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Supersymmetric Yang-Mills Fields as an Integrable System and Connections with other Non-linear Systems*

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ABSTRACT

Integrable properties, i.e., existence of linear systems, infinite number of conservation laws, Reimann-Hilbert transforms, affine Lie algebra of Kac-Moody, and Bianchi-Bäcklund transformation, are discussed for the constraint equations of the supersymmetric Yang-Mills fields. For $N \geq 3$ these constraint equations give equations of motion of the fields. These equations of motion reduce to the ordinary Yang-Mills equations as the spinor and scalar fields are eliminated. These understandings provide a possible method to solve the full Yang-Mills equations. Connections with other non-linear systems are also discussed.

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DISCLAIMER

INTRODUCTION

Beginning with Maxwell's equations and quantum electrodynamics, and the recent spectacular success of the electroweak theory of quantum flavor dynamics [1,2] gauge theories have become an essential part of physics. It has become increasingly clear that, besides its mathematical beauty, the Yang-Mills theory [1,3-5] may provide the key to our understanding of strong interactions. With the recent experimental observation of gluon jets [6], the ideas of Yang-Mills gauge theory for strong interactions is brought one step further to reality. Despite many interesting theoretical and phenomenological observations, like confinement [7], asymptotic freedom and QCD perturbative studies [8], the Yang-Mills equations are far from being solved.

In the past fifteen years or so, powerful mathematical tools have been developed in completely solving many two-dimensional nonlinear differential equations [9-11]. The characteristics of these integrable systems are: existence of linear systems, Bianchi-Bäcklund transformations (BT), infinite number of conversation laws, soliton solutions, and even the construction of S-matrix [12]. It is our intention to see whether these powerful techniques can be applied to solve the Yang-Mills equations.

For the past few years, we have found that amazingly the self-dual Yang-Mills (SDYM) fields, in the J-formulation, possess many characteristics of those integrable systems [13-22]. Recently the affine algebra of Kac-Moody has been found in the SDYM systems [23-25]. It thus provides a beautiful mathematical system in four dimensions [26-28].

Due to the extreme similarity in appearance between the SDYM equations and the two dimensional chiral equations [28-35], and the Ernst equations [36], i.e. the stationary axially symmetric equations of Einstein equations, many fruitful results have come about by investigating both systems hand in hand. Actually some systematic understandings have been made.

In Fig. 1 we give a flow chart of some logic links between various structures of these systems. At the center are the corresponding linear systems for the original non-linear equations. Their existence implies that the original non-linear equations are results of the integrability of the set of linear systems, or the original non-linear equations are equivalent to some generalized curvatureless conditions. Once the linear systems are obtained, many developments indicated in Fig. 1 can be made. The developments indicated by solid lines have been made both for the SDYM equations in four dimensions, as well as for the two dimensional chiral equations and the Ernst equations. The dashed lines indicate developments made only for the latter two dimensional equations. The linear systems are immediately related to the existence of an infinite number of conservation laws. Also from these linear systems, parametric Bianchi-Bäcklund transformations (BT) can be obtained, which are good for generating new solutions (global properties have to be imposed in addition). These BT were originally obtained purely based upon guess work. Now we can show that they are generated by the so called Darboux transformations, which is a special form of the Riemann-Hilbert transforms. From the parameters in the BT, infinite number of local conservation laws can be obtained [30,31]. For the chiral model in two dimensions, we can further show that the linear systems can be derived from the BT via the Riccati equations, [28]. This indicates that the parametric BT given in [30] for the chiral field is probably the most general one, and it is as fundamental as the linear systems, and thus as the original non-linear systems. So far, such development has not been done for the BT [16] of the four dimensional Yang-Mills equations.

To generate new solutions from the linear systems, for the original non-linear systems, is first to generate new solutions to the linear systems, and then via the inverse scattering method to obtain the corresponding new solutions to the original non-linear system. The matrix Reimann-Hilbert (R-H) transforms [37] provide such a method. It is also a way in which the original problem of solving differential equations in the coordinate space being transformed into solving integral equations in the spectrum space. Such methods have been very useful in obtaining many new solutions to the static, axially symmetric Einstein equations, i.e., the Ernst equations [36]. Actually it was shown that all solutions with proper boundary conditions can be obtained via the R-H transform, which was called the Gerosch conjecture [38]. From the finite R-H transforms, the infinitesimal R-H transforms can be derived. We then can show that the algebra structure satisfied by these infinitesimal R-H transforms are of the Kac-Moody type, [23-25,27,28]. This has been the main topic of this workshop. Very interestingly, Dr. Friedan at this workshop has pointed out the link between the values of the central extensions to various two dimensional models in statistical mechanics. However, so far there is no central extensions of the Kac-Moody algebra have been found in many of the non-linear systems, e.g., Sine-Gordon, chiral, and self-dual Yang-Mills. The physical implications of the Kac-Moody algebra for these systems are yet to be understood.

Now we shall motivate how to solve differential equations on curvatures via solving algebraic equations on curvatures. For example, the self-dual, and anti-self-dual Yang-Mills equations are a set of algebraic constraint relations on the curvatures $F_{\mu\nu}$, i.e.,

$$F_{\mu\nu} = \pm \frac{1}{2} \varepsilon_{\mu\nu\gamma\delta} F^{\gamma\delta} \equiv \pm {}^{*}F_{\mu\nu} . \tag{1}$$

Under these constraints, the Bianchi identities

$$\mathcal{D}^{\mu}\, {}^{*}F_{\mu\nu} = 0 , \tag{2}$$

become the equation of motion

$$\mathcal{D}^{\mu} F_{\mu\nu} = 0 \, . \tag{3}$$

Therefore, solving the self-dual and anti-self-dual equations, Eq. (1), can be viewed as a way of finding some special solutions of the Yang-Mills equations, Eq. (3). It was suggested by Witten [39], and by Isenberg, Yasskin, and Green [40] that such concept and procedure can be generalized to more general solutions (even hopefully to all solutions) of the full Yang-Mills equations. It was demonstrated that, formulated in complex d = 8 dimensions, or real d = 4 plus $N \geq 3$ super dimensions, certain algebraic constraint equations do lead to Yang-Mills equations via the Bianchi identities. Recently, it has been shown that these constraints equation process many of the integrable properties. Here we shall concentrate on the discussions of supersymmetric Yang-Mills equations.

