MANIFEST ROTATION SYMMETRIC EXPRESSIONS FOR ANGULAR MOMENTUM EIGENFUNCTIONS

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Abstract:

We give manifest rotation symmetric expressions for eigenfunctions for spin s, orbital angular momentum ℓ and total angular momentum $j = \ell+s, \ldots, |\ell-s|$ in terms of $(2j+1) \times (2s+1)$ multipole transition matrices (MTM). These matrices, which are irreducible tensor matrices, have an algebra together with ordinary spin matrices for spin s and spin j. Explicit expressions for MTM's and their algebra are given for angular momenta ≤ 3. By means of some examples we show that within this formalism angular integrations in central field problems will be simplified considerably. Thus the formalism turns out to be very useful for instance for calculations within the MIT-bag and also within spin-spin interactions in atomic physics.

1. Angular momentum eigenfunctions in terms of multipole transition matrices

The theory for angular moments and their eigenfunctions has been known for a long time[1,2]. If an intrinsic spin $\underset{\sim}{S}$ and an orbital angular momentum $\underset{\sim}{L}$ are coupled to total angular momentum $\underset{\sim}{J} = \underset{\sim}{L} + \underset{\sim}{S}$, the eigenfunctions $\phi(s\ell j;m)$ for $\underset{\sim}{S}^2$, $\underset{\sim}{L}^2$, $\underset{\sim}{J}^2$ and J_3 are given by well known expressions in terms of Clebsch-Gordan coefficients $\langle sm_s \ell m_\ell | jm \rangle$, spherical harmonics $Y_\ell^{m_\ell}(\hat{r})$ and (2s+1) component spin s spinors $\chi_{m_s}^{[s]}$ (- i.e the $\chi^{[\frac{1}{2}]}$'s are Pauli spinors):

$$\varphi(s\ell j;m) = \sum_{m_\ell + m_s = m} \langle s m_s \ell m_\ell | jm \rangle Y_\ell^{m_\ell}(\hat{\underset{\sim}{r}})\, \chi_{m_s}^{[s]} \quad , \tag{1}$$

where the unit vector $\hat{\underset{\sim}{r}} = \underset{\sim}{r}/|\underset{\sim}{r}|$ is the only angular dependent quantity. It is well known[2,3] that for $s=\frac{1}{2}$

$$\varphi(\tfrac{1}{2}\ell j;m) = -\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}}\ \varphi(\tfrac{1}{2}\tilde{\ell} j;m) \quad , \tag{2}$$

where $\tilde{\ell} = j \mp \frac{1}{2}$ if $\ell = j \pm \frac{1}{2}$. $\underset{\sim}{\sigma} \equiv \underset{\sim}{\sigma}^{[\frac{1}{2}]}$ are the Pauli matrices. Trivially, for $\ell = 0$ we have

$$\varphi(\tfrac{1}{2} 0 \tfrac{1}{2};m) = \frac{1}{\sqrt{4\pi}}\, \chi_m^{[\frac{1}{2}]} \quad . \tag{3}$$

Combining (2) and (3) one obtains the well known relation:

$$\varphi(\tfrac{1}{2} 1 \tfrac{1}{2};m) = -\frac{1}{\sqrt{4\pi}}\, \hat{\underset{\sim}{r}}\cdot\underset{\sim}{\sigma}\ \chi_m^{[\frac{1}{2}]} \quad . \tag{4}$$

In contrast to (1), (4) (- and trivially also (3)) has <u>manifest</u> rotation symmetry, because the angles only occur through the factor $\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}}$.

