INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

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INVARIANT BOXES AND STABILITY OF SOME SYSTEMS FROM BIOMATHEMATICS AND CHEMICAL REACTIONS *

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1. INTRODUCTION

In the first part of this paper (Sec.2) we recall a general result (Theorem 2.1) on the flow-invariance of a time-dependent rectangular box with respect to a differential system [1]. In Sec.3.1 we apply Theorem 3.1 to the study of some differential systems from biomathematics and chemical reactions. In Sec.4 we characterize the matrices A for which the corresponding linear system $x' = Ax$ is component-wise positive asymptotically stable (CWPAS). For the proof of Theorem 2.1, as well as for other problems in flow-invariance theory, the reader may consult the papers in the references, in particular Refs.[4], [5] and [7]. In the Appendix we give a partial answer to an open problem in the theory of the flow-invariance of a set with respect to an abstract differential equation in a Banach space.

2. STATEMENT OF THE GENERAL RESULT

Let us consider the differential system

$$x_i' = f_i(t,x_1,\dots,x_m) \quad , \ t \geq 0 \tag{2.1}$$

with the initial condition

$$x_i(t_0) = x_i^0 \quad , \quad x_i^0 \in R \quad , \quad i = 1,\dots,m \tag{2.2}$$

where $f_i : [0, +\infty) \times R^m \to R$ are continuous and guarantee the uniqueness of solution to the Cauchy problems (2.1) and (2.2).

Recall that a time-dependent subset $D(t) \subset R^m$ is said to be flow-invariant (or simply-invariant) with respect to (2.1) if for each $t_0 \geq 0$ and $x^0 = (x_1^0, \dots, x_m^0) \in D(t_0)$, the corresponding solution $x = x(t)$ to (2.1) with $x(t_0) = x^0$ also satisfies

$$x(t) \in D(t) \quad , \quad t \geq t_0 \tag{2.3}$$

(i.e. $x(t)$ remains in $D(t)$ as long as it exists to the right of t_0). The characterization of t-dependent flow-invariant sets in very general cases can be found in Refs.[2], [4] and [5]. Here we recall only the particular case of rectangular boxes below

$$D(t) = \prod_1^m [a_i(t),b_i(t)] = \{ x = (x_1,\dots,x_m) \in R^m \ ; \ a_i(t) \leq x_i \leq b_i(t) \ , \ 0 = t \ , \ i = 1,\dots,m \} \ , \tag{2.4}$$

where a_i , b_i : $[0 , +\infty) \to R$ are continuously differentiable functions, with $a_i(t) < b_i(t)$, for all $i = 1,\dots, m$ and $t \geq 0$. The characterization of the flow-invariance of the above $D(t)$ is given by the following theorem (for its proof see Pavel [3], [4] or Voicu [7]).

<u>Theorem 2.1</u> Let $f_i : [0 , +\infty) \times R^m \to R$ and a_i , $b_i : [0 , +\infty) \to R$ satisfy the above hypotheses. Then for each $t_0 \geq 0$ and $x^0 = (x_1^0 ,\dots, x_m^0)$ with $a_i(t_0) \leq x_i^0 \leq b_i(t_0)$, $i = 1, 2,\dots, m$ the corresponding solution $x = (x_i(t),\dots, x_m(t))$ to the problems (2.1) and (2.2) satisfies

$$a_i(t) \leq x_i(t) \leq b_i(t) \ , \forall t \geq t_0 \ , \ i = 1,\dots,m \tag{2.5}$$

if and only if

$$\begin{aligned} f_i(t,x_1,\dots,x_{i-1},a_i(t),x_{i+1},\dots,x_m) &\geq a_i'(t) \\ f_i(t,x_1,\dots,x_{i-1},b_i(t),x_{i+1},\dots,x_m) &\leq b_i'(t) \end{aligned} \tag{2.6}$$

$\forall t \geq t_0$; $i = 1,\dots, m$; $x_j \in [a_j(t) , b_j(t)]$; $j = 1,\dots, i-1, i+1,\dots m$ (with the convention $x_0 \equiv x_1$, $x_{m+1} \equiv x_m$).