The super symmetric Yang-Mills fields [42] formulated in the coordinate $x \equiv (x^{\alpha\dot{\beta}}, \theta^{\alpha}_{s}, \bar{\theta}^{\dot{\beta}t})$ where $\alpha, \beta = 1,2$; $s,t = 1.2, \cdots N$, have in general six types of curvatures $F_{\mu\nu}$, $F^{st}_{\alpha\beta}$, $F_{\dot{\alpha}s,\dot{\beta}t}$, $F_{\mu,\alpha}^{\;s}$, $F_{\mu,\dot{\beta}t}$, $F^{s}_{\alpha,\dot{\beta}t}$. To eliminate the extra curvatures due to the introduction of super coordinates, constraints [39,42-44] were introduced onto those curvatures involving only the super coordinates, i.e., $F^{st}_{\alpha\beta}$, $F_{\dot{\alpha}s,\dot{\beta}t}$, $F^{s}_{\alpha,\dot{\beta}t}$. Interestingly for $N \geq 3$ supersymmetric Yang-Mills fields, these constraints can put the theory on shell [39,44] i.e., these constraints via the Bianchi identities lead to equations of motion. Recently it has been shown by Volovich [45] that to these constraint equations linear systems can be introduced. And it has been demonstrated by Devchand [46] that many of the formulations [27,28] developed for the self dual Yang-Mills system as an integrable system can also be made for these systems, e.g., infinite number of conservation laws, the existence of affine Lie algebra of Kac-Moody.

With Ge and Popowicz [47], we developed further the integrability properties of supersymmetric Yang-Mills equations. We have constructed the Riemann-Hilbert transforms, the infinitesimal forms of which precisely give the affine Lie Algebra of Kac-Moody. From the linear systems, a two-parameter Bianchi-Bäcklund (BT) transformation can also be constructed. In the special (anti) self-dual cases, the BT transformations involving only the ordinary coordinate become those obtained in Ref. 16.

These new developments give the hope that the full Yang-Mills equations can be solved via the supersymmetric way .

I. Formulation of Supersymmetric Yang-Mills Fields and the Constraint Equations

The superspace coordinates are given by $x = (x^{\mu}, \theta^{\alpha}_{s}, \bar{\theta}^{\dot{\beta}t}) = (x^{\alpha\dot{\beta}}, \theta^{\alpha}_{s}, \bar{\theta}^{\dot{\beta}t})$, where x^{μ}, $\mu = 0,1,2,3$, is the ordinary space coordinate which can also be described as $x^{\dot{\alpha}\beta} = x^{\mu}\sigma^{\dot{\alpha}\beta}_{\mu}$. The θ^{α}_{s}, $\bar{\theta}^{\dot{\beta}t}$ are the super coordinates, with $s,t = 1,2,\cdots N$, characterizing the dimensions of supersymmetry. Corresponding to these coordinates there are three types of derivatives:

$$\partial_{\mu} = \partial/\partial x^{\mu} \text{ or } \partial_{\alpha\dot{\beta}} = \partial/\partial x^{\alpha\dot{\beta}};\ D^{s}_{\alpha} = \partial/\partial\theta^{\alpha}_{s} + i\bar{\theta}^{\dot{\beta}s}\partial_{\alpha\dot{\beta}};\ \bar{D}_{\dot{\beta}t} = -\partial/\partial\bar{\theta}^{\dot{\beta}t} - i\theta^{\alpha}_{t}\partial_{\alpha\dot{\beta}}\ .$$

with their algebraic rules

$$[\partial_{\alpha\dot{\beta}}, \partial_{\gamma\dot{\delta}}] = 0,\ \{D^{s}_{\alpha}, D^{t}_{\beta}\} = 0 = \{\bar{D}_{\dot{\beta}s}, \bar{D}_{\dot{\alpha}t}\},\ \{D^{s}_{\alpha}, \bar{D}_{\dot{\beta}t}\} = -2i\delta^{s}_{t}\partial_{\alpha\dot{\beta}}\ .$$

The covariant derivatives are

$$\nabla_{\mu} \equiv \partial_{\mu} + A_{\mu},\ \text{or}\ \nabla_{\alpha\dot{\beta}} \equiv \partial_{\alpha\dot{\beta}} + A_{\alpha\dot{\beta}};\quad \nabla^{s}_{\alpha} = D^{s}_{\alpha} + A^{s}_{\alpha};\quad \bar{\nabla}_{\dot{\beta}t} = \bar{D}_{\dot{\beta}t} + A_{\dot{\beta}t}.$$

