which are given by asymptotic expansions with $\varepsilon \to 0$ of Eq.(3.14).

The formula (3.14) can be used to derive local currents for Kortveg-de Vries equation, non-linear Schroedinger equation, Sine-Gordon equation, chiral field and so on.

IV. CONSERVATION LAWS FOR NON-LINEAR EQUATIONS.

In this section we shall show that for arbitrary non-linear system the problem of obtaining non-local currents can be reduced to the problem of solving corresponding linear equations. It is possible to represent an arbitrary non-linear system of equations in the form

$$\mathcal{P}[u]u = \sum_{k=0}^{m} P_{\nu_1 \ldots \nu_k}[u](x)\, \partial_{\nu_1} \ldots \partial_{\nu_k} u(x) = 0 \quad , \tag{4.1}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $u(x)$ is a matrix-function and $u \to P_\nu[u](x)$ are some non-linear matrix operators. Let us fix some $u(x)$. In analogy with the linear case [6] introduce a corresponding linear system of differential equations:

$$\mathcal{P}[u] w \equiv \sum_{k=0}^{m} P_{\nu_1 \ldots \nu_k}[u](x)\, \partial_{\nu_1} \ldots \partial_{\nu_k} w(x) = 0 \tag{4.2a}$$

$$v \tilde{\mathcal{P}}[u] \equiv \sum_{k=0}^{m} (-1)^k \partial_{\nu_1} \ldots \partial_{\nu_k} (v(x) P_{\nu_1 \ldots \nu_k}[u](x)) = 0 \tag{4.2b}$$

with respect to ω and v. To the system (4.2) with a fixed U these corresponds the Lagrangian

$$\mathcal{L}(w, v; u) = \mathrm{Tr}\,(v \mathcal{P}[u] w) \quad . \tag{4.3}$$

This Lagrangian is invariant under the transformations $\omega \to \omega c$, $v \to c^{-1} v$, where c is a constant matrix. Therefore by the Noether theorem the system (4.2) has the following conserved current on the solutions ω, v:

$$J_\nu(x) = \sum_{0 \le q+k \le m-1} (-1)^k \cdot \partial_{\nu_1} \ldots \partial_{\nu_k} (v P_{\nu \nu_1 \ldots \nu_k \lambda_1 \ldots \lambda_q}[u]) \partial_{\lambda_1} \ldots \partial_{\lambda_q} w \quad . \tag{4.4}$$

The conservation law follows from the formula

$$\partial_\nu J_\nu = (v \tilde{\mathcal{P}}[u]) w - v (\mathcal{P}[u] w) \tag{4.5}$$

for arbitrary v, ω. In particular, for $\omega = u$ and if v satisfies the Eq.(4.2b), then the current (4.4) is conserved. Really, if we have a system (4.1), we can consider v and ω as some functionals which are determined by Eqs.(4.2) and therefore we have the current (4.4) for the original system (4.1).

Let us consider some specific examples. Other examples will be considered in another publication [12].

4.1 For the equation

$$(P_\nu[u]\partial_\nu + P_0[u])u = 0$$

the corresponding linear system (4.2b) has the form

$$\partial_\nu (v P_\nu[u]) - v P_0[u] = 0$$

and the current

$$J_\nu = v P_\nu [u] u \quad . \tag{4.6}$$

In particular for the non-linear Dirac equation

$$[i \gamma^\nu \partial_\nu + i \gamma^\nu A_\nu(x) + f(\bar\psi \psi)] \psi = 0$$

the linear system (4.2b) has the form

$$i\partial_\nu(v\gamma^\nu) - iv\gamma^\nu A_\nu - vf(\bar\psi\psi) = 0 \qquad (4.7)$$

Let us mention that $v = \bar\psi$ satisfies Eq.(4.6). In this case the current (4.6) is the ordinary electromagnetic current

$$J_\nu = \bar\psi\gamma_\nu\psi \quad .$$

4.2 Boltzman equation has the form

$$\frac{\partial F}{\partial t} + (v, \mathrm{grad}_x F) + (f, \mathrm{grad}_v F) = I(F) \quad ,$$

where

$$I(F)(t,x,v) = \frac{1}{\rho}\int_{\mathbb{R}^3}\int_{\pi(v-u)} [F(v^*)F(u^*) - F(v)F(u)]\,|v-u|\,dy\,du$$

and $F \equiv F(v) \equiv F(t,x,v)$ is a function of variables $t \in \mathbb{R}^1$, $x,v \in \mathbb{R}^3$; f is a given function of t,x ; $v^* = v^*(v,u,y)$ and $u^* = u^*(v,u,y)$ are the given functions and $\pi(v-u)$ is a two-dimensional plane in the y-space which is orthogonal to the vector $v-u$; ρ is a constant density.

