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PAULI-GÜRSEY SYMMETRY IN GAUGE THEORIES

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Abstract

Gauge theories with massless or massive fermions in a selfcontragredient representation exhibit global symmetries of Pauli-Gürsey type. Some of them are broken spontaneously leading to a difermion Goldstone bosons. An example of a boson version of the Pauli-Gürsey symmetry is provided by the Weinberg-Salam model in the limit $\theta_w \to 0$.

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In 1957, Pauli[1] has discovered that the free massless Dirac-field Lagrangian is invariant under the SU(2)-group of canonical transformations

$$\Psi \rightarrow a\Psi + b\gamma_5 \Psi^c, \quad |a|^2 + |b|^2 = 1, \tag{1}$$

where Ψ^c stands for the charge conjugated field. These transformations are generated by the fermion-number current $\bar{\Psi}\gamma_\mu\Psi$ together with two conserved currents $\bar{\Psi}\gamma_\mu\gamma_5\Psi^c$ and $\bar{\Psi}^c\gamma_\mu\gamma_5\Psi$ that change the fermion-number by ± 2. The meaning of these transformations was clarified and the scope of their applications was extended by Gürsey[2]. In an unpublished paper, Heisenberg and Pauli[3] have observed that the four-fermion interaction $(\bar{\Psi}\gamma_\mu\gamma_5\Psi)^2$ remains invariant under (1) and they have speculated about a possible interpretation of isospin as the Pauli-Gürsey symmetry (1). Subsequently, this idea has been used by Heisenberg and collaborators in one version of their "unified theory"[4]. Since that times, the Pauli-Gürsey symmetry (1) has apparently been forgotten (for an exception see ref. [5]) or, at least, it was not generally recognized to play a notable role in more recent renormalizable models of Particle Physics. It is a purpose of this letter to draw attention to the fact that certain standard renormalizable gauge-theories do actually possess symmetries of the Pauli-Gürsey type (1) and to point out some obvious consequences of this fact.

We shall consider left-right symmetric (non chiral) gauge theories with massive fermions that conserve the fermion number. Theories with massless fermions will be considered only if they can be defined as the zero-mass limit of corresponding massive theories. In such theories, fermions may always be described by Dirac fields. Let the Lagrangian be

$$\mathcal{L} = \mathcal{L}_{YM} + \sum_{j=1}^{N} \left\{ \bar{\Psi}_j i\gamma^\mu \overleftrightarrow{D}_\mu \Psi_j - m \bar{\Psi}_j \Psi_j \right\} , \tag{2}$$

where (for each j) Ψ_j denotes a multiplet of Dirac-fermions that transform as a representation t^a of the algebra of the gauge group G : $(D_\mu \Psi_j)_\alpha = (\delta_{\alpha\beta}\partial_\mu - ig A^a_\mu t^a_{\alpha\beta}) \Psi_{j\beta}$. (For definitness, the index $j = 1 \ldots N$ will be called "flavor" and the gauge-group index α will be refered to as "color".) What are all (linear) global symmetries of this Lagrangian, that commute with the gauge-symmetries and with the continuous (proper) space-time symmetries ? It is well known that for $m = 0$ such symmetries involve the chiral group $SU_L(N) \times SU_R(N) \times U_V(1)$ which is broken by the mass term to $U_V(N)$. (The flavor-singlet $U_A(1)$ is broken by the axial anomaly.) We are going to show that provided the fermions are in a self-contragredient representation of the gauge group G, i.e. if there exists a unitary matrix Λ such that

$$t^{a\,T} = - \Lambda^+ t^a \Lambda \quad , \quad \Lambda^+ \Lambda = 1 , \tag{3}$$

the Lagrangian (2) does exhibit a global symmetries of the Pauli-Gürsey type (1), *in addition to the standard symmetries mentioned above.* Selfcontragredient representations involve all representations of the groups SU(2), O(2n+1), O(4n), Sp(n), E_7, E_8, F_4 and G_2 and some representations of the remaining (semi) simple compact Lie groups[6]. For irreducible representations, Λ is unique up to a phase and eq.(3) implies

$$\Lambda^T \Lambda^{-1} = \epsilon = \pm 1 \; . \tag{4}$$

If $\epsilon = +1$, the corresponding selfcontragredient representation is called orthogonal, if $\epsilon = -1$, it is called symplectic. Orthogonal representations are real, but symplectic representations are complex : the simplest example of a symplectic selfcontragredient representation is the fundamental representation of SU(2) - in this case $t^a = \frac{1}{2}\tau^a$, where τ^a are the Pauli matrices and $\Lambda = \tau^2$. In table I we list the lowest symplectic representations of simple compact Lie groups of rank ≤ 8 (this table is extracted from ref.[6]).