3. INVARIANT RECTANGULAR BOXES

We first consider the enzymatic reactions

$$E + S \rightleftarrows C \longrightarrow E + P \quad ,$$

where E, S, C and P denote enzyme, substrate complex and product, respectively. It is known that the evolution of the substrate and complex concentrations (denoted by $x_1(t)$ and $x_2(t)$, respectively) is described by the following system (see Rubinov [6])

$$x_1' = -x_1 + (x_1 + a)x_2 \qquad (3.1)$$
$$r\,x_2' = x_1 - (x_1 + b)x_2 \quad ,$$

where $0 < a \leq b$ and r is a small positive constant.

Another system we want to study here is the system of Field and Noyes (see Ref.[4] for details)

$$x_1' = s(x_2 - x_1x_2 - x_1 - qx_1^2)$$
$$x_2' = s^{-1}(fx_3 - x_2 - x_1x_2) \qquad (3.2)$$
$$x_3' = w(x_1 - x_3) \quad ,$$

where s, w, f and q are positive constants. It is essential to note that q is less than one (precisely $q = 8375 \cdot 10^{-6}$). The differential system (3.2) is the kinetic system for the five irreversible steps model of Field and Noyes

$$A + Y \to X \; ; \quad X + Y \to P \; ; \quad B + X \to 2X + Z \qquad (3.3)$$
$$2X \to Q \; ; \quad Z \to fY$$

where f is a stoichiometric factor, P, Q products and

$X = HBrO_2$, $Y = Br^-$, $Z = Ce$ (IV) , $A = B = BrO_3^-$.

Actually, (3.3) is the quantitative model for the chemical mechanism in the Belousov reaction.

The third non-linear system we want to discuss here is

$$x_1' = x_1 - \frac{1}{10}x_1^2 - x_1x_2 + \frac{1}{10}x_2^2$$
$$x_2' = x_1x_2 - \frac{1}{10}x_2^2 - x_2x_3 + \frac{1}{10}x_3^2 \qquad (3.4)$$
$$x_3' = x_2x_3 - \frac{1}{10}x_3^2 - x_1x_3 + \frac{5}{100}x_1$$

This is the kinetic system for the chemical scheme (of Hanusse see Ref.[4]) of two processes A and B with three intermediate species X , Y , Z. In connection with (3.1) it was proved in Refs.[3] and [4] that all rectangles of the form

$$D_2 = [0,b_1]\times[0,b_2] \quad , \quad b_1, b_2 > 0 \qquad (3.5)$$

are given by

$$ab_2(1-b_2)^{-1} \leq b_1 \leq bb_2(1-b_2)^{-1} \qquad (3.6)$$

for all $b_2 \in (0\,,\,1)$. This can be proved by applying Theorem 2.1 with $m = 2$, $a_1(t) = a_2(t) = 0$, $b_1(t) = b_1$ = const , $b_2(t) = b_2$ = const. for all $t \geq 0$.

It was shown by Murray (see Ref.[4]) that the rectangular boxes

$$D_3 = \left\{ (x_1,x_2,x_3) \; ; 1 \leq x_1 \leq q^{-1},\; y_1 \leq x_2 \leq y_2 \; ; \; 1 \leq x_3 \leq q^{-1} \right\} \qquad (3.7)$$

have the property that every solution to (3.2) in the positive octant R_3^+ eventually enters the box D_3 and no trajectory can leave D, provided that

$$y_1 = qf/(1+q) \quad , \quad y_2 = f/2q \quad . \qquad (3.8)$$

In what follows we shall give more precise informations on the systems above. To do this, let us also consider the rectangular box

$$\tilde{D}_3 = \left\{(x_1,x_2,x_3) ; \; \frac{1}{3} \leq x_1 \leq 10, \frac{1}{8} \leq x_2 \leq 100 \; , \; \frac{1}{20} \leq x_3 \leq 1000 \right\} \; . \qquad (3.9)$$

