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# LINEAR MHD EQUILIBRIA - EXACT AND APPROXIMATE SOLUTIONS

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## ABSTRACT

The linear Grad-Shafranov equation for a toroidal, axi-symmetric plasma is solved analytically. Exact solutions are given in terms of confluent hyper-geometric functions. As an alternative, simple and accurate WKBJ solutions are presented. With parabolic pressure profiles, both hollow and peaked toroidal current density profiles are obtained. As an example the equilibrium of a z-pinch with a square-shaped cross section is derived.

## I. INTRODUCTION

The problem of finding analytic solutions to the ideal MHD equilibrium equation has engaged many authors, see e.g. Refs. [1-10]. Most attempts on toroidal, axisymmetric plasmas have been made on the linear problem, which poses certain constraints on the pressure and current profiles. These constraints still allow for several classes of physically relevant equilibrium solutions.

For mathematical simplicity, authors have in many cases not considered important terms of the profiles, thereby yielding somewhat artificial solutions. The Hill's vortex solution e.g. assumes the simple current density profile dependence $j_\varphi = c_1 r + c_2 r^{-1}$, where $r, \varphi$ are the usual cylinder coordinates and $c_1$, $c_2$ are constants. As will be discussed later, these types of solutions are unsatisfactory from the MHD stability point of view.

This paper presents exact solutions to the full linear problem. The physical relevance of these solutions, however, is accomplished at the expence of simplicity. Consequently approximate solutions become desirable. The WKBJ technique is excellently suitable for this particular problem, as will be demonstrated. The WKBJ solutions computed in this paper are accurate, closed and fairly simple, whereas the exact solutions are given as infinite series.

The equilibrium equations are formulated in Section 2 and the exact solutions are given. The WKBJ approximations are given in Section 3, and are also compared to the exact solutions. In Section 4 the obtained solutions are related to physical parameters. Section 5 contains an application to the z-pinch 'EXTRAP' [11], which features a purely poloidal magnetic field. Applications of one type of the exact solutions to Tokamak equilibria are found in Refs [1-6]. The MHD-stability of the solutions is discussed in Section 6. A few comments on analytical approaches to the non-linear problem are given in Section 7. Finally the paper is concluded with a discussion in Section 8.

## 2. EXACT SOLUTIONS TO THE LINEAR GRAD-SHAFRANOV EQUATION

### 2.1 Preliminaries

The fundamental equilibrium equations of ideal MHD theory

$$\nabla p = \bar{j} \times \bar{B}$$
$$\bar{j} = \nabla \times \bar{B} \tag{1}$$
$$\nabla \cdot \bar{B} = 0$$

together yield the equation [8]

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial \zeta^2} = - \mu_o \rho j_\varphi . \tag{2}$$

Here $\rho, \varphi$ and $\zeta$ are cylinder-coordinates. The scalar poloidal flux function $\psi$ is related to the poloidal magnetic field $\bar{B}_p$ via $\bar{B}_p = (B_r, 0, B_z) = \nabla \times \bar{A}_\varphi = \nabla \times (0, \psi/\rho, 0)$. Assuming the total magnetic field to be composed by both a poloidal and a toroidal field component; $\bar{B} = \bar{B}_p + \bar{B}_T$, we find for the R.H. side of Eq. (2)

$$\mu_o \rho j_\varphi = \mu_o \rho^2 \frac{dp}{d\psi} + \frac{1}{2}\frac{d\chi^2}{d\psi} \tag{3}$$

where the pressure $p$ and the toroidal field function $\chi \equiv \rho B_\varphi = \rho |\bar{B}_T|$ are functions of $\psi$ only. Equations (2) and (3) together constitute the axisymmetric Grad-Shafranov equation.

It is convenient to introduce the transformations

$$r = \frac{\rho}{\rho_o}$$
$$z = \frac{\zeta}{\rho_o} \qquad (4)$$

with $\rho_o$ as the radial distance to the geometric centre of the plasma cross section, see Fig.1. Eqs. (2)-(4) then yield

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = -\rho_o^2\left[\mu_o\rho_o^2 r^2 \frac{dp}{d\psi} + \frac{1}{2}\frac{d\chi^2}{d\psi}\right] . \qquad (5)$$

For the general linear Grad-Shafranov equation we assume $p(\psi) \equiv a + b\psi + c\psi^2$ and $\chi^2 \equiv d + e\psi + f\psi^2$ with a,b,c,d,e and f as arbitrary constants. Before proceeding to the solution of Eq. (5) we shall discuss the possible forms of p and $\chi^2$ in some detail.

First we note that the constants a and d do not enter the equilibrium equation (5) but appear only when determining the boundary conditions. With $c = f = 0$ the equation is greatly simplified and Hill's vortex-like solutions are obtained [8,9,10]. In the remainder of this paper we shall assume $c \neq 0$. If $be \neq 0$ the equation (5) becomes non-homogeneous, i.e. the general solution $\psi$ will consist both of a solution $\psi_h$ to the homogeneous equation and of a particular solution $\psi_p$; $\psi = \psi_h + \psi_p$.

Adopting the conditions (i); $p(\psi_b) = 0$ and (ii); $(dp/d\psi)_{\psi_b} = 0$, where $\psi_b$ is the value of $\psi$ at the plasma boundary, we are led to the following constraints on a,b and c. Condition (ii) yields $\psi_b = -b/2c$, which inserted into (i) gives the relation $b^2 = 4ac$. Writing $\psi_b = \pm\sqrt{a/c}$ there result two cases: $c > 0$, $a \geq 0$ and $c < 0$, $a \leq 0$. The first case yields $p(\psi) = (\sqrt{a} + \sqrt{c}\psi)^2$ and $\psi_b = -\sqrt{a/c}$, which has been treated in Refs. [1,2]. The second case results in $p(\psi) \leqslant 0$ and must be discarded from physical reasons. However, maintaining only condition (i) with $c < 0$ three distinct forms of the pressure profile results; $a < 0$, $b \geqslant 2\sqrt{ac}$; $a = 0$, $b > 0$ and $a > 0$, $b \geqslant 0$. In this Section the exact solution for arbitrary $c$ with the proper conditions on $a$ and $b$ is derived in a somewhat different fashion than that of Refs [1-2].

