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Ling-Lie Chau

Ge Mo-Lin

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Rosy Teh

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REGULAR RIEMANN-HILBERT TRANSFORMS, BÄCKLUND TRANSFORMATIONS AND HIDDEN SYMMETRY ALGEBRA FOR SOME LINEARIZATION SYSTEMS *

Ling-Lie Chau

Physics Department, Brookhaven National Laboratory, Upton, NY 11973, USA.

Ge Mo-Lin ** and Rosy Teh ***

International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

The Bäcklund Transformations and the hidden symmetry algebra for Self-Dual Yang-Mills Equations, Landau-Lifshitz equations and the Extended Super Yang-Mills fields ($N > 2$) are discussed on the base of the Regular Riemann-Hilbert Transform and the linearization equations.

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*** On leave of absence from Institute Technology of MARA, Arau, Perlis, Malaysia.

In recent years much progress has been made in studying non-linear field equations based on their corresponding linearization systems. As is well known there exist linear equations for many physical non-linear systems [1,2]. The remarkable examples are

(i) in 2-dimensions,

(a) Chiral models (the reduced chiral model is related to the Sine-Gordon equation [3])

(b) Landau-Lifshitz equations (which is gauge equivalent to the non-linear Schrödinger equation [4]).

(c) Toda-Lattice (covering Sine-Gordon, Liouville equation etc.).

(d) Ernst equations (Kinnersley-Chitre-Hauser-Ernst formulation [5] and Belinskiy-Zakharov formulation [6]).

(ii) in 4-dimensions

(a) Self-Dual Yang-Mills (SDYM) equations (in R-gauge J- formulation [7,8,9]).

(b) Extended Super Yang-Mills fields with $N > 2$ (The Volovich formulation [10,11].

The integrable systems have many important properties such as Bäcklünd Transformation (BT), infinite number of conserved currents, soliton solutions, Kac-Moody algebraic structures and Riemann Hilbert Transforms (RHT) [11,12,13-16].

As is well known the above properties can be discussed without the use of a particular form of solution of a physical non-linear system. This means that many important properties of an integrable system are determined only by their linearization equations. It has been pointed out that the Regular Riemann-Hilbert Transform (RRHT) is model-independent [6,13] which can be served as a powerful tool to discuss the above problems [14,15]. From the point of view of RRHT, the only difference between any two models is in the different dependence of their linear equations on the spectral parameter. Hence we can deal with the general case before looking into any special models.

In this note we would like first to give a general discussion of any set of linear equations based on the RRHT. This discussion is quite simple but useful. We then use the formula derived to discuss some special models. As examples to show the consequences due to combining both the RRHT and the linear equations, we choose the SDYM fields, Landau-Lifshitz equations and the extended Super Yang-Mills fields $(N \geq 2)$.

II.

For any linearization systems there are two (Lax-pair) or more linear equations each of which can be expressed by the form

$$D(\lambda)\Phi(\lambda) + \mathcal{A}(\lambda)\Phi(\lambda) = 0 \quad , \tag{2.1}$$

where $D(\lambda)$ denotes a linear derivative operator and $\mathcal{A}(\lambda)$ the related potential. In Eq.(2.1) the λ dependence of Φ is emphasized where λ is a complex spectral parameter.

For convenience we list the linearization systems of the equations which we are concerned with:

(a) Lax-pair of the principal chiral model is

$$\partial_\xi \Phi = -\left(\frac{\lambda}{1+\lambda}\right) A_\xi \Phi \tag{2.2}$$

$$\partial_\eta \Phi = \left(\frac{\lambda}{1-\lambda}\right) A_\eta \Phi \, , \tag{2.3}$$

where $A_\xi = g^{-1}\partial_\xi g$, $A_\eta = g^{-1}\partial_\eta g$ with ξ and η being the light-cone coordinates. Here $D(\lambda)$ denotes ∂_ξ or ∂_η and $\mathcal{A}(\lambda)$ is correspondingly $\left(\frac{\lambda}{1+\lambda}\right)A_\xi$ or $\left(\frac{\lambda}{1-\lambda}\right)A_\eta$.