The main result of this section is given by

<u>Theorem 3.1</u> 1) The positive octant R_3^+ is a flow-invariant set with respect to each of the systems (3.1) , (3.2) and (3.4). 2) The necessary and sufficient condition in order for the system (3.1) to admit an invariant set of the form

$$D_2 = [a_1,b_1]\times[a_2,b_2] \quad , \quad 0 < a_i < b_i \quad , \; i = 1,2 \qquad (3.10)$$

is $a = b$. All rectangles of the form (3.10) which are invariant with respect to (3.1) are given by

$$a_2 = a_1 (a_1 + a)^{-1} , \quad b_2 = b_1 (b_1 + a)^{-1} \tag{3.11}$$

with arbitrary $a_1 , b_1 > 0$. 3) All rectangular boxes D_3 of the form (3.7) which are invariant with respect to (3.2) correspond to

$$y_1 \leq qf/(1 + q) , \quad y_2 \leq f/2q . \tag{3.12}$$

4) The set $\hat{D}_3$ (see (3.9) is invariant with respect to the system (3.4).

<u>Proof</u> We shall apply Theorem 2.1. For the part 1), set $x_1 = 0$ in the right-hand side of the first equation of (3.4) (e.g.). We get $\frac{1}{10} x_2^2$ which is greater (or equally) than zero for all x_2 (in particular for $x_2 \geq 0$) and so on, 2) In this case, by Theorem 2.1, the invariances of $\tilde{D}_2$ is equivalent to (3.6) and

$$\begin{aligned} 0 &\leq -a_1 + (a_1 + a)x_2 , \quad \forall x_2 \in [a_2 , b_2] \\ 0 &\leq x_1 - (x_1 + b)a_2 , \quad \forall x_1 \in [a_1 , b_1] . \end{aligned} \tag{3.13}$$

Obviously, (3.13) are equivalent to

$$a_1(a_1 + a)^{-1} \leq a_2 \leq a_1(a_1 + b)^{-1}$$

which implies $a \geq b$. But (3.6) gives $a \leq b$ so $a = b$ and the result follows. 3) One applies Theorem 2.1 with $a_1 = a_3 = 1$, $a_2 = y_1$, $b_1 = b_3 = 1/q$, $b_2 = y_2$ and $n = 3$. Now the conditions (2.6) become

$$fx_3 \geq (1 + x_1)y_1 , \quad fx_3 \leq (1 + x_1)y_2$$

for all $1 \leq x_1 \leq 1/q$ and $1 \leq x_3 \leq 1/q$ which means just (3.12). 4) In this case we apply Theorem 2.1 with $n = 3$, $a_1 = \frac{1}{3}$, $b_1 = 10$, $a_2 = \frac{1}{8}$, $b_2 = 100$, $a_3 = \frac{1}{20}$ and $b_3 = 1000$. The tangential conditions (2.6) become

$$\text{(C1)} \quad \frac{1}{3} - \frac{1}{10}\frac{1}{9} - \frac{1}{3} x_2 + \frac{1}{10} x_2^2 \geq 0$$

$$\text{(C2)} \quad \frac{1}{8} x_1 - \frac{1}{10}\frac{1}{64} - \frac{1}{8} x_3 + \frac{1}{10} x_3^2 \geq 0$$

$$\text{(C3)} \quad \frac{1}{20} x_2 - \frac{1}{10}\frac{1}{400} - \frac{1}{20} x_1 + \frac{5}{100} x_1 \geq 0$$

$$\text{(C4)} \quad -10 x_2 + \frac{1}{10} x_2^2 \leq 0$$

$$\text{(C5)} \quad 100x_1 - 1000 - 100x_3 + \frac{1}{10} x_3^2 \leq 0$$

$$\text{(C6)} \quad 1000x_2 - 10^5 - 10^3 x_1 + \frac{5}{100} x_1 \leq 0$$

for all $x_1 \in \left[\frac{1}{3} , 10\right]$, $x_2 \in \left[\frac{1}{8} , 100\right]$, $x_3 \in \left[\frac{1}{20} , 1000\right]$. The inequality (C1) holds true since

$$\min_{x_2 \in \mathbb{R}} \left\{ \frac{1}{10} x_2^2 - \frac{1}{3} x_2 + \frac{29}{90} \right\} = \frac{4}{40} .$$