Renaming the constants $a - f$ we obtain the following forms for $p$ and $\chi^2$;

$$p(\psi) = p_o + \frac{p_1}{\rho_o^4}\psi + \frac{g}{2\rho_o^4}\psi^2 \tag{6}$$

$$\chi^2(\psi) = \chi_o^2 + \frac{2\chi_1^2}{\rho_o^2}\psi + \frac{h}{\rho_o^2}\psi^2 \tag{7}$$

which yield a convenient form of Eq. (5);

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = -(\mu_o g r^2 + h)\psi - (\mu_o p_1 r^2 + \chi_1^2) \tag{8}$$

or

$$D\psi = -(\mu_o p_1 r^2 + \chi_1^2) \tag{9}$$

$$D \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + (\mu_o g r^2 + h) \ .$$

A particular solution $\psi_p$ to this linear, non-homogeneous partial differential equation is found by inspection to be

$$\psi_p = -\frac{p_1}{g} \tag{10}$$

if the constants are chosen such that the relation

$$\chi_1^2 = \frac{hp_1}{g} \tag{11}$$

holds. Apparently the constants $p_1$ and $\chi_1^2$ are consistent with the fact that a solution is still obtained if an arbitrary constant is added to $\psi$.

With condition (11) as the only constraint on $\chi^2$, we obtain from Eqs. (3), (6), (7) and (10)

$$p = p_c + \frac{g}{2\rho_o^4}\psi_h^2 \tag{12}$$

$$\chi^2 = \chi_c^2 + \frac{h}{\rho_o^2}\psi_h^2 \tag{13}$$

$$j_\varphi = \frac{1}{\mu_o\rho_o^3}\left[\mu_o g r + h r^{-1}\right]\psi_h \tag{14}$$

$$\bar{B}_p = (B_\rho, 0, B_\zeta) = \frac{1}{\rho_o^2 r}\left(-\frac{\partial\psi_h}{\partial z}, 0, \frac{\partial\psi_h}{\partial r}\right) \tag{15}$$

$$B_T = B_\varphi = \frac{1}{\rho_o r}\chi , \tag{16}$$

where $p_c \equiv p_o - p_1^2/(2\rho_o^4 g)$ and $\chi_c^2 \equiv \chi_o^2 - hp_1^2/(\rho_o^2 g^2)$.

We now turn to solutions of the homogeneous equation

$$D\psi_h = 0 . \tag{17}$$

Separation of variables

$$\psi_h = R(r)Z(z) \tag{18}$$

and the transformation

$$R = yr^{1/2} \tag{19}$$

yield

$$Z'' \pm k^2 Z = 0 \tag{20}$$

$$y'' + (-\frac{3}{4} r^{-2} + \mu_o g r^2 + h \mp k^2)y = 0 \tag{21}$$

where the upper or lower signs are to be taken simultaneously. Since we are interested in solutions symmetric with respect to the $z = 0$ plane, we find the two distinct solutions

$$Z_+ = \cos kz \tag{22}$$

$$Z_- = \cosh kz \tag{23}$$

where the indices refer to the signs of Eq. (20).

The form of the corresponding solutions $y_+$ and $y_-$ to Eq. (21) depend on the sign of $g$. Choosing $g = -g$, with $g_1$ being a positive constant, a special case of the general confluent hypergeometric differential equation [12,13] is obtained. We observe that Eqs. (12) and (14) then yield hollow current density profiles. Lehnert [8] has considered this case previously, but with $p_1 = \chi_0 = \chi_1 = h = 0$. Section 2.2 of this paper presents the general solution.

With $g = +g_1$ the equilibrium equation (21) may be transformed into the Coulomb Wave equation, which case with $p_1=\chi_1=0$ has been treated independently by Maschke [1] and Hernegger [2]. The corresponding current profile becomes peaked in this case. An alternative derivation of this solution is given in Section 2.3.

## 2.2. Exact solution with $g = -g_1$

In the case $g = -g_1$ we obtain from Eqs. (19), (21) and from Refs [12,13] the two hypergeometric solutions

$$R_1 = e^{-\frac{\sqrt{\mu_o g_1}}{2} r^2} \sum_{m=0}^{\infty} \frac{(1-\kappa)_m}{(2)_m m!} (\sqrt{\mu_o g_1} r^2)^{m+1} \tag{24}$$

where

$$(x)_m = x(x+1)\ldots(x+m-1), \qquad (x)_o = 1$$

$$\kappa \equiv \frac{h^2 k^2}{4\sqrt{\mu_o g_1}}$$

and

$$R_2 = R_1 \ell n(\sqrt{\mu_o g_1} r^2) + H_{\kappa,1/2} \tag{25}$$

$$H_{\kappa,1/2} = e^{-\frac{\sqrt{\mu_o g_1}}{2} r^2} \left[ \sum_{m=0}^{\infty} \frac{(1-\kappa)_m}{(m+1)!} \frac{(\sqrt{\mu_o g_1} r^2)^{m+1}}{m!} \cdot \Lambda - \kappa^{-1} \right]$$

$$\Lambda \equiv \Omega(m+1-\kappa) - \Omega(m+1) - \Omega(m+2)$$

where

$$\Omega(x) = \frac{d}{dx} \ell n\Gamma(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+x}\right) \quad , \ x \neq \text{neg. integer}$$

so that, after some algebra,

$$\Lambda = \gamma + \frac{(m+1)(m+1-2\kappa)-\kappa}{(m+1)(m+2)(m+1-\kappa)} + \sum_{n=1}^{\infty} \frac{[(m+1)(m+2)-n^2]\kappa - (m+2)(m+n+1)^2}{n(m+n+1)(m+n+2)(m+n+1-\kappa)}$$

where $\Gamma$ denotes the usual gamma-function and $\gamma$ is the Euler constant. The upper sign of $\kappa$ refers to the $y_+$-solution, whereas the lower sign corresponds to the $y_-$-solution.

The sums of Eqs. (24) and (25) converges for most values of $r$ and $\kappa$, although convergence can be extremely slow for large values of $|1-\kappa|$ or $\sqrt{\mu_o g_1}$ (>> 100 terms). We also note that the logarithmic solution $R_2$ is singular at $r = 0$.

## 2.3 Exact solution with $g = +g_1$

Maschke [1] and Hernegger [2] give the solutions for the case $g = +g_1$ in the form of Coulomb Wave functions $F_o(\eta,\rho)$ and $G_o(\eta,\rho)$. The correspondence of these solutions to the hypergeometrical functions suggests, that to obtain simpler mathematics we just perform the substitution $\sqrt{\mu_o g_1} \rightarrow i\sqrt{\mu_o g_1}$ in Eqs. (24)-(25) taking the imaginary part, i.e. formally

$$R_1(+g_1) = \mathrm{Im}\left\{R_1(-g_1, \sqrt{\mu_o g_1} \rightarrow i\sqrt{\mu_o g_1})\right\} \tag{26}$$

$$R_2(+g_1) = \mathrm{Im}\left\{R_2(-g_1, \sqrt{\mu_o g_1} \rightarrow i\sqrt{\mu_o g_1})\right\}. \tag{27}$$

## 2.4. General solution

The general solution in $\psi$ to the linear, non-homogeneous Grad - Shafranov equation as defined by Eqs. (1)-(11) becomes

$$\psi = \psi_p + \psi_h = -\frac{p_1}{g} + \sum_{n=0}^{\infty} \left[A_n R_1(k_n, r) + B_n R_2(k_n, r)\right] \begin{cases} \cos & k_n z \\ \cosh & k_n z \end{cases} \tag{28}$$

where the z-dependence is to be taken in accordance with Eqs. (22)-(23).