(b) For the SDYM fields, the Lax-pair is

$$\partial_y \Phi + B_y \Phi = \frac{1}{\lambda}\partial_{\bar z}\Phi \tag{2.4}$$

$$\partial_z \Phi + B_z \Phi = -\frac{1}{\lambda}\partial_{\bar y}\Phi \tag{2.5}$$

with $B_y = J^{-1}J,_y$, $J = \frac{1}{\Phi}\begin{pmatrix} 1 & \bar\sigma \\ \sigma & \Phi^2 + \sigma\bar\sigma \end{pmatrix}$ [8,12] where Φ, σ and $\bar\sigma$ are as shown by Yang [7]. Here $D = \partial_y - \frac{1}{\lambda}\partial_{\bar z}$ or $\partial_z + \frac{1}{\lambda}\partial_{\bar y}$ and correspondingly $\mathcal{A}$ is B_y or B_z which can be chosen independently of λ

(c) As for the Landau-Lifshitz equations the Lax-pair reads [4]

$$\partial_x \Phi = -i\lambda S \Phi \tag{2.6}$$

$$\partial_t \Phi = \lambda(2i\lambda S - SS_x)\Phi \qquad (S^2 = 1) \, , \tag{2.7}$$

where D can be ∂_x or ∂_t and correspondingly $\mathcal{A}(\lambda)$ represents either $i\lambda\rho$ or $-2i\lambda^2\rho + \lambda\rho\rho_x$

(d) In the case of the extended Super Yang-Mills fields we follow the notations of reference [17] and [18]

$$\begin{aligned} &D^S_\alpha = \frac{\partial}{\partial\theta^\alpha_S} + i\,\bar\theta^{\dot\beta S}\partial_{\alpha\dot\beta} \\ &\bar D_{\dot\beta t} = -\frac{\partial}{\partial\bar\theta^{\dot\beta t}} - i\,\theta^\alpha_t\,\partial_{\alpha\dot\beta} \, , \quad \partial_{\alpha\dot\beta} = \sigma^\mu_{\alpha\dot\beta}\partial_\mu \\ &\alpha, \beta = 1, 2 \, ; \quad \dot\alpha, \dot\beta = \dot 1, \dot 2 \, ; \quad s, t = 3, 4 \, ; \quad \mu = 0, 1, 2, 3. \end{aligned} \tag{2.8}$$

Definitions of covariant derivatives and corresponding curvatures are as usual

$$\nabla^S_\alpha = D^S_\alpha + A^S_\alpha \, , \quad \bar\nabla_{\dot\beta t} = \bar D_{\dot\beta t} + \bar A_{\dot\beta t} \, , \quad \nabla_\mu = \partial_\mu + A_\mu \, ,$$

$$\{\nabla^S_\alpha, \nabla^t_\beta\} = F^{st}_{\alpha\beta} \, , \qquad \{\bar\nabla_{\dot\alpha s}, \bar\nabla_{\dot\beta t}\} = \bar F_{\dot\alpha s, \dot\beta t} \, ,$$

$$\{\nabla^S_\alpha, \bar\nabla_{\dot\beta t}\} = F^S_{\alpha, \dot\beta t} - 2i\nabla_{\alpha\dot\beta}\delta^S_t$$

where one choose the following constraint equations

$$F^{st}_{\alpha\beta} + F^{ts}_{\alpha\beta} = 0$$

$$\bar{F}_{\dot\alpha s,\dot\beta t} + \bar{F}_{\dot\alpha t,\dot\beta s} = 0 \qquad (2.9)$$

$$F^{s}_{\alpha,\dot\beta t} = 0$$

when N=2 the constraint equations just reduce the super-components and guarantee the spin to be not larger than 1. However, when N=3 and 4 these constraints give rise to equations of motion [17,18]. Choosing the gauge as in Ref. [11] so that

$$A^s_1 = g^{-1} D^s_1 g, \qquad A_{\dot 1 t} = g^{-1} \bar{D}_{\dot 1 t} g$$
$$A^s_2 = h^{-1} D^s_2 h, \qquad A_{\dot 2 t} = h^{-1} \bar{D}_{\dot 2 t} h \qquad (2.10)$$

which yields

$$A_{1\dot i} = g^{-1} \partial_{1\dot i} g$$
$$A_{2\dot i} = h^{-1} \partial_{2\dot i} h \qquad (2.11)$$

and defining the matrix $B = gh^{-1}$, the linearization equations can be written in the form

$$(D^s_1 + \lambda D^s_2 + \lambda B D^s_2 B^{-1}) \Phi = 0 \qquad (2.12)$$

$$(\bar{D}_{\dot 2 t} + B \bar{D}_{\dot 2 t} B^{-1} + \lambda^{-2} \bar{D}_{\dot 1 t}) \Phi = 0 \qquad (2.13)$$