To verify (C2) it sufficies to see that for $x_1 = 1/3$ we have

$$\frac{1}{8} \cdot \frac{1}{3} \geq \frac{1}{8} \max\left\{ \frac{1}{640} + \frac{x_3}{8} - \frac{1}{10} x_3^2 \right\} = \frac{1}{8} \frac{13}{45} .$$

The remaining conditions above are obviously verified. The proof is complete.

<u>Corollary 3.1</u> All solutions to the systems (3.1), (3.2) and (3.4) starting from $\tilde{D}_2$ (or D_2) , D_3 and $\tilde{D}_3$ respectively are defined on the whole semiaxis $R_+ = [0 , + \infty)$.

<u>Proof</u> Let x be a solution (e.g.) to (3.4) with $x(0) = x^0 \in \tilde{D}_3$. Then $x(t) \in \tilde{D}_3$ for all $t \in [0 , t_{max})$ and therefore x is bounded on $[0 , t_{max})$. By a classical result, this implies $t_{max} = + \infty$ which concludes the proof.

Note that the assertion of Corollary 3.1 is not trivial. Indeed, even for very simple ordinary differential equations (e.g. $x' = x^2$) the solution (i.e. $x(t) = \frac{1}{1-t}$, $x(0) = 1$) may blows up in finite time. In our example here, $t_{max} = 1$.

4. COMPONENT-WISE POSITIVE ASYMPTOTIC STABILITY (CWPAS). A SPECIAL CLASS OF HURWITZ MATRICES

In this section we will study the flow-invariance of $D(t)$ in the case $a_i(t) = 0$, i.e.

$$D_0(t) = \prod_{i=1}^{m} \left[0, b_i(t)\right], \quad 0 < b_i(t), \ i = 1,\dots,m \ ; \ t \geq 0 \tag{4.1}$$

with respect to the linear homogeneous system with constant coefficients

$$x'(t) = Ax(t) \quad , \quad t = 0 \ , \tag{4.2}$$

where $A = (a_{ij})$, $a_{ij} \in R$, $i,j = 1,\dots, m$ and $x = (x_1,\dots, x_m)$. Clearly, the flow-invariance of $D_0(t)$ means that for each initial condition $x(t_0) = x^0$ with $0 \leqslant x(t_0) \leqslant b(t_0)$, the corresponding solution to (4.2) satisfies

$$0 \leq x(t) \leq b(t) \quad , \quad t \geq t_0 \ ) \tag{4.3a}$$

where $b(t) = (b_1(t),\dots, b_m(t)$. As usual this means

$$0 \leq x_i(t) \leq b_i(t) \quad , \ i = 1,\dots,m \quad . \tag{4.3b}$$

In view of Theorem 2.1 we can prove

<u>Theorem 4.1</u> The necessary and sufficient conditions for the flow-invariance of $D_0(t)$ (i.e. for (4.3a)) to be satisfied, are given by

1°) $a_{ij} \geq 0$, $i \neq j$, $i,j = 1,\dots,m$

2°) $b'(t) \geq Ab(t)$, $\forall t \geq 0$.

In the sequel we keep the condition $b_i(t) > 0$ and in addition we will introduce the asymptotic stability condition

$$\lim_{t\to\infty} b_i(t) = o \quad , \ i = 1,\dots,m \quad . \tag{4.4}$$

<u>Definition 4.1</u> The system $x' = A x$ is said to be (CWPAS) if there exists a m by 1 vector $b(t) > 0$ such that (4.3a) and (4.4) are satisfied.

The theorem below characterizes the matrices A for which $x' = A x$ is (CWPAS).

