COMMENTS ON THE SPONTANEOUS SYMMETRY BREAKING IN SUPERSYMMETRIC THEORIES

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ABSTRACT

The rôle of the complex extension of the symmetry group in supersymmetric theories is revisited. We prove, in particular, that if symmetry breaking occurs at an extremum of the superpotential, then supersymmetry will be preserved if and only if the complex stabilizer of the vacuum is the complexified of its maximal compact part.
In supersymmetric theories with an internal symmetry, the complex extension $G^c$ of the symmetry group $G$ plays an important rôle. This was first remarked by Ovrut and Wess\textsuperscript{1)} and, subsequently, some consequences of this property have been investigated in the context of gauge theories\textsuperscript{2,3,4)} and also in the case of a globally invariant theory\textsuperscript{5)}. Recently, in this line of thought, a detailed study of the Goldstone phenomenon in supersymmetric models has been presented by Lerche\textsuperscript{6)}.

In this letter we discuss some consequences of this peculiar situation when the $G$ symmetry is spontaneously broken to a $H$ subsymmetry. In our discussion it appears that the nature of the complex isotropy group, $K_v$, of the vacuum vector $v$ determines the properties of the theory at the tree level. Hereafter we consider in turn the cases of supersymmetric theories with a global and a local invariance. In the former case we shall comment on the so-called Goldstone boson doubling in connection with the nature of $K$. Then, for a supersymmetric gauge theory, we prove that if the spontaneous breaking of the symmetry is triggered by a vacuum corresponding to an extremum of the superpotential, supersymmetry is unbroken if and only if the complex isotropy group of the vacuum is the complexified $c$ its maximal compact part.

In order to fix the notations let us consider a supersymmetric model with a local invariance under a compact Lie group $G$ (dim $G = n$). The spontaneous breaking of the symmetry is controlled by the $G$ invariant potential, consisting of two terms, the so-called $D$ and $F$ terms

\[
V(z,\bar{z}) = \frac{1}{2} \sum_a |\bar{z} \cdot T_a z|^2 + \sum_i |\frac{\partial W}{\partial z_i}|^2
\]

\[
= \frac{1}{2} \sum_a |D_a|^2 + \sum_i |F_i|^2
\]

The $z_i$'s are complex scalar fields which belong to the chiral multiplets $\Phi_i$ appearing in the Lagrangian and $T_a$ ($a = 1, \ldots, n$) are hermitean generators of $G$. $W(z)$, the superpotential, is a $G$ invariant polynomial in the fields $z_i$, at most of the third degree for a renormalizable theory.

In the case of a global $G$ invariance, the $D$ term is absent of the potential, which is just

\[
V(z,\bar{z}) = \sum_i |\frac{\partial W}{\partial z_i}|^2
\]

$W(z)$ is a $G$ invariant polynomial in $z_i$ and as such is also $G_c$-invariant; if the symmetry breaking respects supersymmetry we have $|\frac{\partial W}{\partial z_i}| = 0$ for any point $z_i$. 
belonging to the orbit of $v$ under $G^c$.

As some scalar fields $z_i$ acquire vacuum expectation values (VEV), grouped into the vector $v$, the $G$ symmetry is broken down to an $H_v$-symmetry, where $H_v$ is the compact isotropy group of the vector $v$ ($\dim H_v = m$).

When we consider the complex extension of $G$, the complex isotropy group of $v$, $K_v$, always contains $H_v$ as its maximal compact subgroup but can be larger than its complex extension $H_v^c$. Indeed in some cases the vector $v$ is also invariant under complex combinations of the broken generators. We call such combinations mixed generators and their set $\mathcal{M}$, so that $K_v = H_v^c + \mathcal{M}$ in this case. A trivial example can be given for $SU(2)$ with $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the symmetry is completely broken and $H_v = \{e\}$, the identity. However if we complexify we immediately realize that $v$ is invariant under the action of $\sigma = \sigma_1 - i\sigma_2$ and $K_v = \{e\} + \{\sigma\}$. When $G = SU(n)$ $\mathcal{M}$ is spanned by some ladder operators.

The very nature of $K_v$ is of prime importance for the physical properties of the theory as we shall see later. We can note in particular that whenever $G/H$ is symmetric then $K_v = H_v^c$ - see also Ref. 6).

Indeed let $\{T_b\}$ denote the broken generators and assume that $\mathcal{M}$ is not empty, then:

$$(T_b + iT_b^*)v = 0$$

so:

$$\bar{v}(T_b - iT_b^*)(T_b + iT_b^*)v = |T_bv|^2 + |T_b^*v| + i\bar{v}[T_b,T_b^*]v = 0$$

if $G/H$ is symmetric $[T_b,T_b^*] \in H$ therefore

$$T_b^*v = 0 \text{ and } T_b^*v = 0 \text{ that is } K_v = H_v^c.$$ 

In the non supersymmetric theories, the spontaneous symmetry breaking, $G \rightarrow H$, creates $(n-m)$ massless Goldstone bosons which are living in $G/H$.