$$\{(\partial_{1\dot i} + g \nabla_{1\dot i} g^{-1}) + \lambda(\partial_{2\dot i} + B \partial_{2\dot i} B^{-1}) + \lambda^{-2} \partial_{1 i} + \lambda^{-1}(\partial_{2i} + g \nabla_{2i} g^{-1})\} \Phi = 0 \; , \qquad (2.14)$$

where for Eq.(2.12), $D(\lambda) = D^s_1 + \lambda D^s_2$, $\mathcal{A} = \lambda B D^s_2 B^{-1}$

for Eq.(2.13) $D(\lambda) = \bar{D}_{\dot 2 t} + \lambda^{-2} \bar{D}_{\dot 1 t}$, $\mathcal{A} = B \bar{D}_{\dot 2 t} B^{-1}$

and for Eq.(2.14) $D(\lambda) = \partial_{1\dot i} + \lambda \partial_{2\dot i} + \lambda^{-2} \partial_{1i} + \lambda^{-1} \partial_{2i}$;

$$\mathcal{A}(\lambda) = g \nabla_{1\dot i} g^{-1} + \lambda B \partial_{2\dot i} B^{-1} + \lambda^{-1} g \nabla_{2i} g^{-1}$$

Let us now consider a transformation which preserves the form of the linear equations such that

$$D(\lambda)\, \Phi'(\lambda) + \mathcal{A}'(\lambda)\, \Phi'(\lambda) = 0 \qquad (2.15)$$

where λ is the same parameter as in Eq.(2.1) and $\mathcal{A}'(\lambda)$ has the same λ dependence as in Eq.(2.1). Obviously the set of equations like Eq.(2.15) gives the same equations of motion with only the difference in prime. When

$$\Phi'(\lambda) = X(\lambda)\, \Phi(\lambda) \qquad (2.16)$$

$$\mathcal{A}'(\lambda) = X(\lambda) \mathcal{A}(\lambda) X(\lambda)^{-1} - [D(\lambda)\, X(\lambda)]\, X(\lambda)^{-1} \qquad (2.17)$$

a Darboux-type transformation is made. From Eq.(2.1) and Eq.(2.15) it is easy to give

$$\mathcal{A}(\lambda) - \mathcal{A}'(\lambda) = D(\lambda) X(\lambda) - W(\lambda) \mathcal{A}(\lambda) + \mathcal{A}'(\lambda) W(\lambda) \; , \qquad (2.18)$$

where

$$W(\lambda) = [\Phi'(\lambda) - \Phi(\lambda)]\, \Phi^{-1}(\lambda) \; . \qquad (2.19)$$

One now is able to introduce the Regular-Riemann-Hilbert Transforms (RRHT)

$$X(\lambda) = 1 - \frac{\lambda}{2\pi i} \int_c \frac{dt}{t(t-\lambda)}\, G(t) \quad \text{with} \quad G(t) = \Phi'(t)\, V(t)\, \Phi^{-1}(t) \qquad (2.20)$$

where $D(\lambda)v(\lambda) = 0$ and Φ and Φ' satisfy Eq.(2.1) and Eq.(2.15), respectively.

In the infinitesimal case λ can take any value. However in the finite case λ has to be outside the contour C so that Eq.(2.20) is regular.

By direct calculation it is easy to prove that

$$D(\lambda)G(\lambda) + \mathcal{A}'(\lambda)G(\lambda) - G(\lambda)\mathcal{A}(\lambda) = 0 \tag{2.21}$$

for any complex λ. Substituting Eq.(2.20) and Eq.(2.21) into Eq. (2.18) we obtain

$$\mathcal{A}(\lambda) - \mathcal{A}'(\lambda) = -\frac{\lambda}{2\pi i}\int_C \frac{dt}{t(t-\lambda)}\{[D(\lambda) - D(t)]G(t) + [\mathcal{A}'(\lambda) - \mathcal{A}'(t)]G(t) - G(t)[\mathcal{A}(\lambda) - \mathcal{A}(t)]\} \ . \tag{2.22}$$

Eq.(2.22) is a formal expression of the transformation from a given $\mathcal{A}(\lambda)$ to $\mathcal{A}'(\lambda)$ for fixed λ outside C_+ for the physical value based on linear equations and RRHT.