<u>Theorem 4.2</u> The system $x' = A x$ is (CWPAS) if and only if 1) $a_{ij} \geq 0$, $\forall i \neq j$, $i,j = 1,\dots, m$. 2) $a_{ii} < 0$, $i = 1,\dots, m$. 3) A has the Hurwitz's property (the real part of each eigenvalue is negative).

<u>Necessity</u> The condition 1) is equivalent to the non-negativity of solutions starting from R^m_+ (One applies (2.6) with $a_i(t) = 0$). Furthermore, from (4.4) it follows that there are $t_i > 0$ such that $b_i'(t) < 0$. By Theorem 4.1 we now have

$$0 > a_{ii}\, b_i(t_i) + \sum_{j=1}^{m} a_{ij} b_j(t_i) \quad , \quad j \neq i \ , \ i = 1,\dots,m \tag{4.5}$$

which yields 2). Clearly, 3) is a consequence of (4.3b) and (4.4) in conjunction with a classical result.

<u>Sufficiency</u> If (1)-(3) are fulfilled, then the system (4.2) is (CWPAS) with $b(t) = x_0\, e^{At}$, for each m by 1 vector $x_0 > 0$. This is because (1)-(3) implies $b(t) > 0$ as well as (4.4) (so we take $b(t) = x(t) = x_0\, e^{At}$). The proof is complete.

<u>Remark 4.1</u> Actually, (CWPAS) of (4.2) is equivalent to the existence of some $m \times 1$ vectors v of components $v_i > 0$ and of some numbers $c > 0$ such that (4.2) is (CWPAS) with $b(t) = v\, e^{-ct}$. We will simmly say that (4.2) is $(\text{CWPAS})_{cv}$. Indeed, Theorem 4.2 can be restated as:

<u>Theorem 4.3</u> The system (4.2) is (CWPAS) iff: 1) $a_{ij} \geq 0$, $\forall i \neq j$, $i,j = 1,\dots, m$. 2) $a_{ii} < 0$, $i = 1,\dots, m$. 3) $Av < 0$,

$$0 < c \leqslant \min_{i=\overline{1,m}} \left\{ -a_{ii} - \frac{1}{v_i} \sum_{j=1}^{m} a_{ij}\, v_j \ ; \ j \neq i \right\} .$$

<u>Remark 4.2</u> It follows that under hypotheses 1) and 2), the matrix A has the Hurwitz's property if and only if there are some m by 1 vectors $v > 0$, such that $Av < 0$. A simple algorithm (which can be readily implemented by computer[4])

to compute such vectors v is given by

Theorem 4.4 Let $A = (a_{ij})$ be a $m \times m$ matrix with the property

(P) $a_{ii} < 0$; $a_{ij} \geq 0$, $i \neq j$, $i, j = 1,\ldots, m$

Then a necessary and sufficient condition for the existence of some $m \times 1$ vectors $v > 0$ such that $Av < 0$ (or equivalently for A to have Hurwitz's property) is that

$$d^{(k)}_{p,p} > 0 , \quad k = 1,\ldots,m-1 ; \; p = 1,\ldots,m-k \tag{4.6}$$

where $d^{(k)}_{p,q}$ are inductively defined by

$$d^{(o)}_{i,j} = a_{i,j} , \quad d^{(k)}_{p,q} = \begin{vmatrix} d^{(k-1)}_{p,q} & d^{(k-1)}_{p,m-k+1} \\ d^{(k-1)}_{m-k+1,q} & d^{(k-1)}_{m-k+1,m-k+1} \end{vmatrix} \tag{4.7}$$

$i,j = 1,\ldots, m$; $k = 1,\ldots, m-1$; $p,q = 1,\ldots, m-k$

Proof We have to find some positive numbers $v_1,\ldots, v_m$ which satisfy the system of inequalities

$$a_{i1}v_1 + \ldots + a_{im}v_m < 0 , \; i = 1,\ldots,m \tag{4.8}$$

(under hypothesis (P)).