The corresponding linear equation (4.2b) has the form

$$\frac{\partial G}{\partial t} + (v, \mathrm{grad}_x G) + (f, \mathrm{grad}_v G) + \frac{I(F)}{F} G = 0 \quad . \qquad (4.8)$$

The conserved currents have the form

$$J_0 = FG \quad , J_\nu = v_\nu FG \; , \; J_{3+\nu} = f_\nu FG \quad , \quad \nu = 1,2,3 \quad .$$

Let us mention that Eq.(4.8) is a linear equation of first order and a solution of this equation can be find for example by the characteristic method.

V. NON-LOCAL CURRENTS FOR THE SUPERSYMMETRIC YANG-MILLS THEORY.

In this section we shall consider the non-local currents for supersymmetric Yang-Mills theory. We will use the mathematical approach to superanalysis which is developed in Ref.[14]. Let $y^A = (x^\mu, \theta^\alpha_s, \bar\theta^{\dot\beta t})$, $\mu = 0,1,2,3$; $\alpha,\beta = 1,2$; $s,t = 1,\dots,K$ are the coordinates in superspace where x^μ are the commuting space-time variables and $\theta^\alpha, \bar\theta^{\dot\beta t}$ are the anticommuting variables. The covariant derivatives have the form $\nabla_A = D_A + \mathcal{A}_A$, where $D_A = (\partial_\mu, D^s_\alpha, \bar D_{\dot\beta t})$, $\partial_\mu = \partial/\partial x\mu$,

$$\partial_{\alpha\dot\beta} = \sigma^\mu_{\alpha\dot\beta}\partial_\mu \quad , \mathcal{D}_\alpha = \frac{\partial}{\partial\theta^\alpha_s} + i\bar\theta^{\dot\beta s}\partial_{\alpha\dot\beta} \; , \bar{\mathcal{D}}_{\dot\beta t} = -\frac{\partial}{\partial\bar\theta^{\dot\beta t}} - i\theta^\alpha_t\partial_{\alpha\dot\beta}$$

$\sigma^\mu = (\sigma^\mu_{\alpha\beta})$ are the Pauli matrices and superpotentials $\mathcal{A}_A(y)$ are Lie algebra valued. The supercurvature F_{AB} is defined by the graded commutator of covariant derivatives

$$[\nabla_A, \nabla_B] = T^C_{AB}\nabla_C + F_{AB} \quad , \qquad (5.1)$$

where T_{AB}^C is the torsion. The superconnection has to satisfy the equations [13]

$$F^{st}_{\alpha\beta} + F^{st}_{\beta\alpha} = 0 \quad , F_{\dot\alpha s,\dot\beta t} + F_{\dot\beta s,\dot\alpha t} = 0 \quad , F^s_{\alpha,\dot\beta t} = 0 \quad .$$

When $K = 3$ Eqs.(4.1) coincides with the equations of motion for the $N = 4$ supersymmetric Yang-Mills theory.

Eq. (5.1) can be derived as the consistency conditions for the following system of linear equations with a parameter λ [15]:

$$\begin{aligned} &(\nabla^s_1 + \lambda\nabla^s_2)\psi = 0 \\ &(\nabla_{\dot 1 t} + \lambda^2\nabla_{\dot 2 t})\psi = 0 \\ &(\nabla_{1\dot 1} + \lambda\nabla_{2\dot 1} + \lambda^2\nabla_{1\dot 2} + \lambda^3\nabla_{2\dot 2})\psi = 0 \quad . \end{aligned} \qquad (5.2)$$