Next we will show how to use Eq.(2.22) to discuss the related BT and Kac-Moody algebra in the infinitesimal case. Only three examples will be discussed here: SDYM field, Landau-Lifshitz equations and extended super Yang-Mills system $(N > 2)$. The discussion for the chiral model will be the same as that of SDYM case.

III.

Substituting Eqs.(2.4) and (2.5) into Eq. (2.22) for the Self-Dual Yang-Mills field we have

$$J^{-1}\partial_y J - J'^{-1}\partial_y J' = -\frac{1}{2\pi i}\int_c \frac{dt}{t^2}\,\partial_{\bar z} G(t) \tag{3.1}$$

$$J^{-1}\partial_z J - J'^{-1}\partial_z J' = \frac{1}{2\pi i}\int_c \frac{dt}{t^2}\,\partial_{\bar y} G(t)\ . \tag{3.2}$$

Putting $t = \frac{1}{\tau}$, Eqs. (3.1) and (3.2) can be rewritten as

$$J^{-1}\partial_y J - J'^{-1}\partial_y J' = -\partial_{\bar z}\frac{1}{2\pi i}\int_{C'} d\tau\, \Phi'(\tau)\, v(\tau)\,\Phi^{-1}(\tau) \tag{3.3}$$

$$J^{-1}\partial_z J - J'^{-1}\partial_z J' = \partial_{\bar y}\frac{1}{2\pi i}\int_{C'} d\tau\, \Phi'(\tau)\, v(\tau)\,\Phi^{-1}(\tau)\ , \tag{3.4}$$

where C' denotes the corresponding contour in complex τ-plane.

We then choose suitable ansatz for $v(\tau)$ and the form $\Phi'(\tau)v(\tau)\Phi^{-1}(\tau)$. From Eqs.(2.4) and (2.5), we observe that

$$\Phi(\tau=0) = J^{-1} f(\bar z - \tau y,\ \bar y + \tau z) \tag{3.5}$$

and
$$\Phi'(\tau=0) = J'^{-1} f'(\bar z - \tau y,\ \bar y + \tau z)\ . \tag{3.6}$$

In the 't Hooft ansatz $f = 1$. here we put $f = f'$ and take the following ansatz:

(i)
$$v(\tau) = \frac{\xi}{\tau}\,\mathbb{1} \tag{3.7}$$

which satisfies $D(\tau)v(\tau) = 0$. Here ξ is an arbitrary parameter. As the BT is independent of any special group, the only choice of the group element must be $\mathbb{1}$. In fact the group dependence will be included in J' as seen below.

$$\Phi'(\tau)\,\Phi^{-1}(\tau)\Big|_{\tau\to 0} = J'^{-1}J + \sum_{n=1}^{\infty}\tau^n\,\Phi^{(n)}\ . \tag{3.8}$$

Eq.(3.8) means that when $\tau \to 0$, Φ satisfies the asymptotic behaviour of Eq.(3.5) [15].

The above ansatz is not only reasonable but also more than sufficient. Substitut... Eq.(3.7) and Eq.(3.8) into Eq.(3.8) and Eq.(3.4) and performing the contour integral we have

$$J^{-1}\partial_y J - J'^{-1}\partial_y J' = -\xi\,\partial_{\bar z}(J'^{-1}J) \tag{3.9}$$

$$J^{-1}\partial_z J - J'^{-1}\partial_z J' = \xi\,\partial_{\bar y}(J'^{-1}J) \tag{3.10}$$

which is the BT discussed in Ref.[12]. In comparison with Eqs.(3.1) and (3.2) we have

$$J' = \left[1 + \frac{\xi}{2\pi i}\int_c \frac{dt}{t^2}\, G(t)\right] J \tag{3.11}$$

which is an integral equation of the Lax-pair Eqs.(2.4) and (2.5). We see that the group dependence has been included in J'.