In order to demonstrate the above claim, it is useful to replace in the Lagrangian (2) each Dirac field Ψ_j by a pair of left-handed Weyl spinors φ_j and χ_j defined as follows. In the chiral representations of γ-matrices

$$\gamma_\mu = \begin{pmatrix} 0 & \tilde{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \tag{5}$$

where $\sigma_\mu \equiv (1, \vec{\sigma})$ and $\tilde{\sigma}_\mu \equiv (1, -\vec{\sigma})$, one writes Ψ_j as

$$\Psi_j = \begin{pmatrix} \varphi_j \\ \hat{\chi}_j \end{pmatrix} . \tag{6}$$

The right-handed component $\hat{\chi}_j$ may be viewed as a "hat-conjugation" of a left-handed Weyl spinor χ_j :

$$\hat{\chi}_{j\alpha\rho} = \Lambda_{\alpha\beta}\, \epsilon_{\rho s}\, \chi^{+}_{j\beta s} \; , \tag{7}$$

where $\epsilon_{\rho s} = -\epsilon_{s\rho}$, $\epsilon_{12} = 1$ and j, α, r stand for "flavor", "color" and spinor indices respectively. The left-handed spinors φ_j and χ_j so defined carry opposite fermion number. Furthermore, due to the factor Λ in eq.(7), φ_j and χ_j transform identically under a self-contragredient representation $\Omega = \exp i\omega^a t^a$ of the gauge group : if $\Psi_j \to \Omega \Psi_j$, i.e. $\varphi_j \to \Omega \varphi_j$ and $\hat{\chi}_j \to \Omega \hat{\chi}_j$, eq.(7) implies $\chi_j \to \Omega \chi_j$, provided the representation t^a satisfies the constraint (3). Similarly, it will prove useful to include the "color" factor Λ into the definition of the charge-conjugated Dirac-spinor Ψ_j^c:

$$\Psi_j^c = (\Lambda \times C)\bar{\Psi}_j^T = \begin{pmatrix} \epsilon \chi_j \\ \hat{\varphi}_j \end{pmatrix}, \tag{8}$$

where $\epsilon = \pm 1$ is the sign defined by eq.(4). This again assures the identical transformation of Ψ_j and Ψ_j^c under the gauge group. It should be noted that with this modified definition of Ψ^c, the Pauli-Gürsey transformation (1) remains canonical only for $\epsilon = +1$. In the <u>symplectic case $\epsilon = -1$</u>, eq.(1) should be replaced by

$$\Psi_j \to a\,\Psi_j - b\,\Psi_j^c, \quad |a|^2 + |b|^2 = 1 \,. \tag{1'}$$

Both Pauli-Gürsey transformations (1) (for $\epsilon = +1$) and (1') (for $\epsilon = -1$) are now canonical, they commute with the gauge transformations and they can be both reexpressed as ϵ-independent SU(2)-rotation [2]

$$\begin{pmatrix} \varphi_j \\ \chi_j \end{pmatrix} \to \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} \varphi_j \\ \chi_j \end{pmatrix}. \tag{9}$$

Using all these definitions, the Lagrangian (2) can be written as

$$\mathcal{L} = \mathcal{L}_{YM} + \sum_{j=1}^{N} \{ \varphi_j^+ i \sigma^\mu \overleftrightarrow{\mathcal{D}}_\mu \varphi_j + \chi_j^+ i \sigma^\mu \overleftrightarrow{\mathcal{D}}_\mu \chi_j - m(\varphi_j^+ \chi_j + \chi_j^+ \varphi_j) \} \qquad (10)$$

