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ITO CALCULUS FOR σ-MODELS AND YANG-MILLS THEORIES

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Abstract : It is pointed out that the effective continuum action for σ-models and Yang-Mills theories may differ from the naive continuum action by terms of order g^2 or higher, which are non-symmetric. The modifications are produced by a generalization of the Ito calculus to dimensions higher than one.

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Consider a non-relativistic particle free to move on a sphere S^2 ($H=L^2/2MR^2$). It can be verified [1] that the Feynman-Kac formula for the propagator is given by

$$\langle \vec{n}_f, T \mid \vec{n}_0, 0 \rangle = \lim_{L\to\infty} \left(\frac{\beta L}{2\pi}\right)^L \times$$

$$\int \prod_{i=1}^{L-1} d\vec{n}_i \quad \exp\left[-\beta L \sum_{i=1}^{L} \frac{(\vec{n}_i - \vec{n}_{i-1})^2}{2}\right] \qquad (1)$$

$$\beta = \frac{MR^2}{kT} \quad , \quad \vec{n}_L = \vec{n}_f \; .$$

It has been known for many years that in spherical coordinates, the correct interpretation of eq.(1) is

$$\langle \Omega_f, T \mid \Omega_0, 0 \rangle = \lim_{L\to\infty} \left(\frac{\beta L}{2\pi}\right)^L \times$$

$$\int \prod_{i=1}^{L-1} d\Omega_i \quad \exp\left\{-\beta L \sum_{i=1}^{L}\left[\frac{(\theta_i - \theta_{i-1})^2}{2}\right.\right. \qquad (2)$$

$$\left.\left. + \sin\theta_i \sin\theta_{i-1} \frac{(\varphi_i - \varphi_{i-1})^2}{2}\right] + \frac{1}{8\beta L}\left(1 + \frac{1}{\sin\theta_i \sin\theta_{i-1}}\right)\right\} \; .$$

The last term in the exponent in eq.(2) (which is strictly properly written only for $\theta_i \neq 0, \pi$ [2]) is an effective potential, induced by the non-trivial metric of integration. Its presence in the functional integral was first noticed by De Witt [3]. A similar effect was noticed earlier by Ito [4] in the study of the Brownian motion and forms the basis of the "Itô calculus". A general formula relating the "Itô terms" to the metric for functional integration in 1-dimension was given by Mc Laughlin and Schulman [5].

In this paper we would like to point out that similar effects occur in higher dimensions too, i.e. that there is an Ito calculus for field theory on non-flat manifolds, such as the σ and Yang-Mills models. Moreoever the presence of these terms of order $1/\beta$ or higher in the effective action is intimately connected to the symmetries present in the problem. (In 1-dimension it is known that the propagator in eq.(2) does not obey the Schrodinger equation written in spherical coordinates unless the Itô term is included [6]).

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We begin our discussion by considering the partition function of the O(N) non-linear σ-model in d-dimensions

$$Z = \lim_{L\to\infty} \int \prod_{i_\mu=0}^{L-1} d\vec{S}_{i_\mu} \exp\left[-\frac{\mu^{d-2}}{g} \sum_{i_\mu=0}^{L-1} a^d \sum_{\nu=1}^{d} \frac{(\vec{S}_{i_\mu+e_\nu} - \vec{S}_{i_\mu})^2}{2a^2}\right] \qquad (3)$$

$$aL = T.$$

The lattice is periodic ($S_0 = S_L$), of spacing a and μ some arbitrary inverse scale (g dimensionless). Eq.(3) is manifestly O(N) invariant, consequently integrating over the spin at some site n_0 ($d\, S_{n_0}$) produces an answer containing only inner products of the neighbouring spins. In particular if all neighbours are parallel, the result of this integration is independent of their common orientation. The question we would like to ask is whether giving eq.(3) its usual naive interpretation in say spherical coordinates for O(3),

$$Z = \lim_{L\to\infty} \int \prod_{i_\mu=0}^{L-1} \sin\theta_{i_\mu}\, d\theta_{i_\mu}\, d\varphi_{i_\mu} \times$$

$$\exp\left\{-\frac{(a\mu)^{d-2}}{2g} \sum_{i_\mu=0}^{L-1} \sum_{\nu=1}^{d} \left[(\theta_{i_\mu+e_\nu} - \theta_{i_\mu})^2 + \sin\theta_{i_\mu+e_\nu} \sin\theta_{i_\mu} (\varphi_{i_\mu+e_\nu} - \varphi_{i_\mu})^2\right]\right\} \qquad (4)$$