Let now us consider the infinitesimal RRHT which will lead to Kac-Moody algebra. In this case Φ' in the integral is equal to Φ. When J' approaches J infinitesimally we have

$$\delta B_y = -\partial_{\bar z}\frac{1}{2\pi i}\int_c \frac{dt}{t^2}\,\Phi(t)\,v(t)\,\Phi^{-1}(t) \tag{3.12}$$

$$\delta B_z = \partial_{\bar y}\frac{1}{2\pi i}\int_c \frac{dt}{t^2}\,\Phi(t)\,v(t)\,\Phi^{-1}(t)\ . \tag{3.13}$$

FollowingEichenherr [16] we introduce $v^m_\alpha(t) = t^{-m}T_\alpha\alpha^a$. Therefore

$$\delta^{(m)}_\alpha B_z = \partial_{\bar y}\frac{1}{2\pi i}\int_c \frac{dt}{t^2}\, t^{-m}\alpha^a\,\Phi(t)\,T_a\,\Phi^{-1}(t) \tag{3.14}$$

$$\delta^{(n)}_\beta\delta^{(m)}_\alpha B_z = \partial_{\bar y}\frac{1}{2\pi i}\int_c \frac{dt}{t^2}\,\alpha^a\beta^b\, t^{-m}\left[\delta_b\Phi(t)\cdot\Phi^{-1}(t),\ \Phi(t)\,T_a\,\Phi^{-1}(t)\right]\ . \tag{3.15}$$

On account of the infinitesimal RRHT we have

$$\delta^{(n)}_\beta\delta^{(m)}_\alpha B_z = \alpha^a\beta^b\,\partial_{\bar y}\frac{1}{2\pi i}\int_c\frac{dt}{t}\int_{c'}\frac{dt'}{t'}\,\frac{1}{t'-t}\,t^{-m}\,t'^{-n}\left[\Phi(t')T_b\Phi(t'),\ \Phi(t)T_a\Phi(t)\right] \tag{3.16}$$

Interchanging α, β and (n), (m) simultaneously is equivalent to interchanging t and t'. Hence be deforming the contour c and c' so that they become rectangular then interchanging c and c' will give rise to a discontinuity of $i\pi\delta(t'-t)$. The principal value of the integral then cancelled by taking the commutation relation of the L.H.S. of Eq.(3.16) so that we get

$$[\delta^{(m)}_b, \delta^{(n)}_a] B_z = \partial_{\bar y}\frac{1}{2\pi i}\int_c\frac{dt}{t^2}\, t^{-m-n}\, C^c_{ab}\,\Phi(t)\,T_c\,\Phi^{-1}(t) \tag{3.17}$$

which from Eq.(3.14) turns out to be the Kac-Moody algebra

$$[\delta^{(m)}_b, \delta^{(n)}_a] B_z = C^c_{ab}\,\delta^{(m+n)}_c B_z \tag{3.18}$$

similar result can be obtained for B_y. If we start from Eq.(3.11) in the infinitesimal case (we put $\xi=1$),

$$\delta J = \frac{1}{2\pi i}\int_c\frac{dt}{t^2}\,\Phi(t)\,v(t)\,\Phi^{-1}(t)\ . \tag{3.19}$$

Repeating the above procedure we again derive the Kac-Moody algebra [19]

$$[\delta^{(m)}_b, \delta^{(n)}_a] J = C^c_{ab}\,\delta^{(m+n)}_c J\ . \tag{3.20}$$

Hence we see the coincidence between "potential" B_y and "field" J. The process discussed is, in fact, on how to derive the transformed relation between J and J' based on the RRHT. It is the extension of the chiral model in Ref.[14] and the SDYM theory in Ref.[19]. However it is not always the case for complicated systems such as the extended super Yang-Mills ($N > 2$) theory which we will discuss in Section V.

IV.

Substituting Eqs.(2.6) and (2.7) into Eq.(2.22) we have for the Landau-Lifshitz equation

$$i\,(S - S') = \partial_x \frac{1}{2\pi i}\int_c \frac{d\lambda'}{\lambda'^2}\, G(\lambda') \tag{4.1}$$

$$-[2i\lambda^2(S-S') - \lambda(SS_x - S'S'_x)] = -\frac{1}{\pi i}\int_c \frac{d\lambda'}{\lambda'}\,\partial_x G(\lambda') + \frac{\lambda}{2\pi i}\int_c \frac{d\lambda'}{\lambda'^2}\{(2i\lambda'^2 S' - S'S'_x)G(\lambda') - G(\lambda')(2i\lambda'^2 S - SS_x)\}\,. \tag{4.2}$$

By virtue of

$$-\partial_t G(\lambda) = (2i\lambda^2 S' - S'S'_x)G(\lambda) - G(\lambda)(2i\lambda^2 S - SS_x)$$

and substituting Eq.(4.1) into Eq.(4.2), we get

$$SS_x - S'S'_x = \partial_t \frac{1}{2\pi i}\int_c \frac{d\lambda'}{\lambda'^2} G(\lambda) \tag{4.3}$$