The curvatures are

$$[\nabla_\mu,\nabla_\nu] = F_{\mu\nu}; \quad \{\nabla^s_\alpha,\nabla^t_\beta\} = F^{st}_{\alpha\beta}; \quad \{\bar\nabla_{\dot\alpha s},\bar\nabla_{\dot\beta t}\} = F_{\dot\alpha s,\dot\beta t};$$

$$[\nabla_\mu,\nabla^s_\alpha] = F_{\mu\alpha}{}^s; \quad [\nabla_\mu,\bar\nabla_{\dot\beta t}] = F_{\mu,\dot\beta t}; \quad \{\nabla^s_\alpha,\bar\nabla_{\dot\beta t}\} = F^s_{\alpha,\dot\beta t} - 2i\delta^s_t\nabla_{\alpha\dot\beta}; \tag{1.1}$$

note that $F^s_{\alpha,\dot\beta t} = D^s_\alpha A_{\dot\beta t} + \bar D_{\dot\beta t}A^s_\alpha + \{A^s_\alpha,A_{\dot\beta t}\} + 2i\delta^s_t A_{\alpha\dot\beta}$. Usually to get rid of the super components of the curvature, constraints are put on the three types of curvatures that involves only the super components:

$$F^{st}_{\alpha\beta} + F^{st}_{\beta\alpha} = 0; \quad F_{\dot\alpha s,\dot\beta t} + F_{\dot\alpha t,\dot\beta s} = 0; \quad F^s_{\alpha,\dot\beta t} = 0. \tag{1.2}$$

It was noted by Volovich [45] and Devchand [46] that very similar to the self-dual Yang-Mills (SDYM) case, some of these curvatureless equations,

$$F^{st}_{11} = 0 = F^{st}_{22}; \quad F_{\dot1 s,\dot1 t} = 0 = F_{\dot2 s,\dot2 t}; \quad F^s_{1,\dot1 t} = 0 = F^s_{2,\dot2 t}, \tag{1.3}$$

can be integrated by the following form of the potentials

$$A^s_1 = g^{-1}D^s_1 g; \quad A_{\dot1 t} = g^{-1}\bar D_{\dot1 t}g; \quad A^s_2 = h^{-1}D^s_2 h; \; A_{\dot2 t} = h^{-1}\bar D_{\dot2 t}h. \tag{1.4}$$

Similarly, by defining a J matrix,

$$J \equiv hg^{-1} , \tag{1.5}$$

in terms of which the curvatures become:

$$F^{st}_{12} = g^{-1}\{D^s_1(J^{-1}D^t_2 J)\}g, \tag{1.6a}$$

$$F_{\dot1 s,\dot2 t} = g^{-1}\{\bar D_{\dot1 s}(J^{-1}\bar D_{\dot2 t}J)\}g, \tag{1.6b}$$

$$F^s_{1,\dot2 t} = g^{-1}\{D^s_1(J^{-1}\bar D_{\dot2 t}J) + 2i\delta^s_t\nabla_{1\dot2}\}g, \tag{1.6c}$$

$$F^s_{2,\dot1 t} = g^{-1}\{D_{\dot1 t}(J^{-1}D^s_2 J) + 2i\delta^s_t\nabla_{2\dot1}\}g. \tag{1.6d}$$

Then the unintegrated constraints on curvatures,

$$F^{st}_{12} + F^{ts}_{12} = 0; \quad F_{\dot1 s,\dot2 t} + F_{\dot1 t,\dot2 s} = 0 \; ; \quad F^s_{1,\dot2 t} = 0 = F^s_{2,\dot1 t}, \tag{1.7}$$

give equations on J

$$D_1^s(J^{-1}D_2^tJ) + D_1^t(J^{-1}D_2^sJ) = 0, \tag{1.8a}$$

$$\bar{D}_{\dot{1}s}(J^{-1}\bar{D}_{\dot{2}t}J) + \bar{D}_{\dot{1}t}(J^{-1}\bar{D}_{\dot{2}s}J) = 0, \tag{1.8b}$$

$$D_1^s(J^{-1}\bar{D}_{\dot{2}t}J) + \delta_t^s 2ig\nabla_{1\dot{2}}g^{-1} = 0, \tag{1.8c}$$

$$\bar{D}_{\dot{1}s}(J^{-1}D_2^tJ) + \delta_s^t 2ig\nabla_{2\dot{1}}g^{-1} = 0. \tag{1.8d}$$

In these equations J^{-1} appears on the left of DJ. We call them the left-formulation.

Another formulation is:

$$F_{12}^{st} = h^{-1}\{D_2^s(JD_1^tJ^{-1})\}h, \tag{1.9a}$$

$$F_{\dot{1}s,\dot{2}t} = h^{-1}\{\bar{D}_{\dot{2}t}(J\bar{D}_{\dot{1}s}J^{-1})\}h, \tag{1.9b}$$

$$F^s_{1,\dot{2}t} = h^{-1}\{\bar{D}_{\dot{2}t}(J\,D_1^sJ^{-1}) + 2i\delta_t^s\nabla_{1\dot{2}}\}h, \tag{1.9c}$$

$$F^s_{2,\dot{1}t} = h^{-1}\{D_2^s(J^{-1}D_{\dot{1}t}J) + 2i\delta_t^s\nabla_{2\dot{1}}\}h. \tag{1.9d}$$

The unintegrated constraints, Eq. (1.7), on these curvatures, give the right-equation on J:

$$D_2^s(JD_1^tJ^{-1}) + D_2^t(JD_1^sJ^{-1}) = 0, \tag{1.10a}$$

$$\bar{D}_{\dot{2}s}(J\bar{D}_{\dot{1}t}J^{-1}) + \bar{D}_{\dot{2}t}(J\bar{D}_{\dot{1}s}J^{-1}) = 0, \tag{1.10b}$$

$$\dot{D}_{\dot{2}s}(JD_1^tJ^{-1}) + \delta_s^t 2ih\nabla_{1\dot{2}}h^{-1} = 0, \tag{1.10c}$$

$$D_2^s(J\bar{D}_{\dot{1}t}J^{-1}) + \delta_t^s 2ih\nabla_{2\dot{1}}h^{-1} = 0. \tag{1.10c}$$

This is what we call the right-formulation.