## 3. WKBJ SOLUTIONS

Since the solutions (24)-(27) are not closed and convergence is poor for cases of interest, it becomes natural to investigate whether or not accurate approximate solutions exist. The WKBJ technique has proven to be very well suited for this problem. In short, the idea is that approximate solutions to the equations

$$y'' - q^2(x)y = 0 \tag{29}$$

$$y'' + q^2(x)y = 0 \tag{30}$$

are given by

$$\tilde{y} = q^{-1/2}\exp\left[\pm \int q\,dx\right] \tag{29'}$$

$$\tilde{y} = q^{-1/2}\exp\left[\pm i\int q\,dx\right], \tag{30'}$$

respectively. The solutions become accurate when the arbitrary function $q(x)$ varies only slightly in the interval of interest, and they are exact if $q(x) = (c_1 + c_2x)^{-2}$, where $c_1$ and $c_2$ are constants.

In this paper the function $q$ is defined by Eq. (21). It is satisfying to note that the integrals of Eqs. (29')-(30') may be exactly evaluated for the present q-function. The following cases are to be distinguished:

For $g = -g_1$ Eqs. (21), (29) and (29')yield the two independent solutions, denoted by $(\pm)$ (the $(\pm)$ signs also correspond to the exact solutions of Eqs. (24) and (25) respectively);

$$R(r) = r^{1(\pm)\frac{\sqrt{3}}{2}} s^{-1/2} \left[2\sqrt{\mu_o g_1}s + 2\mu_o g_1 r^2 + (\pm k^2-h)\right]^{-(\pm)\frac{\pm k^2-h}{4\sqrt{\mu_o g_1}}} \times$$

$$\times \left[\sqrt{3}s + (\pm k^2 - h)r^2 + \frac{3}{2}\right]^{(\mp)\frac{\sqrt{3}}{4}} e^{(\pm)\frac{s}{2}} \tag{31}$$

where

$$s \equiv \sqrt{\mu_o g_1 r^4 + (\pm k^2-h)r^2 + \frac{3}{4}} \quad . \tag{32}$$

The notation $\pm k^2$ still refers to the previously discussed two solutions $y_+$ and $y_-$.

For $g = +g_1$ we define a parameter $\alpha$,

$$\alpha \equiv \frac{1}{2\mu_o g_1} \left[\pm k^2-h + \sqrt{(\pm k^2-h)^2 + 3\mu_o g_1}\right] \tag{33}$$

As a consequence $q = 0$ for $r^2 = \alpha$, i.e. the WKBJ solutions become singular at this point. The singularity lies not in Eq. (21); it is simply an inherent limitation of the WKBJ approximation. When $r^2 < \alpha$ the two WKBJ solutions become (Eqs (29-(29'));

$$R(r) = r^{1(\pm)\frac{\sqrt{3}}{2}} t^{-\frac{1}{2}} \exp\left\{(\mp)\left[\frac{\pm k^2-h}{4\sqrt{\mu_o g_1}}\right]\arcsin\left[\frac{-2\mu_o g_1 r^2+(\pm k^2-h)}{\sqrt{(\pm k^2-h)^2+3\mu_o g_1}}\right]\right\}\times$$

$$\times \left[\sqrt{3}t + (\pm k^2-h)r^2 + \frac{3}{2}\right]^{(\mp)\frac{\sqrt{3}}{4}} e^{(\pm)\frac{t}{2}} \tag{34}$$

where

$$t \equiv \sqrt{-\mu_o g_1 r^4 + (\pm k^2 - h) r^2 + \frac{3}{4}} \quad . \tag{35}$$

For $r^2 > \alpha$ we find (Eqs. (30)-(30'))

$$R(r) = r u^{-1/2} \cdot \begin{cases} \sin w & ; \quad R_1\text{-solution} \\ \cos w & ; \quad R_2\text{-solution} \end{cases} \tag{36}$$

where

$$u \equiv \sqrt{\mu_o g_1 r^4 - (\pm k^2 - h) r^2 - \frac{3}{4}} \tag{37}$$

$$w = u/2 - \left(\frac{\pm k^2 - h}{4\sqrt{\mu_o g_1}}\right) \ln\left[ 2\sqrt{\mu_o g_1}\, u + 2\mu_o g_1 r^2 - (\pm k^2 - h)\right] +$$

$$+ \frac{\sqrt{3}}{4} \arcsin\left[ \frac{(\pm k^2 - h) r^2 + \frac{3}{2}}{r^2 \sqrt{(\pm k^2 - h)^2 + 3\mu_o g_1}} \right] .$$

However, not all values of the parameters $g_1$, h and $\pm k^2$ are permitted. With

$$\beta \equiv -\left[\pm k^2 - h + \frac{3}{4}\right]$$
$$\gamma \equiv \frac{1}{3}\left[\pm k^2 - h\right]^2 \qquad (38)$$

it is found from Eqs. (31)-(32) that e.g. for $r = 1$ the condition $\mu_0 g_1 > \min(\beta,\gamma)$ must be fulfilled for $g = -g_1$.

It is instructive to compare some limiting cases. With $(h \mp k^2)$ as dominating term in Eq. (21) we obtain the two solution pairs $Z=\cos kz$; $y=\cosh\sqrt{h\mp k^2}\,r$, $\sinh\sqrt{h\mp k^2}\,r$ and $Z=\cosh kz$; $y=\cos\sqrt{h\mp k^2}\,r$, $\sin\sqrt{h\mp k^2}\,r$. This is in agreement with the exact solutions of Eqs.(24)-(27), and with the approximate solutions of Eqs.(31)-(37). As an illustration it may be mentioned that for a solution to the linear MHD equilibrium equation it is necessary that $k > 0.45\,\sigma_c/\zeta_b$, where $\zeta_b$ is the $\zeta$-distance to the boundary, in order to obtain at least a 10% variation in $Z_+$ or $Z_-$. Also, if $|(h \mp k^2)/(\mu_0 g r^2 - 0.75 r^{-2})| \gg 1$ the y-solutions approach the simple hyperbolic or trigonometric solutions just mentioned. In the case $k^2$, $|\mu_0 g| \gg 1$ Eq. (20) becomes the Weber differential equation, for which solutions are given in Appendix.

With $|(h \mp k^2)/(\mu_0 g r^2 - 0.75 r^{-2})| \ll 1$ we obtain directly from Eq. (21) the cylinder solutions $Z = \text{const.}$ and $R \approx \exp[(\mp\sqrt{\mu_0 g}/2)r^2]$ for all $g$. This special case is also in agreement with the obtained solutions.