In the infinitesimal case we have

$$i\,\delta S = \partial_x \frac{1}{2\pi i}\int_c \frac{d\xi}{\xi^2}\, \Phi(\xi)\, \vartheta(\xi)\, \Phi^{-1}(\xi) \tag{4.4}$$

$$\delta(SS_x) = \partial_t \frac{1}{2\pi i}\int_c \frac{d\xi}{\xi^2}\, \Phi(\xi)\vartheta(\xi)\Phi^{-1}(\xi)\,. \tag{4.5}$$

Repeating the discussion as in Section III we derive the Kac-Moody algebra

$$[\delta_b^{(m)}, \delta_a^{(n)}]\, S = C_{ab}^{\ c}\, \delta_c^{(m+n)} S \tag{4.6}$$

$$[\delta_b^{(m)}, \delta_a^{(n)}](SS_x) = C_{ab}^{\ c}\, \delta_c^{(m+n)}(SS_x) \tag{4.7}$$

which is the natural extension of the results in Ref.[16].

By direct calculation we can also show the consistency of Eqs.(4.4) and (4.5) in terms of equation of motion.

V.

For the extended Super Yang-Mills fields ($N > 2$) we get

$$B D_2^S B^{-1} - B' D_2^S B'^{-1} = D_1^S \left(\frac{-1}{2\pi i}\right)\int_c \frac{dt}{t^2}\, G(t) \tag{5.1}$$

on substituting Eq.(2.12) into Eq.(2.22). In Eq.(2.13) we have the λ^2 dependence which means that the integral variable in the RRHT can be either λ' or $-\lambda'$. Since for each choice it should give the same consequence, we can take the average of the RRHT related to λ' and $-\lambda'$ to give

$$B \bar{D}_{it} B^{-1} - B' \bar{D}_{it} B'^{-1} = \bar{D}_{it} \left(\frac{-1}{2\pi i}\right)\int_c \frac{dt}{t^3}\, G(t)\,. \tag{5.2}$$

Putting Eq.(2.4) into Eq.(2.22) we get

$$\begin{aligned} &\lambda(B\partial_{2\dot{i}}B^{-1} - B'\partial_{2\dot{i}}B'^{-1}) + \lambda^{-1}(g\nabla_{2i}g^{-1} - g'\nabla'_{2i}g'^{-1}) = \\ &\lambda\frac{1}{2\pi i}\int_c \frac{dt}{t}[\partial_{2\dot{i}}G(t) + B'\partial_{2\dot{i}}B'^{-1}G(t) - G(t)B\partial_{2\dot{i}}B^{-1}] + \lambda^{-1}\left(\frac{-1}{2\pi i}\right)\int_c \frac{dt}{t^2}\partial_{1i}G(t) \\ &+ \frac{1}{2\pi i}\int_c \frac{dt}{t}\{(\partial_{1\dot{i}} + t\partial_{2\dot{i}})G(t) + t[B'\partial_{2\dot{i}}B'^{-1}G(t) - G(t)B\partial_{2\dot{i}}B^{-1}] + [g'\nabla'_{1\dot{i}}g'^{-1}G(t) - G(t)g\nabla_{1\dot{i}}g^{-1}]\} \end{aligned} \tag{5.3}$$

Comparing the different powers of λ gives

$$g\nabla_{2i}g^{-1} - g'\nabla'_{2i}g'^{-1} = \frac{-1}{2\pi i}\int_c \frac{dt}{t^2}\,\partial_{1i}\,G(t) \tag{5.4}$$

$$B\,\partial_{2\dot{2}}B^{-1} - B'\partial_{2\dot{2}}B'^{-1} = \frac{1}{2\pi i}\int_c dt\,[\partial_{2\dot{2}}G(t) + B'\partial_{2\dot{2}}B'^{-1}G(t) - G(t)B\partial_{2\dot{2}}B^{-1}] \tag{5.5}$$

As for the last term in Eq.(5.3) we have, due to Eq.(2.21) sufficiently

$$(\partial_{1\dot{2}} + \lambda\partial_{2\dot{2}})G(\lambda) + \lambda(B'\partial_{2\dot{2}}B'^{-1}G(\lambda) - G(\lambda)B\partial_{2\dot{2}}B^{-1}) + (g'\nabla'_{1\dot{2}}g'^{-1}G(\lambda) - G(\lambda)g\nabla_{1\dot{2}}g^{-1}) = 0 \tag{5.6}$$