II. Linear System to the Constraint Equations ↔ Infinite Number of Conservation Equations

To these equations on J, Eqs. (1.8), a set of linear systems can be constructed [45,46]

$$L^s(\lambda)\Psi(\lambda) \equiv (D_1^s + \lambda D_2^s + \lambda J^{-1}D_2^s J)\,\Psi(\lambda) = 0, \tag{2.1a}$$

$$M_t(\lambda)\Psi(\lambda) \equiv (\bar{D}_{\dot{2}t} + J^{-1}\bar{D}_{\dot{2}t}J + \lambda^{-2}\bar{D}_{\dot{1}t})\,\Psi(\lambda) = 0, \tag{2.1b}$$

$$N(\lambda)\Psi(\lambda) \equiv \{(\partial_{1\dot{2}} + g\nabla_{1\dot{2}}g^{-1}) + \lambda(\partial_{2\dot{2}} + J^{-1}\partial_{2\dot{2}}J) + \lambda^{-1}(\partial_{2\dot{1}} + g\nabla_{2\dot{1}}g^{-1}) + \lambda^{-2}\partial_{1\dot{1}}\}\,\Psi(\lambda) = 0. \tag{2.1c}$$

The integrability conditions of these linear equations give equations on J, Eqs. (1.8): $\{L^s, L^t\} = 0$, implies Eq. (1.8a); $\{M_s, M_t\} = 0$ gives Eq. (1.8b); and $\{L_s, M_t\} = -2i\delta_t^s N$ gives the last two equations, Eqs. (1.8 c,d) according to two different powers of λ.

The right-formulation for the linear equation can be obtained in the following way: multiplying Eqs. (2.1) with J on the left hand side and replacing Ψ by $J^{-1}J\Psi$, after simple manipulations we obtain

$$\hat{L}^s(\lambda^{-1})\hat{\Psi}(\lambda^{-1}) \equiv (D_2^s + \lambda^{-1}D_1^s + \lambda^{-1}JD_1^sJ^{-1})\,\hat{\Psi}(\lambda^{-1}) = 0, \tag{2.2a}$$

$$\hat{M}_t(\lambda^{-1})\hat{\Psi}(\lambda^{-1}) \equiv (\lambda^2\bar{D}_{\dot{2}t} + \bar{D}_{\dot{1}t} + J\bar{D}_{\dot{1}t}J^{-1})\,\hat{\Psi}(\lambda^{-1}) = 0, \tag{2.2b}$$

$$\hat{N}(\lambda^{-1})\hat{\Psi}(\lambda^{-1}) \equiv \{\lambda(\partial_{1\dot{2}} + h\nabla_{1\dot{2}}h^{-1}) + \lambda^2\partial_{2\dot{2}} + \lambda^{-1}(\partial_{1\dot{1}} + J\partial_{1\dot{1}}J^{-1}) + \partial_{2\dot{1}} + h\nabla_{2\dot{1}}h^{-1}\}\,\hat{\Psi}(\lambda^{-1}) = 0. \tag{2.2c}$$

One can show that the equations (1.10) are integrability consequence of Eqs. (2.2), i.e., $\{\hat{L}^s, \hat{L}^t\} = 0$, $\{\hat{M}_s, \hat{M}_t\} = 0$, and $\{\hat{L}^s, \hat{M}_t\} = -2i\delta^s_t \hat{N}$.

We now construct infinite numbers of conservation laws. Note that Eqs. (1.8 a,b) can be solved in terms of two functions $X^{(1)}$, $X^{(2)}$, defined by

$$J^{-1}D^s_2 J = D^s_1 X^{(1)}, \tag{2.3a}$$

$$J^{-1}\bar{D}_{\dot{2}t} J = \bar{D}_{\dot{1}t} X^{(2)}. \tag{2.3b}$$

Then an infinite number of X's can be generated iteratively in the following way: define $X^{(n)}$ by

$$D^s_1 X^{(n)} = [D^s_2 + J^{-1}D^s_2 J]\ X^{(n-1)}, \tag{2.4a}$$

$$\bar{D}_{\dot{1}s} X^{(n)} = [D_{\dot{2}s} + J^{-1}\bar{D}_{\dot{2}s} J]\ X^{(n-2)}. \tag{2.4b}$$

Note that we define $X^{(0)} = 1$. It is easy to show that such defined $D^s_1 X^{(n)}$, $\bar{D}_{\dot{1}t} X^{(n)}$ satisfy the continuity-like equation, Eqs. (1.8 a,b).

Following the procedure given in Ref. [18], i.e., multiplying Eq. (2.4 a,b) by λ^n, λ being an arbitrary complex number, summing n = 0 to ∞, and defining

$$\Psi \equiv \sum_{n=0}^{\infty} \lambda^n X^{(n)}, \tag{2.5}$$

we obtain the linear equations (2.1 a,b). Furthermore, one can show that the relation Eqs. (2.3 a,b) solve all Eqs. (1.8 a-d) if Ψ satisfies Eq. (2.1c).

Similarly for the right-formulation we have

$$JD_1^t J^{-1} = D_2^t \hat{X}^{(1)}, \tag{2.6a}$$

$$J\bar{D}_{\dot{1}t} J^{-1} = D_{\dot{2}t} \hat{X}^{(2)}, \tag{2.6b}$$

and

$$D_2^s \hat{X}^{(n)} \equiv (D_1^s + JD_1^s J^{-1})\, \hat{X}^{(n-1)}, \tag{2.7a}$$

$$\bar{D}_{\dot{2}s} \hat{X}^{(n)} = [D_{\dot{1}s} + J\bar{D}_{\dot{1}s} J^{-1}]\, \hat{X}^{(n-2)}, \tag{2.7b}$$

multiplying Eqs. (2.7) by λ^n and summing, and defining

$$\hat{\Psi} \equiv \sum_{n=0}^{\infty} \lambda^n \hat{X}^{(n)}, \tag{2.8}$$

we obtain the linear systems of Eqs. (2.2 a,b), and Eqs. (1.10 a-d) are solved by (2.6 a,b) if $\hat{\Psi}$ satisfies Eq. (2.2c).