In Fig. 2 is displayed a numerical comparison of the exact and WKBJ solutions for various values of the parameters $k$ and $g$. As an illustration a case with a singularity, where the WKBJ approximation fails, is included.

## 4. THE SOLUTION RELATED TO PHYSICAL PARAMETERS

For applications, it is advantageous to rewrite Eq. (12);

$$p(\psi_h) = p_c \pm \frac{g_1}{2\rho_o^4}\,\psi_h^2 \equiv p_a \mp (p_a - p_b)\,\frac{\psi_h^2 - \psi_{ha}^2}{\psi_{hb}^2 - \psi_{ha}^2} \tag{39}$$

where $p_a$ and $p_b$ are the scalar plasma pressures at the magnetic axis and at the boundary, respectively. Eq. (39) may also be interpreted as a definition of the physical meaning of the parameter $g$.

From Eqs. (13) and (16) we arrive at

$$h = \rho_o^4\,\frac{r_a^2 B_{Ta}^2 - r_i^2 B_{Ti}^2}{\psi_{ha}^2 - \psi_{hb}^2} \quad . \tag{40}$$

Here $B_{Ta}$ and $B_{Ti}$ are the toroidal magnetic field components at the axis and at the inner plasma boundary, respectively. It has been shown [5] that for a case with $p_b = \psi_{hb} = 0$ the plasma is paramagnetic for $h > 0$, and diamagnetic for $h < 0$. Assuming $B_T \sim r^{-\alpha}$ we note that $h = 0$ for $\alpha = 1$. Generally, since $r_a > r_i$, the sign of $h$ is determined by the sign of the denominator in Eq. (40). The solutions of this paper consequently yield $h > 0$ for $g = +g_1$ and $h < 0$ for $g = -g_1$.

On inserting the expressions for $g$ and $h$ from Eqs. (39)–(40) into Eq. (14), we find for the toroidal current density

$$j_\varphi = \frac{\rho_o \mu_o^{-1}}{\psi_{ha}^2 - \psi_{hb}^2} \left[ 2\mu_o (p_a - p_b) r + (r_a^2 B_{Ta}^2 - r_i^2 B_{Ti}^2) r^{-1} \right] \psi_h. \quad (41)$$

A convenient definition of the poloidal beta is given by

$$\beta_p \equiv \frac{2\mu_o p_a}{\langle B_z \rangle^2} \quad (42)$$

with $\langle B_z \rangle$ as the average value of the vertical magnetic field component $B_z$ in the median plane $z = 0$, between $r = r_i$ and $r = r_a$. $\langle B_z \rangle$ is computed from

$$B_z = \frac{1}{\rho_o^2 r} \frac{\partial \psi_h}{\partial r}$$

$$\langle B_z \rangle = \int_{r_i}^{r_a} B_z r dr \Big/ \int_{r_i}^{r_a} r dr \quad (43)$$

assuming no current reversal within the plasma region. We obtain

$$\beta_p = \frac{\mu_o \rho_o^4 p_a (r_a^2 - r_i^2)^2}{2(\psi_{ha} - \psi_{hb})^2} \quad . \quad (44)$$

## 5. Z-PINCH EQUILIBRIUM

As a particular example on an application of the WKBJ solutions we choose the EXTRAP z-pinch, which features a purely poloidal magnetic field, yielding $\chi^2 \equiv 0$ in this case. The pressure and current profiles (39) and (41) become simply

$$p(\psi_h) = p_a(\psi_h/\psi_{ha})^2 \tag{45}$$

$$j_\varphi(r,\psi_h) = \frac{2\rho_o p_a}{\psi_{ha}^2} r\psi_h \tag{46}$$

if we assume $g = +g_1$ (i.e. peaked profiles), $p_b = 0$ and $\psi_{hb} = 0$. It is, in fact, possible to derive an equilibrium flux function $\psi_h$, which suits the proper boundary conditions, with the $n = 0$ coefficients of Eq. (28) only. We will now follow a procedure of Galvão [4] and outline the derivation of the equilibrium.

One type of EXTRAP equilibria is characterized by a square plasma cross section with corners located at regions of vanishing magnetic field [11]. At these stagnation points we must have

$$\frac{\partial\psi_h}{\partial r} = 0$$
$$\frac{\partial\psi_h}{\partial z} = 0 \,, \tag{47}$$

where

$$\psi_h = \left[AR_1 + BR_2\right] \cos kz \quad . \tag{48}$$

Defining

$$\delta \equiv \frac{\partial^2\psi_h}{\partial r^2}\frac{\partial^2\psi_h}{\partial z^2} - \left[\frac{\partial^2\psi_h}{\partial r\partial z}\right]^2 \tag{49}$$

the stagnation point is elliptic if $\delta > 0$ and hyperbolic if $\delta < 0$. It is easily demonstrated that the corner stagnation points must be hyperbolic, and for these the solution to Eq. (17) becomes

$$\zeta = (2n + 1)\,\frac{\pi}{2}\,\frac{\rho_o}{k_n}\,, \quad n = 0,1,2... \tag{50}$$

$$R_1 + (B/A)R_2 = 0\ . \tag{51}$$

With the four corners located at $(\rho,\zeta) = (\rho_o \pm a, \pm a)$, $k_n$ is determined from Eq. (50). We here choose $n = 0$, avoiding current inversion inside the plasma boundary, and let $k_o \equiv k$. The parameters $g_1$ and $B/A$ are then obtained from the eigenvalue equation

$$\frac{R_1(\rho_o - a)}{R_2(\rho_o - a)} = \frac{R_1(\rho_o + a)}{R_2(\rho_o + a)} = -\,B/A\ . \tag{52}$$

The magnetic axis $r = r_a$, finally, corresponds to an elliptic stagnation point $(\delta > 0)$ and is obtained via

$$\zeta = m\pi \frac{\rho_o}{k} \quad , \; m = 0,1,2\ldots$$

$$\left[ \frac{dR_1}{dr} + B/A \frac{dR_2}{dr} \right]_{r=r_a} = 0 \quad , \tag{53}$$

where we choose $m = 0$ to position the magnetic axis in the midplane, and from the necessary condition

$$\mu_o g_1 r_a^2 \geqslant k^2 \; . \tag{54}$$

The latter condition is a mathematical consequence of the behaviour of the confluent hypergeometric functions [3,13].

In Fig.3 the flux plot and profiles of an equilibrium with $\rho_o = 0.45$ m and $a = 0.05$ m are shown. We have here used the WKBJ solutions (36)-(37). For this case $k^2 = 200.0$, $\sqrt{\mu_o g_1} = 20.06$ and $B/A = 3.353$. We find $r_a = 1.01$.