The covariant derivatives acting on φ and χ are <u>identical</u> : $(\overleftrightarrow{\mathcal{D}}_\mu)_{\alpha\beta} = \delta_{\alpha\beta} \overleftrightarrow{\partial}_\mu - ig\, A_\mu^a t_{\alpha\beta}^a$. In this form it becomes manifest that <u>for m = 0 the Lagrangian (2) exhibits the global symmetry SU(2N)</u>. The standard chiral-symmetry group $SU_L(N) \times SU_R(N) \times U_V(1)$ generated by the usual vector and axial currents ($a = 1 \ldots N^2-1$, and λ_a are "flavor" SU(N) - matrices)

$$\begin{aligned} V_a^\mu &= \bar{\Psi} \gamma^\mu \tfrac{\lambda_a}{2} \Psi = \varphi^+ \sigma^\mu \tfrac{\lambda_a}{2} \varphi - \chi^+ \sigma^\mu \tfrac{\lambda_a^T}{2} \chi \\ j^\mu &= \bar{\Psi} \gamma^\mu \Psi \;\; = \varphi^+ \sigma^\mu \varphi \;\; - \chi^+ \sigma^\mu \chi \\ A_a^\mu &= \bar{\Psi} \gamma^\mu \gamma^5 \tfrac{\lambda_a}{2} \Psi = \varphi^+ \sigma^\mu \tfrac{\lambda_a}{2} \varphi + \chi^+ \sigma^\mu \tfrac{\lambda_a^T}{2} \chi , \end{aligned} \qquad (11)$$

is obviously a subgroup of the whole symmetry-group SU(2N). In addition, SU(2N) contains the Pauli-Gürsey transformations (1) or (1') and their flavor-changing generalization : the corresponding Noether-currents are (i,j = 1...N)

$$\begin{aligned} \mathcal{V}_{ij}^\mu &= \bar{\Psi}_i \gamma^\mu \gamma^5 \Psi_j^c = \epsilon \varphi_i^+ \sigma^\mu \chi_j + \varphi_j^+ \sigma^\mu \chi_i = \epsilon\, \mathcal{V}_{ji}^\mu \\ \mathcal{A}_{ij}^\mu &= \bar{\Psi}_i \gamma^\mu \Psi_j^c = \epsilon \varphi_i^+ \sigma^\mu \chi_j - \varphi_j^+ \sigma^\mu \chi_i = - \epsilon\, \mathcal{A}_{ji}^\mu \end{aligned} \qquad (12)$$

together with their hermite-conjugates. The $4N^2-1$ SU(2N) currents (11) and (12) are all gauge invariant and they are free of any gauge-group anomaly. The currents $\mathcal{V}^\mu$ are vectors, while $\mathcal{A}^\mu$ are axial-vectors, (note that under parity $\varphi \to \hat{\chi}$ and $\chi \to -\epsilon\hat{\varphi}$). Under the (modified) charge-conjugation (8) ($\varphi \to \epsilon\chi$, $\chi \to \varphi$) one has $\mathcal{V}^\mu \to \mathcal{V}^{\mu+}$ and $\mathcal{A}^\mu \to -\mathcal{A}^{\mu+}$.