aL = T fixed,

agrees with the above stated test. More specifically we want to check that as a $\longrightarrow$ 0

$$I = \int_0^{\pi} \sin\theta_0\, d\theta_0 \int_{-\pi}^{\pi} d\varphi_0 \exp\left\{-\frac{(a\mu)^{d-2}}{2g}\left[(\theta_0-\theta)^2 + \sin\theta_0 \sin\theta (\varphi_0-\varphi)^2\right]\right\} \qquad (5)$$

is independent of (θ, φ) at least to order a^d. Let us consider first the case d<2. Since a $\longrightarrow$ 0, we can ignore the limits of integration and obtain

$$I = \left[\frac{2\pi g}{(\mu a)^{d-2}}\right]^{1/2} \int_0^{\pi} \sqrt{\frac{\sin\theta_0}{\sin\theta}}\, d\theta_0 \exp\left[-\frac{(a\mu)^{d-2}}{2g}(\theta_0-\theta)^2\right]$$

$$= \left[\frac{2\pi g}{(\mu a)^{d-2}}\right]^{1/2} \int_{-\infty}^{+\infty} dx \left[1 + \frac{\cos\theta}{\sin\theta}\,\frac{x}{2} - \left(1 + \frac{1}{\sin^2\theta}\right)\frac{x^2}{8} + \cdots\right] e^{-\frac{(a\mu)^{d-2}}{2g}x^2}$$

$$= \frac{2\pi g}{(\mu a)^{d-2}} \left[1 - \frac{1}{8}\left(1 + \frac{1}{\sin^2\theta}\right) g\,(a\mu)^{2-d} + \cdots\right] \tag{6}$$

We see immediately that I is not independent of θ to the desired order. In fact if d=1, the violation is precisely the negative of the Itô term (eq.(2)), hence it will not occur with the Itô action (eq.(2)). For d<2 there are a finite number of terms in eq.(6) of order at least a^d. For d>2 the integral I (eq.(5)) depends on (θ, φ) to arbitrary order in $g(a\mu)^{2-d}$, hence an infinite number of Itô terms must be introduced into eq.(4) to recover the O(3) invariance. Unfortunately for d>2 the simple procedure developed by Mc Laughlin and Schulman [5] for computing the Itô terms in 1 dimension is not applicable ; indeed it relies on Gaussian dominance for a --> 0, which is no longer true for d>2.

Next let us consider lattice gauge models. They can be regarded as σ models, with the spins attached to the links and the action a special invariant function of the four spins belonging to each plaquette:

$$Z = \lim_{L\to\infty} \int \prod_{i_\mu=0}^{L-1} d\vec{S}_{i\mu}\, \exp\left[-\frac{(a\mu)^{d-4}}{g} \sum_{plqs} f_{plq.}[\vec{S}_{i\mu}]\right\} \tag{7}$$

The notable difference from the σ-models eq.(3) is the power of $(a\mu)$ changed from d-2 to d-4 (so as to formally obtain the usual action as a --->0). For d<4 one can use the Mc Laughlin-Schulman [5] gaussian estimates and compute the Itô terms. In fact, it turns out that the Ito calculus for gauge models in d=2 is precisely the same as the one for σ-models in d=1. (An easy way to see that is to notice that in the axial gauge, the gauge model becomes a product of uncoupled 1-dimensional σ-models). For d=4 the situation with gauge models is analogous to the case d=2 for σ-models.

Final remarks

We have presented arguments to show that the effective continuum action of certain lattice models may be different from the naive continuum action. Our discussion has ignored the important fact that the limit L ---> ∞ in eqs.(3) and (7) may not define physically interesting theories unless g is allowed to vary appropriately with L. Such a dependence would change the

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naive counting of powers of a and thus affect the Itô calculus. Until the proper continuum limit of these lattice theories is constructed, all we can say is that a priori the effective continuum action may be different from the naive continuum action by Itô terms. These terms, while manifestly breaking the symmetries of the problem, are in the action precisely to insure that the true Green's functions of the theory possess the desired symmetries. Finally let us notice that the Itô terms, if at all present, are non-coercive, hence enhance the field fluctuations. For example for S^2 $d=1$, the Itô term in eq.(2) insures that the ground state probability is flat over the sphere.

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