For the physical parameters we choose $p_a = 2 \cdot 10^4$ Nm$^{-2}$ and adjust the parameter A in Eq. (48) so that $\beta_p = 1.0$. This leaves no free parameters in the solution, and the labelling of the ordinates in Fig.3 thereby becomes absolute.

It is not clear, though, at what extent ideal MHD theory is applicable for EXTRAP, due to the extended magnetic field-null center region of the plasma.

The above equilibrium derivation holds good also for z-pinches with a finite toroidal magnetic field, or for Tokamaks e.g., in which cases the substitution $k^2 \rightarrow k^2 - h$ is performed.

## 6. STABILITY

The unstable MHD-spectrum of EXTRAP equilibria has been investigated by Dalhed and Hellsten [14], using the GATO code. With the pressure profile given by $p(\psi) = p_o(\psi - \psi_b)^\alpha$ D-shaped, square or inverse D-shaped equilibria are obtainable due to variations of the external field. Only the square-shaped equilibria proved stable against axisymmetric perturbations.

For zero toroidal mode number axisymmetric modes are stable provided $\alpha > 2$. For $\alpha < 2$ a radial and a vertical unstable mode appears. This is due to the shape of the current profile at the boundary; in the large aspect ratio limit we have approximately $dj_\varphi/d\psi \sim d^2p/d\psi^2 \sim (\psi - \psi_b)^{\alpha-2}$. Consequently the current density has an edge at the boundary for $\alpha < 2$. In this paper we have $\alpha = 2$, indicating that the solutions are marginally stable to axisymmetric perturbations.

When the toroidal mode number exceeds zero two gross kink modes appear for $\alpha < 2$. The sausage or interchange modes are stabilized by a sufficiently small pressure gradient. The free-boundary modes are stabilized when $\alpha > 2$. Internal kink modes have large growth rates for large mode numbers. They are not observed and are probably stabilized by FLR effects.

It is obvious that Hill's vortex-like solutions ($\alpha = 1$) are unsatisfactory. The marginal MHD-stability of the present solutions represent a first step towards an analytical understanding of experimentally obtained equilibria.

Discussions of the stability of Tokamak equilibria with different forms on $p(\psi)$ and $\chi^2(\psi)$ are found in Refs [15-17]. It is concluded that the $\alpha = 2$ solutions become unstable to internal modes for $\beta_p = 1$ and an average $\beta$ of about 2%. Stability is improved, maintaining $\beta$ at about 12 %, by using forms $p(\psi) = c_0 + c_1\psi^2 + c_2\psi^3$ and $\chi^2(\psi) = c_3\psi + c_4\psi^3 + c_5\psi^5$, where the $c_i$:s are constants.

## 7. ON NON-LINEAR METHODS

In Eq. (2) the $\zeta$-dependence is implicit. To treat arbitrary pressure profiles it is suggestive to somehow transform the resulting non-linear equation into an ordinary differential equation in $\rho$.

By integrating Eq. (2) with respect to $\zeta$ and by taking the Laplace transform, we find

$$- s^2\tilde{\psi} + s\psi(\rho,o) = \tilde{\psi}_{\rho\rho} - \frac{1}{\rho}\tilde{\psi}_\rho + \mu_o\rho^2\tilde{f}_1 + \tilde{f}_2 \qquad (55)$$

where the transform is denoted by $\sim$, $s$ is the transform parameter, index $\rho$ denotes differentiation with respect to $\rho$, $f_1 \equiv dp/d\psi$ and $f_2 \equiv d\chi^2/d\psi$. With $p$ and $\chi^2$ given as power expansions in $\psi$, the procedure to solve Eq. (55) may be as follows.

First $\psi(\rho,o)$ is specified. Incidentally, this condition is similar as for free-boundary computer codes. Having transformed $f_1$ and $f_2$, an ordinary but non-linear differential equation for $\tilde{\psi}$ in $\rho$ results, with $s$ as a parameter. This equation is then solved and the solution $\psi(r,z)$ is obtained via the inverse Laplace transform.

The difficulty with this method lies in transforming $\psi^n$, with n a positive integer, and it is evident that further studies are necessary for determining the applicability of this formal method.

Indeed, after completion of the present Section, it was brought to the author's attention the work of Mazzucato [6], who successfully used a Laplace-transform method for the case $p(\psi) = c_0 + c_1\psi$, $\chi^2 = c_2 + c_3\psi + c_4\psi^2$ with the $c_i$:s being constants.

## 7. DISCUSSION

Toroidal equilibria with suitable boundary conditions are generally obtained numerically with computer codes. This is time-consuming due to the nature of the problem. Lately a semi-analytical approach using variational moments [19] has been developed, reducing the computer time by a factor 10. This method is mathematically quite complex, though.

Naturally analytical methods are preferable. However, they are fully developed for linear equilibria only. Therefore it becomes crucial to determine which devices and what parameter regimes are correctly described by theory, as presented in this paper. For this task equilibrium codes including complete transport models are unavailable.

The solutions presented here naturally needs some numerical effort to adapt to the boundary conditions. Furthermore the exact logarithmic solution $R_2$ is quite involved, but may on the other hand be neglected, choosing $\psi(0,z)$ finite. The exact $R_1$ solutions are simple in cases when only a few terms of the infinite series are required. For large $|\pm k^2 - h|$ values e.g. many terms are needed. As has been discussed in this paper, there are some limiting cases where indeed simple solutions are obtained, but to cover a wide range of parameters $g,h$ and $\pm k^2$ the WKBJ solutions are clearly preferable, assuming the singularity at $q = 0$ is avoided.

The solutions are marginally MHD-stable and should therefore be helpful in studying basic features of equilibria.

ACKNOWLEDGEMENTS

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APPENDIX

The first term within the parenthesis of Eq. (21) is of order unity. When this term becomes negligible, i.e. when $|3/4r^{-2}| << |\mu_o gr^2 + h \mp k^2|$, Eq. (21) becomes the Weber equation [18] for $g = -g_1$. If $g = +g_1$ the solution may be given in terms of Bessel wave equations.

(i) $g = -g_1$.

The general solution in this case is a linear combination of the two Weber functions [18];

$$y(r) = AW_e(p,qr) + BW_o(p,qr) \tag{A1}$$

$$p \equiv (\pm) \left[ \frac{h \mp k^2}{2\sqrt{\mu_o g_1}} \right] - \frac{1}{2}$$

$$q \equiv (\mp)\sqrt{2}\ (\mu_o g_1)^{1/4} ,$$

where A and B are arbitrary constants. The parameter p should be chosen positive and the positive or the negative signs are to be taken simultaneously. $W_e$ *is an even function and* $W_o$ is odd. The Weber functions are generally absolute convergent, infinite series. However, the series terminate for certain values of p; with p being an even integer, we obtain

$$W_e(p,qr) = (-2)^{p/2} \frac{(p/2)!}{p!} e^{-\frac{(qr)^2}{4}} H_p(qr) \tag{A2}$$

and for p being an odd integer

$$W_o(p,qr)=(-2)^{(p-1)/2}\,\frac{[(p-1)/2]!}{p!}\,e^{-\frac{(qr)^2}{4}}\,H_p(qr). \tag{A3}$$

$H_p(qr)$ are Hermite polynomials;

$H_0 = 1$

$H_1 = qr$

$H_2 = (qr)^2 - 1$

$H_3 = (qr)^3 - 3(qr)$

$H_4 = (qr)^4 - 6(qr)^2 + 3$

$H_5 = (qr)^5 - 10(qr)^3 + 15(qr)$

$H_6 = (qr)^6 - 15(qr)^4 + 45(qr)^2 - 15, \ldots$

Obviously the solutions (A1) - (A3) are mathematically simple.