Besides the chiral-symmetry group $SU_L(N) \times SU_R(N) \times U_V(1)$, the group SU(2N) contains two important subgroups that are related to the symmetry properties of the mass-term $\bar{\Psi}\Psi$: one is generated by the set of charges $G_{\mathcal{V}}^{(\epsilon)} = (\mathcal{Z}, V_a, \mathcal{V}_{ij}, \mathcal{V}^+_{ij})$, the other is generated by the set $G_{\mathcal{A}}^{(\epsilon)} = (\mathcal{Z}, V_a, \mathcal{A}_{ij}, \mathcal{A}^+_{ij})$. If the fermion representation is orthogonal (ϵ =+1), the Pauli-Gürsey charges $\mathcal{V}_{ij}$ are symmetric and $\mathcal{A}_{ij}$ are anti-symmetric (cf. eq. (12)). In this case, $G_{\mathcal{V}}^{(\epsilon=+1)}$ consists of N(2N+1) charges and it is identified[7] to be the symplectic group of rank N, $G_{\mathcal{V}}^{(\epsilon=+1)} \equiv Sp(N) \subset SU(2N)$, while $G_{\mathcal{A}}^{(\epsilon=+1)}$ contains N(2N-1) charges generating[7] the orthogonal group $G_{\mathcal{A}}^{(\epsilon=+1)} \equiv O(2N) \subset SU(2N)$. If, on the other hand, the fermions are in a symplectic representation (ϵ =-1), the role of the groups $G_{\mathcal{V}}$ and $G_{\mathcal{A}}$ is reversed : $G_{\mathcal{V}}^{(\epsilon=-1)} \equiv O(2N)$ and $G_{\mathcal{A}}^{(\epsilon=-1)} \equiv Sp(N)$. The actual symmetry group of $\bar{\Psi}\Psi$ is now easily obtained, if one calculates the divergences of the Pauli-Gürsey currents (12) in the presence of the fermion mass. One obtains

$$\partial_\mu \mathcal{A}^\mu_{ij} = 0 \quad , \quad \partial_\mu \mathcal{V}^\mu_{ij} = 2im\,\bar{\Psi}_i \gamma_5 \Psi^c_j \qquad (13)$$

valid to all orders of perturbation theory in a complete analogy with the conservation of the vector currents $\mathcal{Z}^\mu$, V^μ_a and with the partial conservation of the axial-vector currents A_a^{μ}. (In particular, the currents $\mathcal{V}^{\mu}$ and $\mathcal{R}^{\mu}$ have vanishing anomalous dimensions). We thus see that even for massive fermions the actual global symmetry of the Lagrangian (2) is larger than the usual $U_V(N)$: it is generated by the set $G_{\mathcal{R}}^{(\epsilon)}$ that contains, in addition to the usual vector charges, the Pauli-Gürsey charges $\mathcal{R}_{ij}$, $\mathcal{R}^{+}_{ij}$. In the case of an orthogonal representation (ϵ =+1), the symmetry group of the mass term is $O(2N) \supset U_V(N)$ and in the symplectic case (ϵ =-1) it is $Sp(N) \supset U_V(N)$. (If the fermions are in a symplectic representation, the theory exhibits a non-trivial global symmetry even if one has a single "flavor" (N=1) : $Sp(1) \simeq SU(2)$ is then a symmetry of the Lagrangian (2) irrespectively of whether $m=0$ or $m\neq 0$.)

We note in passing that all statements made above concerning the orthogonal case ϵ =+1 also apply to the system of N free Dirac-fields : such a system may be viewed as a gauge theory with fermions in the trivial representation and the trivial representation of any group is certainly selfcontragredient with $\Lambda \equiv 1$, i.e. ϵ =+1. Consequently, the system of N free Dirac-fermions of a common mass m exhibits a symmetry O(2N) that becomes SU(2N) as $m \to 0$. (The symmetry $SU(2) \times SU(2) \simeq O(4)$ of two free massive fermions has already been remarked by Gürsey[21]). It might be interesting to investigate theories obtained by gauging these Pauli-Gürsey symmetries of the free Lagrangian. If one gauges the symmetry O(2N) of the mass-term, one obtains a left-right symmetric theory of N massive Dirac-fields that is anomaly free and for which the fermion-number current $\mathcal{Q}^{\mu}$ is no longer conserved since there exist difermion gauge-fields : the fermion-number $\mathcal{Q}$ becomes one of the non-Abelian generators of the gauge-group O(2N).

(If, on the other hand, one would gauge a subgroup of SU(2N) larger than O(2N), thereby forbidding a mass term to develop, one might - at least in some cases - end up with theories whose quantum version presumably suffers from the disease recently discovered by Witten[8].)

Returning to the global Pauli-Gürsey symmetries of the Lagrangian (2), let us exhibit an interesting application : the spectrum of zero-mass bound-states of the corresponding massless theory that results from the spontaneous breakdown of the m = 0 symmetry SU(2N). One may first state the following

Theorem : Let the fermions (N>1) be in a symplectic representation of the gauge group and let the theory (2) confine its "color". Then the corresponding massless theory necessarily contains zero-mass spinless difermion bound-states that couple to the Pauli-Gürsey currents $\mathcal{V}^{\mu}$ and/or $\mathcal{A}^{\mu}$.