Introducing the operators (see [16])

$$\begin{aligned} &\mathcal{D}^s = D^s_1 + \lambda D^s_2 \quad , \bar{\mathcal{D}}_t = D_{\dot 1 t} + \lambda^2 D_{\dot 2 t} \\ &\mathcal{D} = \partial_{1\dot 1} + \lambda\partial_{2\dot 1} + \lambda^2\partial_{1\dot 2} + \lambda^3\partial_{2\dot 2} \\ &\mathcal{O}\!\ell^s = \mathcal{A}^s_1 + \lambda\mathcal{A}^s_2 \; , \quad \bar{\mathcal{O}\!\ell}_t = \mathcal{A}_{\dot 1 t} + \lambda^2\mathcal{A}_{\dot 2 t} \\ &\mathcal{O}\!\ell = \mathcal{A}_{1\dot 1} + \lambda\mathcal{A}_{2\dot 1} + \lambda^2\mathcal{A}_{1\dot 2} + \lambda^3\mathcal{A}_{2\dot 2} \end{aligned}$$

we write the system (5.1) in the form

$$(\mathcal{D}^s + \alpha^s)\psi = 0$$
$$(\bar{\mathcal{D}}_t + \bar{\alpha}_t)\psi = 0 \qquad (5.3)$$
$$(\mathcal{D} + \alpha)\psi = 0 \quad .$$

The consistency conditions for the system of equations (5.3) are

$$\mathcal{D}^s\alpha^t + \mathcal{D}^t\alpha^s + \{\alpha^s, \alpha^t\} = 0$$
$$\bar{\mathcal{D}}_s\bar{\alpha}_t + \bar{\mathcal{D}}_t\bar{\alpha}_s + \{\bar{\alpha}_t, \bar{\alpha}_s\} = 0$$
$$\bar{\mathcal{D}}_t\alpha^s + \mathcal{D}^s\bar{\alpha}_t + \{\alpha^s, \bar{\alpha}_t\} - 2i\delta_t^s\alpha = 0 \quad .$$

We can write Eqs.(5.3) in the form analogous to Eq.(4.1):

$$(\xi_s\mathcal{D}^s + \xi_{K+t}\bar{\mathcal{D}}_t + \xi_{2K+1}\mathcal{D} + \xi_s\alpha^s + \xi_{K+t}\bar{\alpha}_t + \xi_{2K+1}\alpha)\psi = 0 ,$$

where ξ_n is a column, which has identity matrix on the n place and the other' places are zero.

The corresponding linear system (4.2b) in this case for the anticommuting $v = (\chi,\varphi,\tau)$, $\chi = (\chi_1,\dots,\chi_K)$, $\varphi = (\varphi^1,\dots,\varphi^K)$, $\tau = (\tau)$ has the form:

$$\mathcal{D}^s(v\xi_s) + \bar{\mathcal{D}}_t(v\xi_{K+t}) + \mathcal{D}(v\xi_{2K+1}) + v\xi_s\alpha^s + v\xi_{K+t}\bar{\alpha}_t + v\xi_{2K+1}\alpha = 0$$

or

$$\mathcal{D}^s\chi_s + \bar{\mathcal{D}}_t\varphi^t + \mathcal{D}\tau + \chi_s\alpha^s + \varphi^t\bar{\alpha}_t + \tau\alpha = 0 \quad . \qquad (5.4)$$

The conserved current has the form (J_s, J^t, J) , where

$$J_s = \chi_s\psi \ , \quad J^t = \varphi^t\psi \ , \quad J = \tau\psi$$

and the conservation law is

$$\mathcal{D}^s J_s + \bar{\mathcal{D}}_t J^t + \mathcal{D}J = 0 \quad .$$

The particular solutions of Eqs.(5.3] are

$$\chi_q = \alpha^p \ , \ \chi_p = \alpha^q \ , \ \chi_s = 0 \ , \ q \neq s \neq p \ ; \quad \varphi^t = 0 \ , \ \tau = 0$$

or

$$\chi_s = 0 \ ; \quad \varphi^q = \bar{\alpha}_p \ , \quad \varphi^p = \bar{\alpha}_q \ , \ \varphi^t = 0 \ , q \neq t \neq p \ ; \ \tau = 0$$

and correspondingly the currents

$$J_q = \alpha^p\psi \ , \ J_p = \alpha^q\psi \ ; \ J_s = 0 \ , \ q \neq s \neq p \ , \ J^t = 0 \ ; \ J = 0$$

and

$$J_s = 0 \ ; \ J^q = \bar{\alpha}_p\psi \ , \ J^p = \bar{\alpha}_q\psi \ , J^t = 0 \ , q \neq t \neq p \ ; \ J = 0 \quad .$$

The solutions of the linear system can be constructed like in Sec.II. In particular, if we chose the gauge $\mathcal{A}_1^s = \mathcal{A}_{1t} = \mathcal{A}_{11} = 0$ which corresponds to the gauge (2.8), it is possible to get an explicit series solution of Eqs.(5.3).

Another method of obtaining the currents for the supersymmetric Yang-Mills theory was considered in [17,18].

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