For the case $j=3/2$, $\ell=1$, $s=\frac{1}{2}$ it turns out that ϕ can be written similar to (5):

$$\phi(\tfrac{1}{2}1\tfrac{3}{2};m)=\sqrt{\tfrac{3}{4\pi}}\,\hat{\underset{\sim}{r}}\cdot\underset{\sim}{\sigma}^{[\frac{1}{2},\frac{3}{2}]}\,\chi_m^{[\frac{3}{2}]}\qquad, \tag{5}$$

where $\underset{\sim}{\sigma}^{[\frac{1}{2},3/2]}$ are 2×4 matrices for dipole transition, transforming as a vector in the same sense as the spin matrices $\underset{\sim}{S}$ [4,5]. Explicit expressions for these are given in Appendix A. Eqs. (3), (4), (5) are special cases of the more general formula:

$$\phi(s\ell j;m)=c(s\ell j)\hat{r}_{k_1}\hat{r}_{k_2}\cdots\hat{r}_{k_\ell}\,T^{[s,j]}_{k_1k_2\cdots k_\ell}\,\chi_m^{[j]}\qquad, \tag{6}$$

where the matrices $T^{[s,j]}_{k_1--k_\ell}$ are $(2s+1)\times(2j+1)$ 2^ℓ-pole transition matrices having ℓ vector indices k_1--k_ℓ ($\ell=0$ is monopole, $\ell=1$ dipole, $\ell=2$ quadrupole, and so on). The $c(s\ell j)$'s are numerical coefficients depending on s, ℓ and j. Special cases of (6) are explicitly given in Appendix B. Explicit expressions for some matrices are given in Appendix A for $s,j\leq 3$. The generally nonquadratic $(2s+1)\times(2j+1)$ $T^{[s,j]}$-matrices satisfy an algebra together with spin matrices for spin s and j respectively. Some examples are[5]:

$$\sigma_k^{[0,1]}\,S_j^{[1]}=i\,\varepsilon_{kjn}\,\sigma_n^{[0,1]}\qquad, \tag{7a}$$

$$\sigma_k\,\sigma_j^{[\frac{1}{2},\frac{3}{2}]}=K_{kj}^{[\frac{1}{2},\frac{3}{2}]}-\tfrac{1}{2}\,i\,\varepsilon_{kjn}\,\sigma_n^{[\frac{1}{2},\frac{3}{2}]}\qquad, \tag{7b}$$

where the quadrupole transition matrices $K_{kj}^{[\frac{1}{2},3/2]}$ are defined by:

$$K_{kj}^{[\frac{1}{2},\frac{3}{2}]}\equiv\tfrac{1}{2}\left[\sigma_k\,\sigma_j^{[\frac{1}{2},\frac{3}{2}]}+\sigma_j\,\sigma_k^{[\frac{1}{2},\frac{3}{2}]}\right]\qquad. \tag{7c}$$

More algebra of this type is given in Appendix C. The multipole transition matrices (MTM) $T^{[s,j]}$ are irreducible tensor operator matrices, which implies that they are completely symmetric in all vector indices. Moreover, they are traceless in the sense that when two vector indices are contracted we obtain zero:

$$T^{[s,j]}_{kkk_3\cdots k_\ell} = 0 \quad ; \ell \geq 2 \quad . \tag{8}$$

For $s=j$ they are also traceless as matrices:

$$\mathrm{Tr}\left\{ T^{[s,s]}_{k_1\cdots k_\ell} \right\} = 0 \quad ; \ell \geq 1 \quad . \tag{9}$$

Moreover, they are "quasi-hermitian" in the sense that

$$\left(T^{[s,j]}_{k_1\cdots k_\ell} \right)^{\dagger} = T^{[j,s]}_{k_1\cdots k_\ell} \quad , \tag{10}$$

i.e. they are hermitian for $s=j$. $T^{[s,s]}_k = S^{[s]}_k$ are the spin matrices for spin s.

$$T^{[s,s]}_{kn} \equiv K^{[s,s]}_{kn} = \frac{1}{2}\left(S^{[s]}_k S^{[s]}_n + S^{[s]}_n S^{[s]}_k \right) - \frac{1}{3} s(s+1)\, \mathbb{1}^{[s]} \delta_{kn} \tag{11}$$

are the quadrupole matrices for spin s, and so on.