Hence, we get,

$$[(\partial_{1\dot{2}} + g\nabla_{1\dot{2}}g^{-1}) + \lambda(\partial_{2\dot{2}} + B\partial_{2\dot{2}}B^{-1})]\Phi = 0 \tag{5.7}$$

and from Eq.(2.14) we have

$$[\partial_{1i} + \lambda(\partial_{2i} + g\nabla_{2i}g^{-1})]\Phi \tag{5.8}$$

Eqs.(5.7) and (5.8) show that Eq.(2.14) has been split into 2 sub-linear equations. Defining

$$\mathcal{M} = 1 + H = 1 + \frac{1}{2\pi i}\int_c \frac{dt}{t}G(t)\ . \tag{5.9}$$

Eq.(5.5) can be recast into

$$B'\partial_{2\dot{2}}B'^{-1} = \mathcal{M}B\partial_{2\dot{2}}B^{-1}\mathcal{M}^{-1} - (\partial_{2\dot{2}}\mathcal{M})\mathcal{M}^{-1} \tag{5.10}$$

that is

$$B' = \mathcal{M}B \tag{5.11}$$

From Eqs. (2.13) and (2.21)

$$(\bar{D}_{\dot{2}t} + \lambda^{-2}\bar{D}_{\dot{1}t})G(\lambda) + B'\bar{D}_{\dot{2}t}B'^{-1}G(\lambda) - G(\lambda)B\,\bar{D}_{\dot{2}t}B^{-1} = 0\ . \tag{5.12}$$

Eq.(5.2) can then rewritten in the form

$$B'\bar{D}_{\dot{2}t}B'^{-1} = \mathcal{M}B\bar{D}_{\dot{2}t}B^{-1}\mathcal{M}^{-1} - (\bar{D}_{\dot{2}t}\mathcal{M})\mathcal{M}^{-1} \tag{5.13}$$

with coincide with Eq.(5.11).

Hence the set of BT is

$$BD_2^S B^{-1} - B'D_2^S B'^{-1} = D_1^S\left(\frac{-1}{2\pi i}\right)\int_c \frac{dt}{t^2}G(t) \tag{5.14}$$

$$g\nabla_{2i}g^{-1} - g'\nabla'_{2i}g'^{-1} = \partial_{1i}\left(\frac{-1}{2\pi i}\right)\int_c \frac{dt}{t^2}G(t) \tag{5.15}$$

$$B' = \mathcal{M}B = \left\{1 + \frac{1}{2\pi i}\int_c \frac{dt}{t}G(t)\right\}B\ . \tag{5.16}$$

In the infinitesimal case if we take the ansatz [14,20]

$$\upsilon(t) = \frac{t}{t-\lambda}T_a\alpha^a \tag{5.17}$$

and defining $S(\lambda)$ by

$$S(\lambda) = \Phi(\lambda)\,T\,\Phi(\lambda)^{-1} \tag{5.18}$$

we obtain the results in Ref.[11]. However here we would like to discuss the Kac-Moody algebraic structure in a different way. In the infinitesimal case Eqs.(5.14) and (5.15) become

$$\delta_\alpha(BD_2^S B^{-1}) = D_1^S\left(\frac{-1}{2\pi i}\right)\int_c \frac{dt}{t^2}\Phi(t)\,\upsilon_\alpha(t)\,\Phi^{-1}(t) \tag{5.19}$$

$$\delta_\alpha (g \nabla_{2i} g^{-1}) = \partial_{1i}\left(\frac{-1}{2\pi i}\right)\int_C \frac{dt}{t^2}\, \Phi(t)\, \upsilon_\alpha(t)\, \Phi^{-1}(t) \quad . \tag{5.20}$$

Taking the second derivative δ_β to Eqs.(5.19) and (5.20) and repeating the discussions in deriving the Kac-Moody algebra as in the SDYM case, we obtain

$$[\delta_b^{(m)}, \delta_a^{(n)}](B D_2^S B^{-1}) = -C_{ab}^{\ \ c}\, \delta_c^{(m+n)}(B D_2^S B^{-1}), \tag{5.21}$$

$$[\delta_b^{(m)}, \delta_a^{(n)}](g \nabla_{2i} g^{-1}) = -C_{ab}^{\ \ c}\, \delta_c^{(m+n)}(g \nabla_{2i} g^{-1}) \quad . \tag{5.22}$$

However we cannot get the same relation for δB because from Eq.(5.16)

$$\delta B = \frac{1}{2\pi i}\int_C \frac{dt}{t}\, \Phi(t)\, \upsilon(t)\, \Phi^{-1}(t) B \tag{5.23}$$

which has a different power of t.