(ii) $g = +g_1$.

With $q = (\pm)\sqrt{h \mp k^2}$ and $\chi = (\pm)\sqrt{\mu_o g_1}$, the solution to the modified Eq. (21) becomes

$$y = \sqrt{r}\left[A\phi_{1/2}(\chi,qr) + B\phi_{-1/2}(\chi,qr)\right]. \tag{A4}$$

For parameters of interest the Bessel wave functions $\phi$ are infinite series, and the solution (A4) is not less complex than either of the solutions (26)-(27) or (33)-(37).

## FIGURE CAPTIONS

Fig.1. The geometry of the axially symmetric plasma.

Fig.2. Comparisons of the exact radial solutions $R_1^E$ and $R_2^E$ (Eqs. (24)-(25)) with the approximate WKBJ solutions $R_1^A$ and $R_2^A$ (Eqs. (31)-(37)) for various values of the parameters $g$ and $k$. For $g = +g_1$ and $(k^2, \mu_0 g_1) = (1,1)$ the singularity of the $R^A$-solutions at $r = 1.22$ is apparent. All solutions are normalized at $r = 1.0$.

Fig.3.a) Flux surface contours of the square-shaped z-pinch equilibrium discussed in Section 5. The solutions of Eqs. (36)-(37) are used with $\rho_o = 0.45$, $\zeta_b = 0.05$, $A = 0.00566$, $B/A = 3.353$, $g_1 = 3.202 \cdot 10^8$, $k = 14.14$, $p_a = 2.0 \cdot 10^4 \text{Nm}^{-2}$. Also plotted are the corresponding profiles in the median plane ($z = 0$) of:

b) Poloidal flux function $\psi$

c) Scalar plasma pressure $p$

d) Toroidal current density $j_\varphi$

for $\beta_p = 1$.

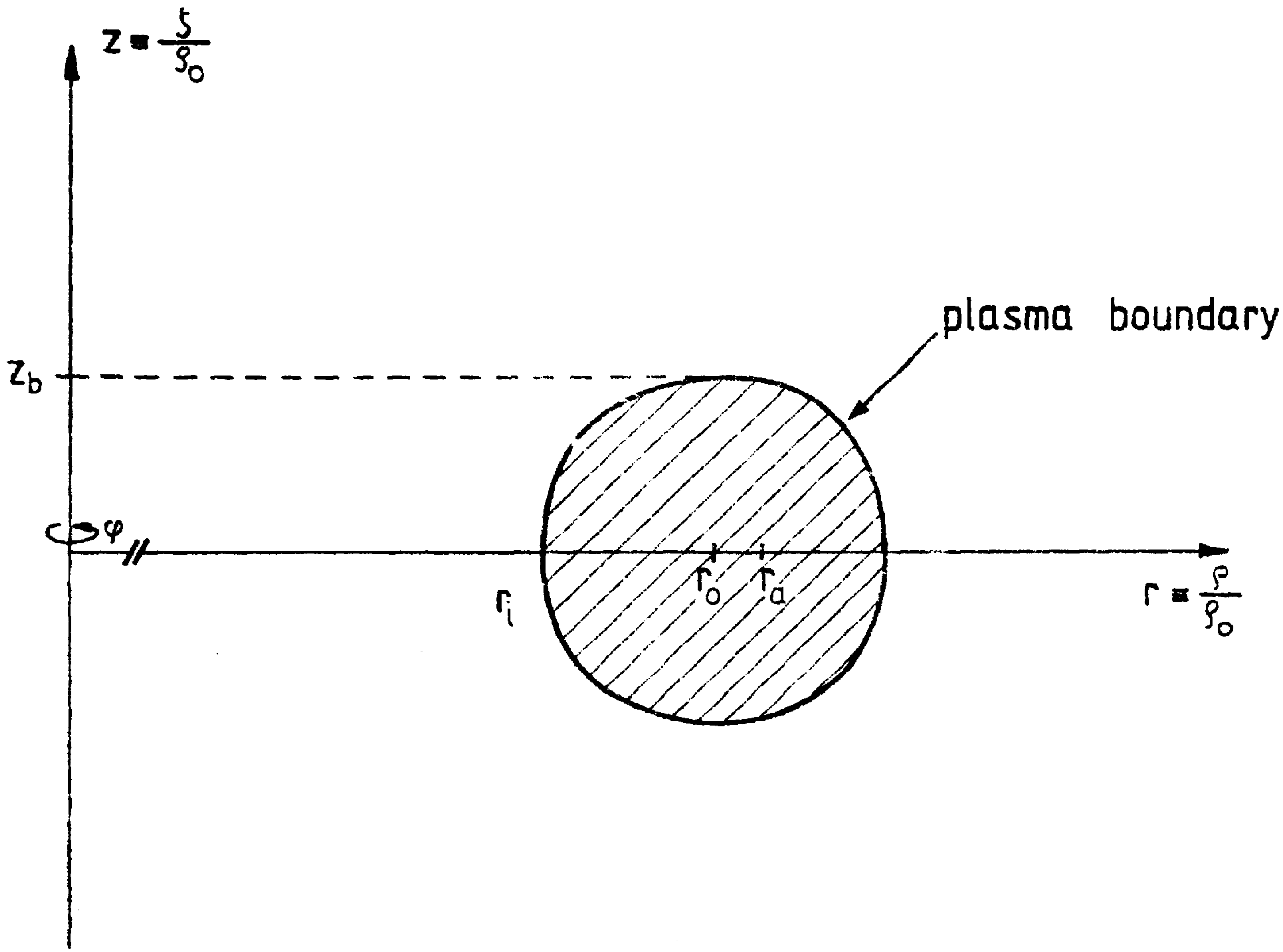



Fig.1

Fig. 2 a)