Note that these linear equations are given by Devchand [46], which are related to those first obtained by Volovich [45] by rewriting the linear equation for a gauge transformed wave function $\psi = g\Psi$. This situation is very similar to that in the self-dual Yang-Mills field [21,22].

III. Finite Reimann-Hilbert Transforms

Now we construct the Riemann-Hilbert transform (R-H) for these linear equations, which can all be abstractly written in the following form

$$\mathscr{D}\psi + \mathscr{B}\psi = 0 , \tag{3.1}$$

where $\mathscr{D}$ denotes the differenciations in Eqs. (2.1): $D_1^s + \lambda D_2^s$, $\lambda^{-2}\bar{D}_{\dot{1}t} + \bar{D}_{\dot{2}t}$, or $\partial_{1\dot{2}} + \lambda\partial_{2\dot{2}} + \lambda^{-2}\partial_{1\dot{1}} + \lambda^{-1}\partial_{2\dot{1}}$;

$\mathscr{B}$ can denote respectively their corresponding potential terms in Eqs. (2.1), $\lambda J^{-1}D^{s}_{2}J$, $J^{-1}\bar{D}_{\dot{2}t}J$, or $g\nabla_{1\dot{2}}g^{-1} + \lambda J^{-1}\partial_{2\dot{2}}J + \lambda^{-1}g\nabla_{2\dot{1}}g^{-1}$. First note that if a new solution Ψ' can be constructed from a given solution Ψ by a Darboux-type transformation [48]

$$\Psi'(\lambda) = R(\lambda)\Psi(\lambda) \ , \tag{3.2}$$

a new potential is generated through

$$\mathscr{B}' = R\mathscr{B}R^{-1} - (\mathscr{D}R)\ R^{-1}. \tag{3.3}$$

Now consider a contour C in the complex λ-plane dividing the plane into two regions C_+ and C_-. Let $R_\pm$ be analytic in the $C_\pm$ regions respectively. Then Eq. (3.3) becomes $\mathscr{B}'_\pm = R_\pm \mathscr{B} R_\pm^{-1} - (\mathscr{D}R_\pm)\ R_\pm^{-1}$. Requiring $\mathscr{B}'_+ = \mathscr{B}'_-$ on C, we can show, after some calculation,using the linear equations,

$$\mathscr{D}\,(\Psi^{-1}R_+^{-1}R_-\Psi) = 0, \text{ which implies} \tag{3.4}$$

$$\Psi^{-1}R_+^{-1}R_-\Psi = U(\lambda,\cdots), \tag{3.5}$$

where U is a group element of the theory and independent of the three differentiations. In general U is still a function of linear combinations of $x_{\alpha\dot{\beta}}$, θ^{s}_{α}, $\bar{\theta}^{\dot{\beta}t}$ i.e.,

$$\lambda(\theta^1_s - i\bar{\theta}_{\dot{\beta}s}\, x^{1\dot{\beta}}) - (\theta^2_s - \bar{\theta}_{\dot{\beta}s}x^{2\dot{\beta}});\quad \lambda^{-2}(\bar{\theta}^{\dot{2}t} - i\theta^t_\alpha x^{\alpha\dot{2}}) - (\bar{\theta}^{\dot{1}t} - i\theta^t_\alpha x^{\alpha\dot{1}});$$

$$x^{1\dot{2}} - \lambda^{-1}x^{2\dot{2}} + \lambda^2 x^{1\dot{1}} - \lambda x^{2\dot{1}};\quad x^{1\dot{2}} + \lambda^{-1}x^{2\dot{2}} - \lambda^2 x^{1\dot{1}} - \lambda x^{2\dot{1}};\ \text{and}$$

$$x^{1\dot{2}} - \lambda^{-1}x^{2\dot{2}} - \lambda^2 x^{1\dot{1}} + \lambda x^{2\dot{1}}. \tag{3.6}$$

From Eq. (3.5) we obtain $R_- = R_+\Psi U\Psi^{-1}$ or alternative $R_+ - R_- = R_+\Psi(1 - U)\Psi^{-1}$, from which a dispersion relation can be written.

$$R(\lambda) = 1 - \frac{\lambda}{2\pi i} \int_C \frac{d\lambda'}{\lambda'(\lambda' - \lambda)} R(\lambda')\Psi(\lambda') \; [U(\lambda') - 1] \; \Psi^{-1}(\lambda'), \qquad (3.7a)$$

where we assume the boundary condition $R(0) = 1$, and that one subtraction is needed in the dispersion integral. Using Eq. (3.2), we obtain from Eq. (3.7a) the integral equation for Ψ',

$$\Psi'(\lambda) = \Psi(\lambda) - \frac{\lambda}{2\pi i} \left\{ \int_C \frac{d\lambda'}{\lambda'(\lambda' - \lambda)} \Psi'(\lambda') \; [U(\lambda') - 1] \; \Psi^{-1}(\lambda') \right\} \Psi(\lambda). \qquad (3.7b)$$

These are the R-H transforms. It is well known that R-H transforms have been very useful in generating new solutions in many non-linear systems[36-38]. This is a very fascinating prospect for the supersymmetric Yang-Mills fields.