As a result of the linear equation split into Eqs.(5.7) and (5.8), we still have the self-duality condition in the super expression such that

$$F_{yz} = 0 , \quad F_{\bar{y}\bar{z}} = 0 \qquad F_{y\bar{y}} + F_{z\bar{z}} = 0 \quad , \tag{5.24}$$

where $1\dot{2} \to \bar{y}$, $\dot{2}\dot{2} \to -z$, $1i \to z^-$ and $2i \to y$.

To guarantee Eq.(5.24), the gauge

$$g \nabla_{1\dot{2}} g^{-1} = 0 \quad , \quad g \nabla_{2i} g^{-1} = B \partial_{2i} B^{-1} \tag{5.25}$$

can be used [11]. upon such a consideration we have the simplified set of equations

$$B D_2^S B^{-1} - B' D_2^S B'^{-1} = D_i^S \left(\frac{-1}{2\pi i}\right)\int_C \frac{dt}{t^2}\, G(t) \tag{5.26a}$$

$$B \bar{D}_{\dot{2}t} B^{-1} - B' \bar{D}_{\dot{2}t} B'^{-1} = \bar{D}_{it} \left(\frac{-1}{2\pi i}\right)\int_C \frac{dt}{t^3}\, G(t) \tag{5.26b}$$

$$B\, \partial_{2i} B^{-1} - B' \partial_{2i} B'^{-1} = \partial_{1i} \left(\frac{-1}{2\pi i}\right)\int_C \frac{dt}{t^2}\, G(t) \tag{5.26c}$$

$$B\, \partial_{2\dot{2}} B^{-1} - B' \partial_{2\dot{2}} B'^{-1} = \partial_{1\dot{2}} \left(\frac{-1}{2\pi i}\right)\int_C \frac{dt}{t^2}\, G(t) \quad . \tag{5.26d}$$

The corresponding linear equations of Eqs.(5.7) and (5.8) become

$$[\partial_{1\dot{2}} + \lambda(\partial_{2\dot{2}} + B\partial_{2\dot{2}} B^{-1})]\, \Phi = 0 \tag{5.27}$$

$$[\partial_{1i} + \lambda(\partial_{2i} + B\partial_{2i} B^{-1})]\, \Phi = 0 \tag{5.28}$$

Eqs.(5.26c) and (5.26d) are identify with the ordinary SDYM Lax-pair of Eqs.(2.4) and (2.5).

VI.

(i) For the chiral model we can repeat the discussion mode in the SDYM case. In addition to the Kac-Moody algebra we have based on the RHHT, the finite BT

$$g^{-1}\partial_\xi g - g'^{-1}\partial_\xi g' = \beta\ \partial_\xi (g'^{-1} g) \tag{6.1}$$

$$g^{-1}\partial_\eta g - g'^{-1}\partial_\eta g' = -\beta\ \partial_\eta (g'^{-1} g) \tag{6.2}$$

which were derived in Ref.[9]. It has been shown that they can be reduced to the Riccati equations [2]. However we cannot prove such a point in the SDYM case.

(ii) We cannot get the BT for the Landau-Lifshitz equations in the same way as in chiral model because the asymptotic behaviour at a special value of λ in the Lax-pair of Eqs.(2.6) and (2.7) is very complicated. However the derivation of Kac-Moody algebra is very simple.

(iii) In deriving Eqs.(5.7) and (5.8) of the extended Super Yang-Mills fields ($N>2$) we require Eq.(5.6). In general we only need the whole integral, that is the last term of Eq.(5.3) to vanish. In such a consideration it may perhaps gives the non-self-duality condition. Since no special gauge is needed for deriving Eqs.(5.14)-(5.16) we have to deal with equations including g and h separately and not Eqs.(5.27) and (5.28) where only B appears.

(iv) In certain cases the new "potentials" $\mathcal{A}'(\lambda)$ may have new λ dependence. However it has been shown [21] that they still satisfy the same equations of motion in terms of direct verification. Here we consider all the new "potentials" $\mathcal{A}'(\lambda)$ to be having the same λ dependence as $\mathcal{A}(\lambda)$. In the discussion of section IV the constarint $D_{it}\bar{\Phi} = 0$ will not change the results.

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