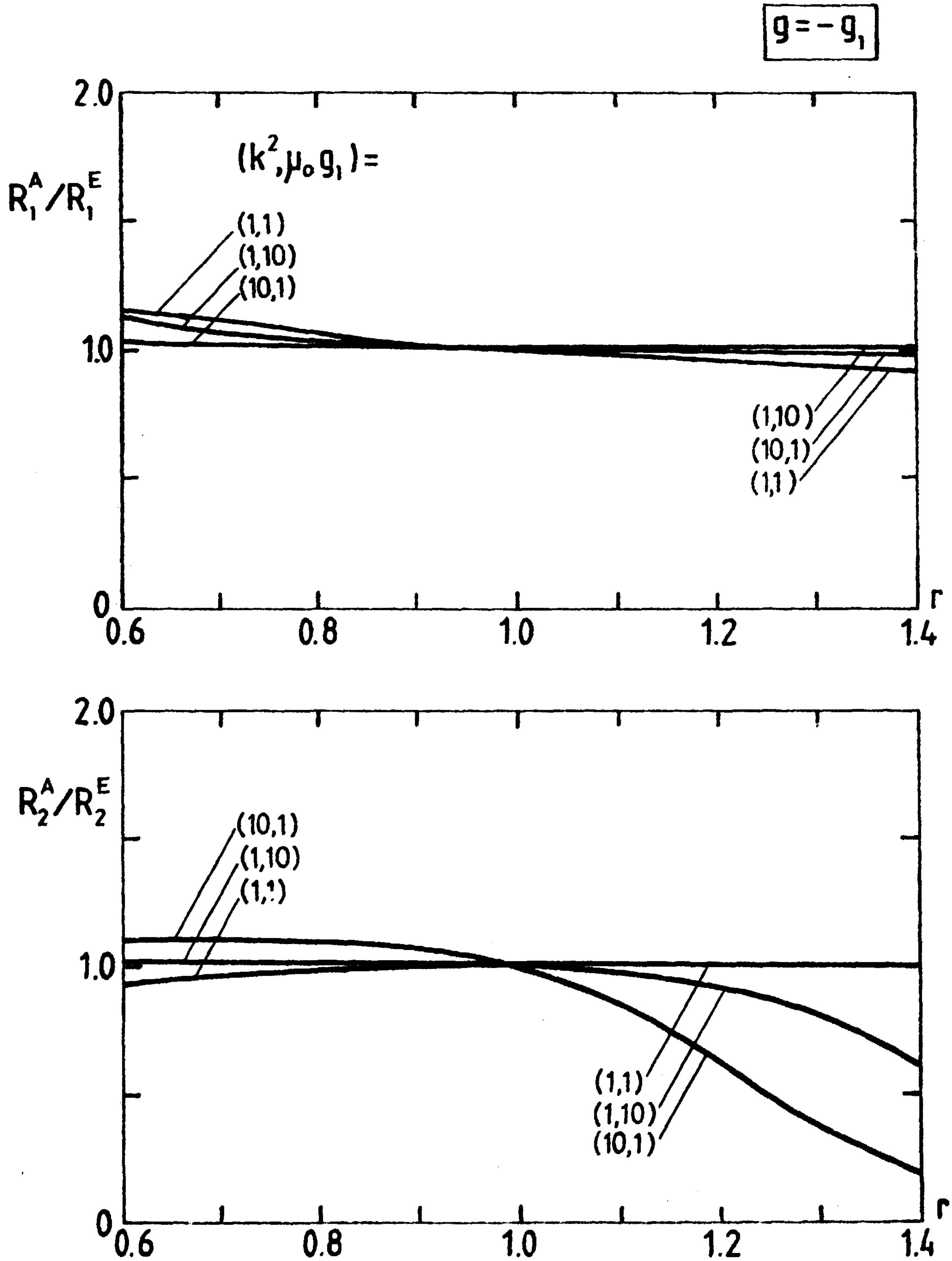

g = −g1
2.0
(k2, μo g1) =
R1A / R1E
(1,1)
(1,10)
(10,1)
1.0
(1,10)
(10,1)
(1,1)
0
0.6
0.8
1.0
1.2
1.4
r
2.0
R2A / R2E
(10,1)
(1,10)
(1,1)
1.0
(1,1)
(1,10)
(10,1)
0
0.6
0.8
1.0
1.2
1.4
r

Fig. 2 b)