Following the same procedure as for the left-formulation linear equations, we can formulate R-H transform for the right-formulation linear equations,

$$\hat{R}(\lambda^{-1}) = 1 - \frac{1}{2\pi i} (\lambda^{-1}) \int_{C'} \frac{d(\lambda'^{-1})}{(\lambda'^{-1})(\lambda'^{-1} - \lambda^{-1})} \hat{R}(\lambda'^{-1})\hat{\Psi}(\lambda'^{-1})[\hat{U}(\lambda'^{-1}) - 1]\hat{\Psi}^{-1}(\lambda'^{-1}),$$

and

$$\hat{\Psi}'(\lambda^{-1}) = \hat{\Psi}(\lambda^{-1}) - \frac{1}{2\pi i} (\lambda^{-1}) \left\{ \int_{C'} \frac{d(\lambda'^{-1})}{(\lambda'^{-1})(\lambda'^{-1} - \lambda^{-1})} \hat{\Psi}'(\lambda'^{-1})[\hat{U}(\lambda'^{-1}) - 1] \right.$$

$$\left. \hat{\Psi}^{-1}(\lambda'^{-1}) \right\} \cdot \hat{\Psi}(\lambda^{-1}), \qquad (3.8b)$$

the boundary conditions here is $\hat{R}(0) = 1$, $\hat{\Psi}'(0) = \hat{\Psi}(0)$ at $\lambda \to \infty$.

Comparing the left-formulation and the right formulation linear equations, Eqs. [2.1,2.2], we find the $\hat{\Psi}$ and Ψ are related by the following relation [49]

$$\hat{\Psi}(\lambda^{-1}) = J\Psi(\lambda) \; \rho(\lambda, \cdots), \qquad (3.9)$$

where ρ is an arbitrary function and "$\cdots$" denotes those variables of Eq. (3.6), which are independent of the differentiations in Eqs. (2.2).

IV. Infinitesimal Reimann-Hilbert Transform and the Affine Algebra of Kac-Moody

Now lets discuss the infinitesimal R-H transform, i.e., $U(\lambda',\cdots) - 1 \equiv v(\lambda',\cdots)$ with $v(\lambda',\cdots)$ belonging to the algebra. Keeping only the first order in variation, Eq. (3.7b) becomes

$$[\delta_v \Psi(\lambda)]\Psi^{-1}(\lambda) = -\frac{\lambda}{2\pi i}\int_C \frac{d\lambda'}{\lambda'(\lambda'-\lambda)}\Psi(\lambda')v(\lambda',\cdots)\Psi^{-1}(\lambda'), \tag{4.1a}$$

Similarly for Eq. (3.8b),

$$[\delta_{\hat{v}}\hat{\Psi}(\lambda^{-1})]\hat{\Psi}^{-1}(\lambda^{-1}) = -\frac{\lambda}{2\pi i}\int_{C'} \frac{d\lambda'^{-1}}{\lambda'^{-1}(\lambda'^{-1}-\lambda^{-1})}\hat{\Psi}(\lambda'^{-1})\hat{v}(\lambda'^{-1},\cdots)\hat{\Psi}^{-1}(\lambda'^{-1}).$$

Take different form of v, $\hat{v}$ we can obtain different transforms. Using the appropriate v, $\hat{v}$ we can obtain the affine Lie algebra of Kac-Moody [23-28], choosing

$$V(\lambda',\cdots) = V_a^m(\lambda') = \eta(\lambda)^m t_a, \tag{4.2}$$

where $[t_a, t_b] = C_{ab}^c t_c$, and $\eta(\lambda) = \prod_i(\lambda - \lambda_i)/(\lambda - \lambda_i^*)$,

one obtains

$$\begin{aligned}\delta_a^m\Psi(\lambda) &= -\frac{\lambda}{2\pi i}\left\{\int\frac{d\lambda'}{\lambda'(\lambda'-\lambda)}\eta^m(\lambda')\left[\Psi(\lambda')\; t_a\Psi(\lambda')^{-1}\right]\right\}\Psi(\lambda)\\ &\equiv -\frac{\lambda}{2\pi i}\left\{\int\frac{d\lambda'}{\lambda'(\lambda'-\lambda)}\eta^m(\lambda')\; S_a(\lambda')\right\}\Psi(\lambda)\ ,\end{aligned} \tag{4.3}$$

where $S_a(\lambda') \equiv \Psi(\lambda')\, t_a\Psi^{-1}(\lambda')$. To obtain the algebraic structure of such variation, we can make another infinitesimal transformation to obtain $\delta_b^n\delta_a^m\Psi(\lambda)$. After lengthy calculations and manipulations involving contour deformations, [50,51], we obtain

$$(\delta^m_a \delta^n_b - \delta^n_b \delta^m_a)\ \Psi(\lambda)$$

$$= -\frac{\lambda}{2\pi i}\int \frac{d\lambda'}{\lambda'(\lambda' - \lambda)}\ \eta^{m+n}(\lambda')\ [S_a(\lambda'),\ S_b(\lambda')]$$

$$= -\frac{\lambda}{2\pi i}\int \frac{d\lambda'}{\lambda'(\lambda' - \lambda)}\ \eta^{m+n}(\lambda')\ C^c_{ab}\ S_c(\lambda')]$$

$$= C^c_{ab}\ \delta^{m+n}_c\ \Psi(\lambda)\ , \tag{4.4}$$

i.e., it is an affine lie algebra of Kac and Moody.

The implications of such infinite dimensional algebra on the physical systems are yet to be understood.

V. Parametric Bianchi-Bäcklund Transformations

Now we derive a two-parameter Bianchi-Bäcklund transformation (BT). From the Darboux-type transformation Eq. (3.2), and specifying R in the following form [48]

$$R = \xi I \pm f(\lambda) J'^{-1} J, \tag{5.1}$$

where ξ is a constant parameter and $f(\lambda)$ is an arbitrary function of λ.