The proof may be sketched in three steps. i) Let $H \subset SU(2N)$ be the subgroup that (as $m \to 0$) remains unbroken spontaneously. If all Pauli-Gürsey charges $\mathcal{V}_{ij}$, $\mathcal{A}_{ij} \in H$, then $H = SU(2N)$, i.e. the whole symmetry SU(2N) remains unbroken : this follows from the fact that successive commutators of $\mathcal{V}$, $\mathcal{A}$, $\mathcal{V}^{+}$ and $\mathcal{A}^{+}$ yield the whole algebra of SU(2N). In particular, the spontaneous breakdown of the chiral symmetry $[SU_L(N) \times SU_R(N) \times U_V(1)] \subset SU(2N)$ cannot occur without some spontaneous breakdown of the Pauli-Gürsey symmetry. ii) Certain three-point functions made from the SU(2N) currents (11) and (12) develop the Adler-Bell-Jackiw anomaly. In addition to the well known anomalies that concern the chiral-symmetry currents (11), one now has new anomalies in the three-point functions $\langle T(\mathcal{V}\mathcal{V}^{+}A)\rangle$, $\langle T(\mathcal{A}\mathcal{A}^{+}A)\rangle$ and $\langle T(\mathcal{V}\mathcal{A}^{+}V)\rangle$. ($\langle T(\mathcal{V}\mathcal{A}^{+}\mathcal{Z})\rangle$ is anomaly free). The corresponding (m=0) Ward-identities imply that the theory contains massless states[9] of spin 0 and/or 1/2[10] .

iii) The product of an odd number of symplectic representations cannot contain the singlet, (cf. the spinorial representations of SU(2)). Consequently, provided the theory we consider confines its "color" it cannot contain (bound) states of half-integer spin. It follows that the anomalous Ward-identities have to be saturated by spin-0 Goldstone bosons that result from a spontaneous breakdown of the symmetry SU(2N). According to i), this set of Goldstone bosons must contain difermions coupled to $\mathcal{A}^\mu$ and/or $\mathcal{V}^\mu$.

If one makes the usual assumptions that, like in QCD, the spontaneous breakdown of chiral symmetry is flavor independent, i.e. $U_V(N) \subset H$ (cf. ref. [11]), then one remains with only three possibilities : i) H = Sp(N), ii) H = O(2N) and iii) H = $U_V(N)$. (This is a consequence of irreducibility of $\mathcal{V}_{ij}$ and $\mathcal{A}_{ij}$ under $SU_V(N)$). If, furthermore, the spontaneous breakdown proceeds via formation of the fermion-antifermion condensate $<\bar{\Psi}\Psi>$, the case ii) is excluded (for fermions in a symplectic representation) and H = Sp(N) is most likely to happen. In this case the set of Goldstone bosons spans the (N-1)(2N+1) dimensional symmetric space SU(2N)/Sp(N). It consists from the usual (N^2-1) $J^P=0^-$ "pions" and from N(N-1) $J^P=0^+$ massless difermions. Let us conclude this discussion with an example of model "QCD" with two colors and two flavors, (the gauge group is SU(2) with fermions in the fundamental representation that is symplectic). As $m \to 0$, one expects the symmetry SU(4) = O(6) to break down to Sp(2) ≅ O(5). Under this group the Goldstone bosons form an irreducible quintuplet, whose explicit content may be read off from the partial conservations of the axial-currents A_a^μ (a=1,2,3) and of the Pauli-Gürsey current $\mathcal{V}^\mu_{12} = - \mathcal{V}^\mu_{21}$, cf. eq. (13) : the corresponding local operators representing this quintuplet are

$$B = \bar{\Psi}_1 i\gamma_5 \Psi_2^c , \quad \pi^a = \bar{\Psi} i\gamma_5 \frac{\tau^a}{2} \Psi , \quad B^+ = \bar{\Psi}_2^c i\gamma_5 \Psi_1 . \tag{14}$$

We thus see that in "QCD" with two colors, the lowest baryon-states are 0^+ Goldstone bosons arising from the spontaneous breakdown of the Pauli-Gürsey symmetry.