For $s=0$, $j=\ell$, we have only 2^j-pole transitions, and (6) reads in this case:

$$Y^m_\ell(\hat{\underset{\sim}{r}}) = c(0\ell\ell)\, \hat{r}_{k_1} \hat{r}_{k_2} \cdots \hat{r}_{k_\ell}\, T^{[0,\ell]}_{k_1 k_2\cdots k_\ell}\, \chi^{[\ell]}_m \quad . \tag{12}$$

For $s=\frac{1}{2}$, $j=\ell\pm\frac{1}{2}$ we have $2^{(j-\frac{1}{2})}$ and $2^{(j+\frac{1}{2})}$-pole transitions, and we have two versions of (6):

$$\varphi(\tfrac{1}{2}\ell j;m)=c(\tfrac{1}{2}\ell j)\,\hat{r}_{k_1}\cdots\hat{r}_{k_\ell}\,\sigma^{[\frac{1}{2},j]}_{k_1\cdots k_\ell}\,\chi^{[j]}_m \quad , \tag{13a}$$

for odd values of ℓ, and for even values we obtain:

$$\varphi(\tfrac{1}{2}\ell j;m)=c(\tfrac{1}{2}\ell j)\,\hat{r}_{k_1}\cdots\hat{r}_{k_\ell}\,K^{[\frac{1}{2},j]}_{k_1\cdots k_\ell}\,\chi^{[j]}_m \quad . \tag{13b}$$

The matrices $T^{[s,j]}$ can be found[5] from a formula due to Barut et al.[6]. They gave a general formula for $(2s+1)\times(2j+1)$ matrices which transform as four tensors in the same sense as $\sigma_\mu=(1,\underset{\sim}{\sigma})$ transforms as a four vector. Within our context we need only the space part. For $s=0$, $\ell=j=1$, the matrices are given by[4,5,7]

$$\left(\sigma^{[0,1]}_k\right)_{0\beta}=\sum_{\gamma,\delta,\lambda}\langle\tfrac{1}{2}\gamma\tfrac{1}{2}\delta|00\rangle\langle\tfrac{1}{2}\lambda\tfrac{1}{2}\delta|1\beta\rangle\left(\sigma^{[\frac{1}{2}]}_k\right)_{\gamma\lambda} \quad . \tag{14a}$$

For $s=0$ and $s=\frac{1}{2}$ the matrices are constructed for higher spin j by means of the following procedure: For given half integer j; 2^ℓ with $\ell=j-\frac{1}{2}$ is the <u>lowest</u> multipole moment. Then

$$\left(T^{[\frac{1}{2},j]}_{k_1\cdots k_\ell}\right)_{\alpha\beta}=\sum_{\lambda,\delta}\langle 00\tfrac{1}{2}\delta|\tfrac{1}{2}\alpha\rangle\langle\ell\lambda\tfrac{1}{2}\delta|j\beta\rangle\left(T^{[0,\ell]}_{k_1\cdots k_\ell}\right)_{0\lambda} \quad . \tag{14b}$$

The highest $2^{(\ell+1)}$-multipole moment matrices $T^{[\frac{1}{2},j]}_{k_1--k_{\ell+1}}$ can then be obtained from the lowest ones in (14b) by multiplying from left with Pauli matrices and symmetrize with respect to all vector indices, like in (7c). Then for $s=0$, $j=\ell+1$ we obtain:

$$\left(T^{[0,\ell+1]}_{k_1\cdots k_{\ell+1}}\right)_{0\beta}=\sum_{\gamma,\delta,\lambda}\langle\tfrac{1}{2}\gamma\tfrac{1}{2}\delta|00\rangle\langle(\ell+\tfrac{1}{2})\lambda\tfrac{1}{2}\delta|(\ell+1)\beta\rangle\left(T^{[\frac{1}{2},\ell+\frac{1}{2}]}_{k_1\cdots k_{\ell+1}}\right)_{\gamma\lambda} \quad . \tag{14c}$$

Similar construction procedures can be used for $s\geq 1$ [5,6].