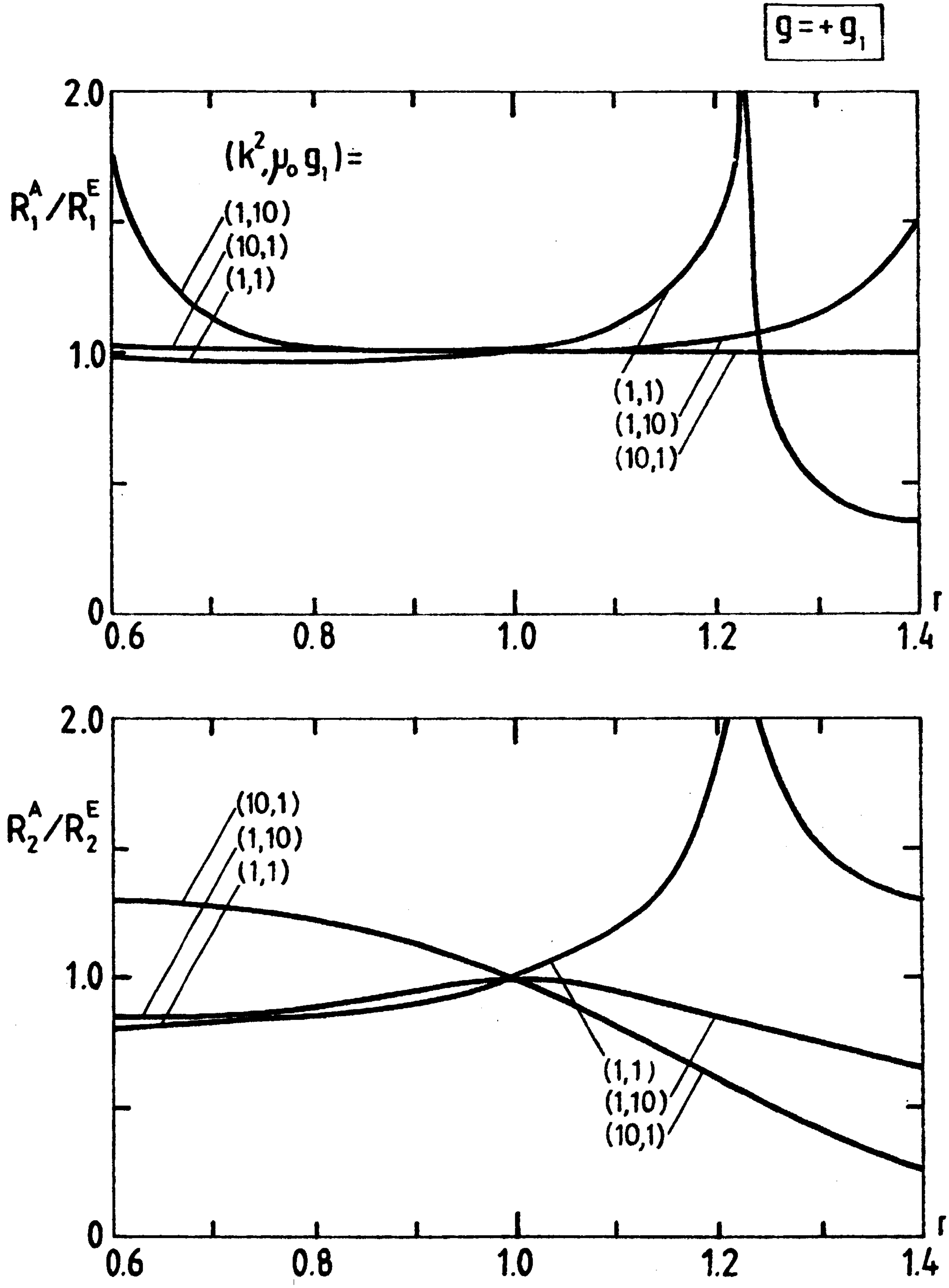
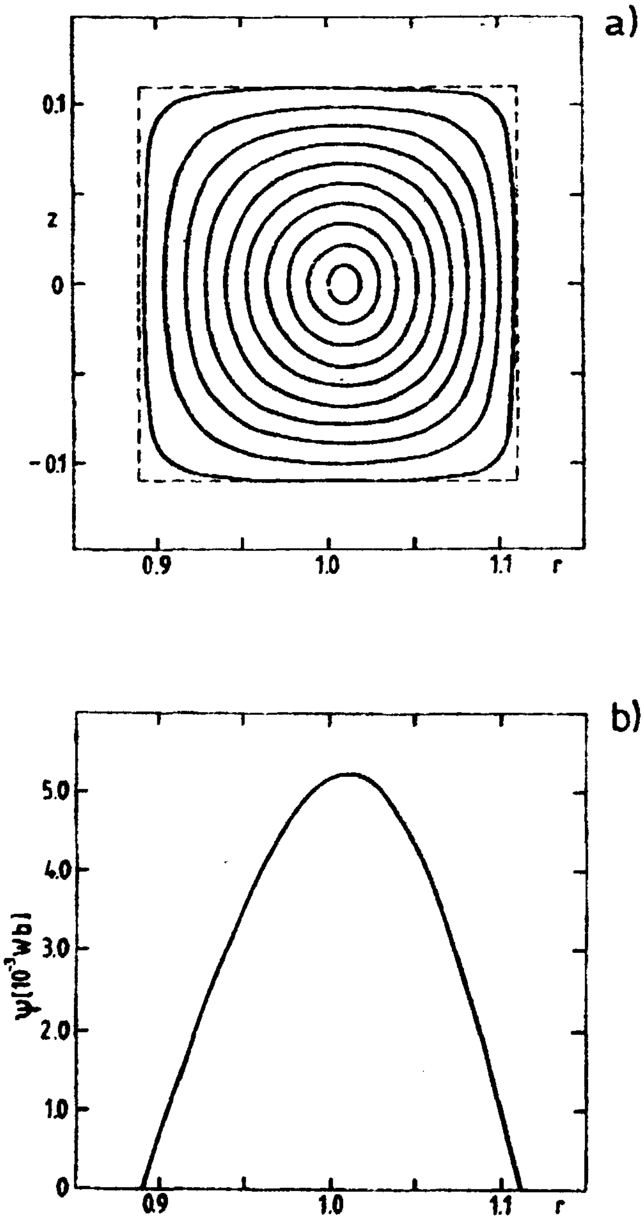
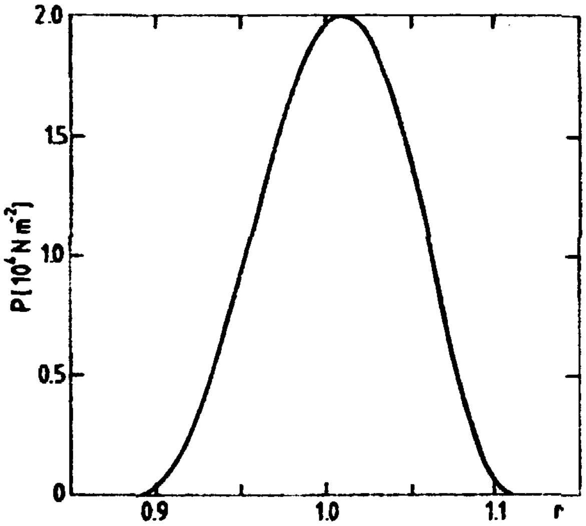
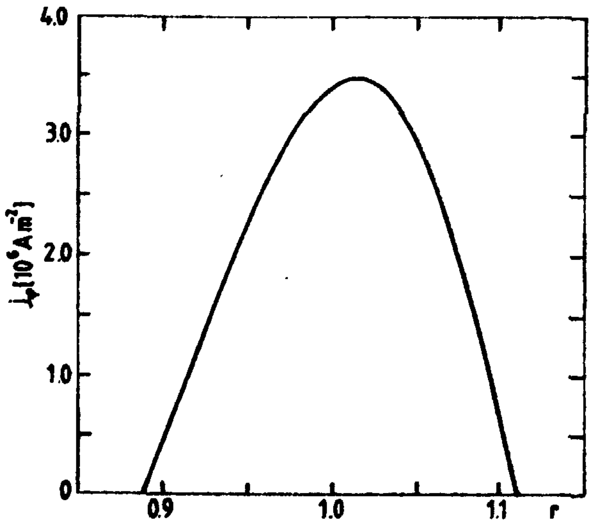

g = + g_1
2.0
1.0
0
0.6
0.8
1.0
1.2
1.4
r
R_1^A / R_1^E
(k², μ_0 g_1) =
(1,10)
(10,1)
(1,1)
(1,1)
(1,10)
(10,1)
2.0
1.0
0
0.6
0.8
1.0
1.2
1.4
r
R_2^A / R_2^E
(10,1)
(1,10)
(1,1)
(1,1)
(1,10)
(10,1)

Fig.3






c)




d)

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LINEAR MHD EQUILIBRIA - EXACT AND APPROXIMATE SOLUTIONS

J. Scheffel, March 1984, 30 p. in English

The linear Grad-Shafranov equation for a toroidal, axi-symmetric plasma is solved analytically. Exact solutions are given in terms of confluent hyper-geometric functions. As an alternative, simple and accurate WKBJ solutions are presented. With parabolic pressure profiles, both hollow and peaked toroidal current density profiles are obtained. As an example the equilibrium of a z-pinch with a square-shaped cross section is derived.

Key words: Equilibrium, MHD equilibrium, EXTRAP, Grad-Shafranov equation.