Substituting Eq. (5.1) into Eq. (3.3) we obtain the following BT transformations

$$f(\lambda) D^s_1 (J'^{-1} J) = \lambda\xi (J^{-1} D^s_2 J - J'^{-1} D^s_2 J\ ), \tag{5.2}$$

$$\lambda^{-2} f(\lambda) \bar{D}_{1\dot{t}} (J'^{-1} J) = \xi (J^{-1} \bar{D}_{2\dot{t}} J - J'^{-1} \bar{D}_{2\dot{t}} J')\ , \tag{5.3}$$

$$f(\lambda)[\partial_{1\dot{2}}(J'^{-1}J) + \lambda^{-2}\partial_{1\dot{1}}(J'^{-1}J) + \lambda^{-1}\partial_{2\dot{1}}(J'^{-1}J)$$

$$= \xi\{(g\nabla_{1\dot{2}}g^{-1} - g'\nabla'_{1\dot{2}}g'^{-1}) + \lambda(J^{-1}\partial_{2\dot{2}}J - J'^{-1}\partial_{2\dot{2}}J')$$

$$+ \lambda^{-1}(g\nabla_{2\dot{1}}g^{-1} - g'\nabla'_{2\dot{1}}g'^{-1})\} + f(\lambda)[(J'^{-1}J)(g\nabla_{1\dot{2}}g^{-1} + \lambda^{-1}g\nabla_{2\dot{1}}g^{-1})$$

$$- (g'\nabla'_{1\dot{2}}g'^{-1} + \lambda^{-1}g'\nabla'_{2\dot{1}}g'^{-1})(J'^{-1}J)]\ . \tag{5.4}$$

We next restrict oursleves to the self-dual case, i.e., under the additional imposed constraints [45,46] $g\nabla_{1\dot{2}}g^{-1} = 0$, $A_{1\dot{2}} = g^{-1}\partial_{1\dot{2}}g$, $A_{2\dot{1}} = h^{-1}\partial_{2\dot{1}}h$. By requiring the constraints Eq. (1.2, 1.10) to be held in the primed quantities, Eq. (5.4) can be reduced to

$$f(\lambda)\partial_{1\dot{2}}(J'^{-1}J) + f(\lambda)\lambda^{-2}\partial_{1\dot{1}}(J'^{-1}J)$$

$$=\lambda\xi(J^{-1}\partial_{2\dot{2}}J - J'^{-1}\partial_{2\dot{2}}J') + \lambda^{-1}\xi(J^{-1}\partial_{2\dot{1}}J - J'^{-1}\partial_{2\dot{1}}J') \ . \tag{5.5}$$

Now specifying $f(\lambda) = \lambda$ and comparing the same powers in λ, we obtain from Eqs(5.2, 5.3)

$$D_1^s(J'^{-1}J) = \xi(J^{-1}D_2^sJ - J'^{-1}D_2^sJ'), \tag{5.6a}$$

$$D_{\dot{1}t}(J^{\bar{\prime}}\ \ J) \bar{=}^{1}\eta(J\ \ D_{\dot{2}t}\bar{J}^{1}\bar{-}\ J'\ \ D_{\dot{2}t}\bar{J}^{\bar{1}}\bar{)}, \tag{5.6b}$$

and Eq. (5.5) yields

$$\partial_{1\dot{2}}(J'^{-1}J) = \xi(J^{-1}\partial_{2\dot{2}}J - J'^{-1}\partial_{2\dot{2}}J'), \tag{5.6c}$$

$$\partial_{1\dot{1}}(J'^{-1}J) = \xi(J^{-1}\partial_{2\dot{1}}J- J'^{-1}\partial_{2\dot{1}}J') \tag{5.6d}$$

where $\eta = \xi\lambda^1$, another free-parameter. Similarly if we choose $g\nabla_{2\dot{1}}g^{-1} = 0$, $A_{1\dot{2}} = h^{-1}\partial_{1\dot{2}}h$ and $A_{2\dot{1}} = g^{-1}\partial_{2\dot{1}}g$ and same for the primed qualitites we find that necessarily $f(\lambda) = \lambda^2$, and correspondingly Eqs. (5.6) become

$$D_1^s(J'^{-1}J) = \zeta(J^{-1}D_2^sJ - J'^{-1}D_2^sJ'), \tag{5.7a}$$

$$\bar{D}_{\dot{1}t}(J'^{-1}J) = \xi(J^{-1}\bar{D}_{\dot{2}t}J - J'^{-1}\bar{D}_{\dot{2}t}J'), \tag{5.7b}$$

$$\partial_{2\dot{1}}(J'^{-1}J) = \xi(J^{-1}\partial_{2\dot{2}}J - J'^{-1}\partial_{2\dot{2}}J'), \tag{5.7c}$$

$$\partial_{1\dot{1}}(J'^{-1}J) = \xi(J^{-1}\partial_{1\dot{2}}J- J'^{-1}\partial_{1\dot{2}}J') \tag{5.7d}$$

where $\zeta\eta = \xi^2$.

Note that these BT transformations given by Eqs. (5.7) coincide precisely with those given in Ref. 16 for the ordinary SDYM fields.

VI. Discussions on the Constraint Equations and Equations of Motion

The main purpose of the constraints Eqs. (1.2) are to eliminate fields that are unphysical, or to make sure that fields in the super dimensions are to be determined in terms of fields in the ordinary space. Witten [39] gave a geometric description of how these constraint equations come about: let's consider light like lines in the ordinary space

$$x^{\alpha\dot{\alpha}} = C^{\alpha\dot{\alpha}} + t\, \lambda^{\alpha}\bar{\lambda}^{\dot{\alpha}}, \tag{6.1}$$

where λ^{α} with $\alpha = 1, 2$ is an arbitrary pair of complex numbers, $c^{\alpha\dot{\alpha}}$ and λ^{α}'s are given but t arbitrary. Translation in the light-like direction $\lambda^{\alpha}\bar{\lambda}^{\dot{\alpha}}$ are generated by

$$D = i \quad \lambda^{\alpha}\bar{\lambda}^{\dot{\alpha}}\, \partial/\partial x^{\alpha\dot{\alpha}}. \tag{6.2}$$

In super space, translations in light-like directions are square roots of D. In fact

$$T \equiv (1/\sqrt{2})\ (\lambda^{\alpha}D^{s}_{\alpha} - \bar{\lambda}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}s}) \tag{6.3}$$

gives

$$T^2 = D. \tag{6.4}$$

So we can view

$$T^{s} = \lambda^{\alpha}D^{s}_{\alpha}, \qquad \bar{T}_{s} = -\,\bar{\lambda}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}s}, \tag{6.5}$$

as the accompanying light lines in the super space.