2. Angular integrals

Angular integrals involving $\phi(s\ell j;m)$'s and $\hat{\underset{\sim}{r}}$'s can be found in terms of Clebsch-Gordan coefficients, 6-j symbols, 9-j symbols, --- by standard methods[1]. Using (6) however, the pure angular integrals will always at the end be of the following form:

$$\int d^2\hat{\underset{\sim}{r}} = 4\pi \quad , \int d^2\hat{\underset{\sim}{r}}\, \hat{r}_i \hat{r}_j = \frac{4\pi}{3}\delta_{ij} \quad ,$$

$$\int d^2\hat{\underset{\sim}{r}}\, \hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_n = \frac{4\pi}{15}\left[\delta_{ij}\delta_{kn} + \delta_{ik}\delta_{jn} + \delta_{in}\delta_{kj}\right] \; , \; \cdots \; ,$$

$$\int d^2\hat{\underset{\sim}{r}}\, \hat{r}_{i_1}\hat{r}_{i_2}\cdots\hat{r}_{i_{2n-1}}\hat{r}_{i_{2n}} = \frac{4\pi}{(2n+1)!!}\left[\delta_{i_1}\delta_{i_2}\cdots\delta_{i_{2n-1}}\delta_{i_{2n}} + \cdots\cdots \right] \; , \tag{15}$$

where the number of terms in the last parenthesis is $(2n-1)!!$ $((2n+1)!!=1\cdot3\cdot5\cdot\ \ldots\ (2n+1))$. For an odd number of $\hat{\underset{\sim}{r}}$'s such integrals are zero. The advantage of the formalism presented here is therefore the following: When expressions involving ϕ, $\phi^\dagger$, $\hat{\underset{\sim}{r}}$, $\underset{\sim}{S}^{[s]}$, $\hat{\underset{\sim}{r}}\cdot\underset{\sim}{S}^{[s]}$ ..... are integrated over $d^2\hat{\underset{\sim}{r}}$, the indices of the Kroenecker δ's obtained from (15) have to be contracted with the vector indices of (products of) $T^{[s,j]}$ matrices. Many contributions will then cancel due to Eq.(8). Moreover, we can often use the matrix algebra of $T^{[s,j]}$-matrices (- see (7) and Appendix C) to obtain a simple expression in terms of matrix elements of $T^{[s,j]}$ matrices.

As an example, consider integrals of the form:

$$I_{ij\cdots n}(\ell m) = \int d^2\hat{\underset{\sim}{r}}\, \hat{r}_i \hat{r}_j \cdots \hat{r}_n\, Y_\ell^m(\hat{r}) \quad . \tag{16}$$

If the expression (12) is used for Y_ℓ^m we see that the integral in (16) will vanish due to (8) if $\ell >$ the number $\hat{\underset{\sim}{r}}$'s in $\hat{r}_i \hat{r}_j -- \hat{r}_n$. Furthermore, the number of $\hat{\underset{\sim}{r}}$'s plus ℓ must be an even number because angular integrals of the form in (15) are zero for an odd number of $\hat{\underset{\sim}{r}}$'s.

To illustrate these general remarks let us consider the simple case

$$I_{ij}(\lambda m) = \int d^2\hat{\underset{\sim}{r}}\; \hat{r}_i \hat{r}_j\; Y_\lambda^m(\hat{\underset{\sim}{r}}) \quad . \tag{17}$$

This integral will vanish unless $\ell=0$ or 2. Using the expression (12) for Y_ℓ^m and using eq. (15) one finds

$$I_{ij}(\lambda m) = \frac{\sqrt{4\pi}}{3}\delta_{\lambda 0}\delta_{m0}\delta_{ij} + \sqrt{\frac{8\pi}{15}}\,\delta_{\lambda 2}\, K_{ij}^{[0,2]}\, \chi_m^{[2]} \quad . \tag{18}$$

The explicit values of $I_{ij}(\ell m)$ can then be found by using the quadrupole matrices given in Appendix A.

Often one is not interested in $I_{ij}(\ell m)$ itself, but rather the expression

$$\sum_m Y_\lambda^{m*}(\hat{\underset{\sim}{r}}) \int d^2\hat{\underset{\sim}{r}}'\; \hat{r}_i' \hat{r}_j'\; Y_\lambda^m(\hat{\underset{\sim}{r}}') \quad . \tag{19}$$