Because $a_{mm} < 0$ and $a_{pm} \geq 0$, $\forall p = 1,\ldots, m-1$ (4.8) is equivalent to (if $a_{pm} > 0$)

$$\frac{a_{m1}}{a_{mm}} v_1 + \ldots + \frac{a_{mm-1}}{a_{mm}} v_{m-1} < v_m < \frac{-a_{p1}}{a_{pm}} v_1 + \ldots + \frac{-a_{pm-1}}{a_{pm}} v_{m-1} . \tag{4.9}$$

We may conclude that (even $a_{pm} = 0$) $v_1,\ldots, v_{m-1}$ have actually to satisfy (necessarily and sufficient)

$$-d^{(1)}_{p,1} v_1 - \ldots - d^{(1)}_{p,m-1} v_{m-1} < 0 , \quad p = 1,\ldots,m-1 \tag{4.10}$$

where

$$d^{(1)}_{p,q} = \begin{vmatrix} a_{pq} & a_{pm} \\ a_{mq} & a_{mm} \end{vmatrix} , \quad p,q = 1,\ldots,m-1 \tag{4.11}$$

hence $d^{(1)}_{p,q} \leq 0$, for $p \neq q$. Accordingly, the conditions $d^{(1)}_{p,q} > 0$ $(p = 1,\ldots, m)$ are necessary. The main remark now is that the matrix in (4.10) also satisfies a condition of type (P) (with $-d^{(1)}_{p,q}$ in place of a_{pq}). Therefore we can eliminate v_{m-1} from (4.10) as v_m was eliminated from (4.8), and so on. Thus, after the elimination of v_m , $v_{m-1},\ldots, v_3$ the remaining variables v_1, v_2 satisfy

$$\frac{-d^{(m-2)}_{2,1}}{d^{(m-2)}_{2,2}} v_1 < v_2 < \frac{d^{(m-2)}_{1,1}}{-d^{(m-2)}_{1,2}} v_1 \tag{4.12}$$

with $d^{(m-2)}_{1,2}$, $d^{(m-2)}_{2,1} \geq 0$; $d^{(m-2)}_{1,1}$, $d^{(m-2)}_{2,2} > 0$. In these conditions, (4.12) is equivalent to

$$\begin{vmatrix} d^{(m-2)}_{1,1} & d^{(m-2)}_{1,2} \\ d^{(m-2)}_{2,1} & d^{(m-2)}_{2,2} \end{vmatrix} v_1 = -d^{(m-1)}_{1,1} v_1 < 0 \tag{4.13}$$

hence $d^{(m-1)}_{1,1} > 0$ is a necessary and sufficient condition for the existence of some $v_1 > 0$,satisfying (4.13). Obviously, in (4.13) v_1 may be assigned positive values at pleasure. Then we take a $v_2 > 0$ satisfying (4.12). In such a manner, going back from (4.13) to (4.10) and (4.9) we prove the existence (sufficiency part) of $v_1,\ldots, v_{m-1}$, $v_m > 0$ (with $v_1 > 0$, arbitrary) satisfying (4.8). This completes the proof.

Remark 4.3 A fortran programme to compute such $v_1, \ldots, v_m$ was given by Gherasim (see the references in [4]).

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APPENDIX

On the extendability of continuous and dissipative functions.

The theory of differential equations on closed subsets on which this paper is based, leads to the following problem (see author's paper [4]).

(P) Let D be a non-empty closed subset of the Banach space X and let $A : D \to X$ be a continuous and dissipative function, which is tangent to D (see [4], (3.9), Ch.2). Can A be extended to X preserving both continuity and dissipativity ? The answer to this question is given by

Theorem 1 Let $A : D \to X$ be as in (P). If $X = R$ (real axis) then A can be extended to X preserving both continuity and dissipativity. This fact may not be true if the dimension of X is greater than one.