Finally, it is worth mentioning that the Pauli-Gürsey symmetry may be extended to complex scalar fields that transform as a selfcontragredient symplectic representation of a gauge group. Let us consider a gauge theory

$$\mathcal{L} = \mathcal{L}_{YM} + \frac{1}{2}(D_\mu \Phi)^+ (D^\mu \Phi) - V(\Phi^+ \Phi) , \tag{15}$$

where $\Phi \equiv \{\Phi_\alpha\}$ form a single gauge multiplet : $(D_\mu \Phi)_\alpha = (\delta_{\alpha\beta}\partial_\mu - i g A_\mu^a t^a_{\alpha\beta})\Phi_\beta$. Provided the representation t^a is symplectic, the Lagrangian (15) is invariant under the SU(2)-group of global transformations

$$\Phi \to a\Phi - b\Phi^c , \quad |a|^2 + |b|^2 = 1 , \quad \Phi^c_\alpha = \Lambda_{\alpha\beta} \Phi^+_\beta \tag{16}$$

that may be viewed as an analog of the Pauli-Gürsey transformations (1') for fermions in a symplectic representation : i) The transformations (16) are indeed canonical if and only if $\Lambda^T = -\Lambda$. ii) They obviously commute with the gauge transformations $\Phi \to (\exp i\omega^a t^a)\Phi$. iii) The manifest U(1)-global symmetry of (15), cf. $\Phi \to e^{i\omega}\Phi$, is contained in the SU(2) group (16). It is seen that, indeed, the analogy is complete. An interesting illustration of this boson-version of the Pauli-Gürsey symmetry is provided by the

Weinberg-Salam model in the limit of vanishing Weinberg angle : in this case one has a doublet of complex scalars and the gauge group is SU(2), (the U(1)-gauge boson decouples as $\theta_w \to 0$). This model exhibits a symmetry $SU(2)_{Local} \times SU(2)_{Global}$ and actually, it may be rewritten as the Gell-Mann-Levy chiral σ -model with the "left-handed" SU(2)-symmetry gauged. Since in the Higgs phase all Goldstone bosons are eaten up by the gauge-fields, the global Pauli-Gürsey SU(2)-symmetry survives the spontaneous breakdown and it becomes a classification symmetry group of the physical states of the theory. (In Nature, this symmetry is, of course, slightly explicitly broken by $\theta_w \neq 0$). We have argued elsewhere[12] that this "hidden" Pauli-Gürsey symmetry makes it possible and natural to reinterpret the Weinberg-Salam model as an effective low-energy description of bound states of a "more fundamental preon-gauge-theory".

The question whether some of the gauge theories that are concerned by the present work will actually prove usefull in Particle Physics (beyond their use as toy-models) remains, of course, completely open. The routes to simplicity and/or naturalness[9], especially those that introduce substructures of quarks, leptons and gauge bosons, still require an examination of a large variety of possible models. In this context, some generalisation of the idea of Heisenberg and his collaborators[4] to use Pauli-Gürsey symmetries as a gain of economy in explaining the inflation of particle-species might eventually find its application.

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Rank	Group	Symplectic representations
1	SU(2)	all spinorial
2	Sp(2) ≃ O(5)	{4}, {16}, {20}, {40}, {56}, ...
3	Sp(3)	{6}, $\{14\}_1$, {56}, {64}, ...
4	Sp(4)	{8}, {48}, {120}, ...
5	SU(6)	{20}, {540}, ...
	O(11)	{32}, {320}, ...
	Sp(5)	{10}, {110}, ...
6	O(12)	{32}, {352}, ...
	O(13)	{64}, {768}, ...
	Sp(6)	{12}, {208}, ...
7	E_7	{56}, ...
	Sp(7)	{14}, {560}, ...
8	Sp(8)	{16}, {544}, ...

Table 1

Table caption : Lowest selfcontragredient symplectic representations of simple compact Lie groups of rank ≤ 8 (following ref. [6])