Integrability on a line means that the translation operators along that line satisfy a commutation relation unmodified in the presence of gauge fields. Integrability along a line in the ordinary space is trivial. But integrability along a light-like line including the super directions puts restrictions on the curvature. Integrability along the generalized light-like line means [39],

$$\{T^s, T^t\} = 0 = \{\bar{T}_s, \bar{T}_t\}, \quad \{T^s, \bar{T}_t\} = 2\delta^s_t D, \tag{6.6}$$

which is true for arbitrary λ^α, $\bar{\lambda}^{\dot{\alpha}}$, thus

$$\{D^s_\alpha, D^t_\beta\} + \{D^t_\alpha + D^s_\beta\} = 0, \tag{6.7a}$$

$$\{\bar{D}_{\dot{\alpha}s}, \bar{D}_{\dot{\beta}t}\} + \{\bar{D}_{\dot{\alpha}t} + \bar{D}_{\dot{\beta}s}\} = 0, \text{ and} \tag{6.7b}$$

$$\{D^s_\alpha, \bar{D}_{\dot{\beta}t}\} = -2\delta^s_t\, \partial/\partial x^{\alpha\dot{\beta}}. \tag{6.7c}$$

In the presence of gauge fields, the unalteration of these algebras precisely gives the constraint equations Eqs. (1.2).

We can see that as the super dimension N increases, the number of constraints increases. It was pointed out by Sohnius [44] and Witten [39] that at N = 3, these constraints on curvatures via Bianchi identities actually give equations of motion [52]. This feature was considered not so desirable due to the lack of Lagrangian formulation.

However, from our point of view, for the N = 3,4 super-symmetric Yang-Mills theories, solving these constraint equations might lead to the solutions of the full supersymmetric N = 3,4 Yang-Mills theories.

VII. Outlook

Equations of motion for N = 3,4 supersymmetric Yang-Mills fields are results of integrable conditions in the generalized light-like directions in super coordinates. This gives the hope that the theory can be solved using the integration technique for non-linear systems. The important directions to pursue are: firstly, to see what kind of classical solutions the finite R-H transforms, and the BT can provide; secondly, to study the quantum inverse scattering for these systems, [53].

Figure 1. Some Generic Structures of integrable non-linear systems

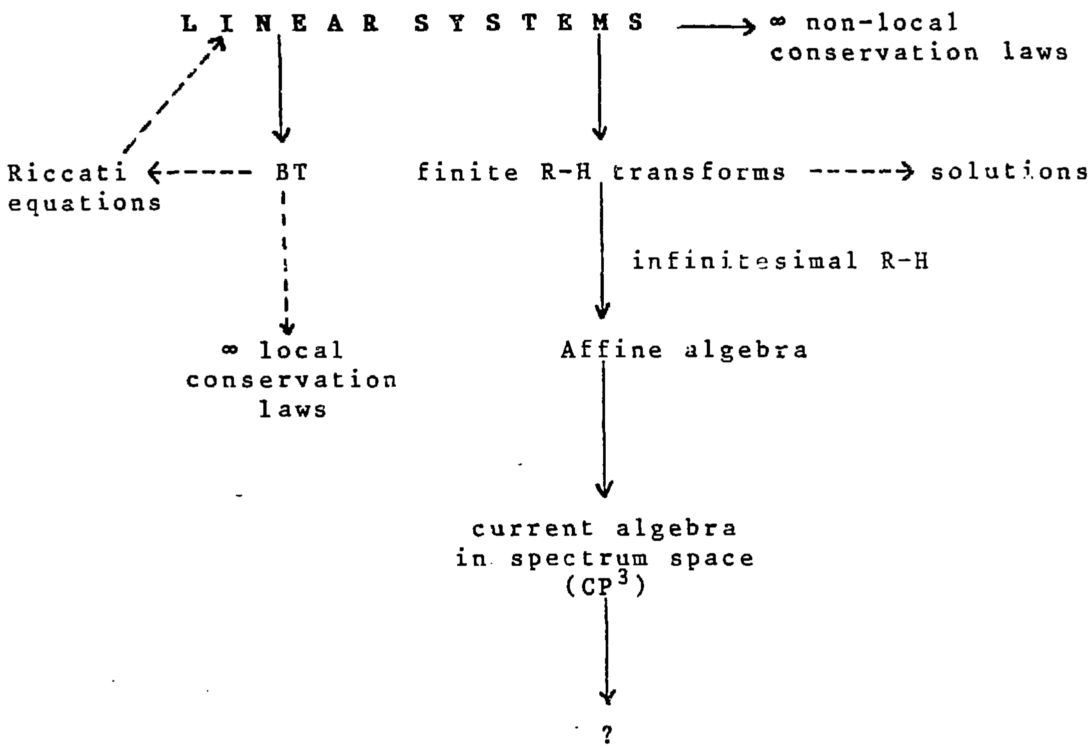

Figure Caption: The solid lines indicate developments made for the four dimensional self-dual Yang-Mills Fields, and for the two dimensional chiral fields, and the static axially symmetric Einstein equations (i.e., the Ernst equations). The dashed lines indicate developments made only for the latter two dimensional systems.

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