Proof Let $X = R$. In this case the fact that $A : D \subset R \to R$ is dissipative means that A is non-increasing. Denote by D_1 the complementary $R \setminus D$ of D. Take an arbitrary open interval $]a,b[\subset D_1$ with $-\infty < a < b < +\infty$. Hence $a, b \in D$ and $Aa \geqslant Ab$. Set

$$Ax = Aa + (b-a)^{-1}(Ab-Aa)(x-a) , \ x \in [a,b] \quad . \tag{1}$$

In such a manner, A can be extended to R as a continuous and non-increasing function, q.e.d. If the dimension of X is greater than one (i.e. $\dim X \geqslant 2$), then we have the following counterexample to the extendability of A to X, namely:

$$X = R^2 , \ D = \left\{(x,y) \ ; \ x \in R \ , \ y \geqslant 1/2\right\} , \ A(x,y) = \left(-\frac{2x}{y} , \frac{x^2}{y^2}\right) \quad . \tag{2}$$

Clearly, $-A$ is the gradient of the convex function

$$f(x,y) = \frac{x^2}{y} , \ x \in R , \ y > 0 \tag{3}$$

and therefore A is dissipative. Moreover, A is tangent to D. Indeed, we have $(x,1/2) + hA(x,1/2) \in D$ for all $x \in R$ and $h > 0$, hence (3.9)' in [4] Chapter 2 is verified. We now prove that A cannot be extended to X as a dissipative function. To this goal, assume by contradiction that there exists a dissipative extension (denoted also by A) of A to the whole X. Set $A(0 , -1) = (p,q)$ with $p , q \in R$. Then the dissipativity of A yields

$$\langle A(x,y) - A(0\ ,\ -1)\ ,\ (x,y+1) \rangle \leq 0\ ,\ \forall x \in R\ ,\ y \geq 1/2\ , \tag{4}$$

where $\langle .,. \rangle$ denotes the inner product of R^2. In view of (2), the inequality (4) becomes

$$\frac{x^2}{y}\left(\frac{1}{y} - 1\right) + px + q(y+1) \leq 0\ ,\ \forall x \in R\ ,\ y \geq 1/2\ . \tag{5}$$

Substituting $y = 1/2$ into (5) we reach the absurdity

$$4x^2 + 2px + 3q \leq 0\ ,\ \forall\, x \in R \tag{6}$$

which completes the proof.

Remark 1 We have actually proved that grad f cannot be extended to R^2. It follows that if $F : R^2 \to R^2$ is a function of class C^1, such that its restriction F/D to a closed convex subset $D \subset R^2$ is convex, then F/D cannot be extended to R^2 as a convex function of class C^1. We have to recall that even if $X = R$, a continuous and convex function $F : D \subset R \to R$ (with D closed and convex) cannot be extended to R preserving both continuity and convexity. A standard counterexample in this direction is the following: $D = [-1, 1]$, $F(x) = -(1-x^2)^{1/2}$, $x \in D$.

Remark 2 Obviously, A given by (2) is continuous and dissipative on the open subsets $U = \{(x,y)\ ;\ y > 0\} \supset D$. Consequently, the following problem arises:

(P_1) Let D and A be as in (P) and let $\dim X \geq 2$. Is it true that A cannot be extended to any open subset $U \supset D$ preserving both continuity and dissipativity ?

REFERENCES

[1] N.H. Pavel, "Analysis of some non-linear problems" in Banach Spaces and Applications", University of Iasi (Romania) (1982).

[2] N.H. Pavel, "Semilinear evolution equations with t-dependent domain perturbations", Israel J. Math. (1983).

[3] N.H. Pavel, "Positivity and stability of some differential systems from biomathematics", ICTP, Trieste, internal report IC/83/101 (1983).

[4] N.H. Pavel, Differential Equations, Flow-invariance and Applications, (Pitman Publishing Ltd., London) (to appear).

[5] N.H. Pavel and I.I. Vrabie, "Differential equations with t-dependent domain operators", Lectures Notes in Mathematics 737, 236-249 (1979).

[6] S.I. Rubinov, Introduction to Mathematical Biology, Wiley-Interscience, New York, 1975).

[7] M. Voicu, "Component-wise asymptotic stability of the linear constant dynamical systems", Preprint Series in Mathematics 5, University of Iasi (Romania) (1982).