As stated above, if supersymmetry survives the symmetry breaking, the manifold of the vacua is $G^c$ invariant, whereas the potential is only $G$ invariant. Because of this extra invariance - which may also be accidentally larger \(^7\) - we expect to have $(n-m)$ quasi Goldstone bosons (QGB) \(^{fl}\); altogether the number of massless bosons is then $2(n-m)$, the dimension of $G^c/H_v^c$. This may also be understood, given the unbroken supersymmetry, since there are $(n-m)$ fermion

\(^{fl}\) We do not use the terms pseudo Goldstone bosons (PGB) given the special features of these massless excitations.
partners (Quasi Goldstone fermions or QGF) of the Goldstone bosons so that the matching between bosonic and fermionic degrees of freedom calls for (n-m) more bosons, the QGB's5). Actually these massless fields may be arranged into (n-m) Goldstone superfields, each corresponding to a broken generator. There is therefore a "doubling" of the number of Goldstone boson fields as compared with the non supersymmetric case. Let us remark also that if supersymmetry is unbroken, the QGB's will not acquire a mass by radiative corrections because of the non renormalization theorems8), despite the G\textsuperscript{C} non invariance of the kinetic term - for a global symmetry. In this respect they are a special class of PGB's.

The above mentioned doubling always occurs when $K_{\nu} = H_{\nu}$, on the other hand if $K_{\nu} = H_{\nu} + \mathfrak{m}_{\nu}$, then dim $G/C/K_{\nu} < \text{dim } G/C/H_{\nu}$ and the number of Goldstone superfields is reduced by one unit for each element of $\mathfrak{m}_{\nu}$. Such a case seems to be excluded when the symmetry is gauged, because we need as many Goldstone superfields as broken generators to give masses to the vector superfields of the broken sector. If there is no full doubling some of these vector superfields remain massless and the desired breaking is not achieved. This strongly indicates that we need to have $K_{\nu} = H_{\nu}$ to correctly break the local symmetry.

Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras associated to the local symmetry groups $G$ and $H$ (subgroup of $G$), it is well known that $\mathfrak{g}$ is a representation space for $H$, on which $H$ acts in a complete reducible way. This implies that any broken generator $T_{b}$ is such that

$$\text{either } [T_{b}, T_{h}] = 0 \quad \text{or} \quad [T_{b}, T_{h}] \sim T_{b}, \quad T_{h} \in \mathfrak{h}$$

therefore the generators for which

$$D_{c} = \nabla_{c} T_{c} \neq 0$$

are such that $[T_{c}, \mathfrak{h}] = 0$.

Such a property gives a first selection on the possible symmetry breaking patterns which have been studied in Ref. 9) using the compact group representations.

We denote by $G^{C}\nu$ the orbit of the vacuum vector $\nu$, then we can prove the following which strictly qualifies the situation described in Ref. 2) (f2).

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(f2) There seems to be very little to add to the elementary analytical proof constructed by J. Wess and B. Zumino, hitherto unpublished in order to make it complete, but somehow, the geometrical conditions insisted upon here have not so far appeared there in a natural way.
Theorem: \( \exists \, \omega \in G^C_v \text{ such that } D_a = \tilde{\omega} T_a \omega = 0 \text{ if and only if } K_v = H^C_v. \)

First assume that \( \tilde{v} T_a v = 0 \) and that \( K = H^C_v + M_v \), this means that there are 2 broken generators \( T_b \) and \( T_b' \), which verify

\[
(T_b + iT_b')v = 0
\]

Using the commutation relations, we write:

\[
\bar{v} [T_b, T_b'] v = f_{bb'}, \quad \tilde{v} T_a v = 0 = -2i |T_b v|^2
\]

that is \( T_b v = T_b' v = 0. \)

This proves that \( D_a = 0 \) implies \( K_v = H^C_v. \)

In order to prove the converse property, which states that: if \( K_v = H^C_v \) then \( D_a \) vanishes at some point of the orbit \( G^C_v \) we need further properties.

(a) Saying that \( \tilde{v} T_a v = 0 \) is equivalent to say that \( |v|^2 \) is extremum on the orbit of \( v \). This is easily proved if one calculated \( \delta |v|^2 \) on this orbit.

(b) There is a mathematical theorem due to G.W. Schwartz which establishes that the orbit \( G^C_v \) is closed if and only if \( K_v = H^C_v \); in addition if the orbit is not closed, its closure contains a single closed orbit.

Given these properties, assume that \( K = H^C \) and consider the intersection of the closed ball of radius \( |v|^2 \), with the orbit \( G^C_v \) which is closed to \( (b) \). This intersection is compact, so the \( C^0 \) function \( |v|^2 \) will have at least one minimum on it, say, \( |v|^2 \).

Using property (a) this establishes that \( \tilde{\omega} T_a \omega = 0 \). We thus proved that \( D_a = 0 \iff K_v = H^C_v \), it follows in the pathological case where the stabilizer \( K_v \) is larger than \( H^C_v \), say \( K_v = H^C_v + M_v \), that the \( D \) term does not vanish on the \( v \)-orbit. It might seem that one could break supersymmetry by the \( D \) term alone in this case. In fact this is not true because if we insist to keep the \( F_1 \) term at zero, the minimum for the \( D \) term is not a stable one. We know from (b) that in the closure of the orbit there is a single closed orbit, on which therefore \( D_a = 0 \), which will be a stable minimum. To this closed orbit will correspond another isotropy group \( H^C \) but one still has \( 3W/3z_i = 0 \) on this closed orbit. Indeed, the manifold \( 3W/3z_i = 0 \) is an algebraic manifold, \( \mathcal{O}' \), which is closed so when we take the closure of the orbit \( G^C_v \), we stay in the manifold \( \mathcal{O}' \), however on a different orbit.
In conclusion we can say that for supersymmetry to be unbroken, the vacuum must be such that its complex isotropy group is of the form $H^c$. This, in turn implies that the doubling of Goldstone fields occurs whenever supersymmetry is preserved in the spontaneous gauge symmetry breaking.

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