Inserting the value (18) for the integral, using $\sum_m \chi_m^{[\ell]^+} \chi_m^{[\ell]} = 1$, and in the end using the expression in appendix C for the product $K_{ab}^{[0,2]} K_{cd}^{[2,0]}$, one finds that

$$\sum_m Y_\lambda^{m*}(\hat{\underset{\sim}{r}}) \int d^2\hat{\underset{\sim}{r}}'\; \hat{r}_i' \hat{r}_j'\; Y_\lambda^m(\hat{\underset{\sim}{r}}') = \frac{1}{3}\delta_{ij}\delta_{\lambda 0} + \left(\hat{r}_i\hat{r}_j - \frac{1}{3}\delta_{ij}\right)\delta_{\lambda 2} \quad . \tag{20a}$$

Similarly one can also show that

$$\sum_m Y_\lambda^{m*}(\hat{\underset{\sim}{r}}) \int d^2\underset{\sim}{r}'\; \hat{r}_i' \hat{r}_j' \hat{r}_k'\; Y_\lambda^m(\hat{\underset{\sim}{r}}')$$

$$= \frac{1}{5}\left(\delta_{\lambda 1} - \delta_{\lambda 3}\right)\left(\delta_{ij}\hat{r}_k + \delta_{ik}\hat{r}_j + \delta_{kj}\hat{r}_i\right) + \delta_{\lambda 3}\,\hat{r}_i\hat{r}_j\hat{r}_k \quad . \tag{20b}$$

As another example, let us consider the integral

$$I_k(m,m') = \int d^2\hat{r}\; \hat{r}_k\, Y_{\ell=1}^{m*}(\hat{r})\, Y_{\ell=2}^{m'}(\hat{r}) \qquad . \tag{21}$$

This integral is also easily evaluated within our formalism. First of all, express the Y_ℓ^m's by using Eq.(12) (see also Appendix B). Then, integrate by using Eq.(15). At last, use the algebra of Appendix C to show that $\sigma_j^{[1,0]} K_{jk}^{[0,2]} = \sigma_k^{[1,2]}$, thus giving the answer

$$I_k(m,m') = \sqrt{\frac{2}{5}}\; \chi_m^{[1]\dagger}\, \sigma_k^{[1,2]}\, \chi_{m'}^{[2]} \qquad . \tag{22}$$

Using the explicit expressions for the different $\sigma_k^{[1,2]}$'s of Appendix A, one may for instance show that $I_1(1,2) = -1/\sqrt{5}$.

As a similar example, we obtain for $s=\frac{1}{2}$ and $j=3/2$:

$$\begin{aligned} J_k(m',m) &\equiv \int d^2\hat{r}\; \varphi^\dagger(\tfrac{1}{2}1\tfrac{3}{2};m')\, \hat{r}_k\, Q\, \varphi(\tfrac{1}{2}2\tfrac{3}{2};m) \\ &= -\frac{2}{5}\, \chi_{m'}^{[\frac{3}{2}]}\, \sigma_n^{[\frac{3}{2},\frac{1}{2}]}\, Q\, K_{nk}^{[\frac{1}{2},\frac{3}{2}]}\, \chi_m^{[\frac{3}{2}]} \qquad , \end{aligned} \tag{23a}$$

where Q is some 2×2 matrix independent of $\hat{r}$. For $Q=1$ we obtain (see C.7):

$$J_k(m',m) = -\frac{2}{15}\, \chi_{m'}^{[\frac{3}{2}]\dagger}\, S_k^{[\frac{3}{2}]}\, \chi_m^{[\frac{3}{2}]} \qquad . \tag{23b}$$

Results for angular integrals of type (16)-(23) obtained from the present formalism must of course be equivalent with results obtained by standard methods[1]. From Eqs.(1), (6) and (14) we see in which way the methods are equivalent: The sums of products of n-j symbols - which in general are obtained[1] for integrals of the type in Eqs.(16) to (23) - are built into the MTM's. The MTM's pick out the spherical components of the $\hat{r}$'s in Eq.(6) and combine them in the right way to $Y_\ell^{m_\ell}$'s. The elements of the matrix $[\hat{r}_{k_1}\cdots\hat{r}_{k_\ell}\, T^{[s,j]}_{k_1\cdots k_\ell}]$ are then $Y_\ell^{m_\ell}$'s multiplied with the right Clebsch-Gordon coefficients to obtain $\phi(s\ell j)$.

When the matrices $T^{[s,j]}$ are constructed by means of (14) and their algebra (- see Appendix C) is known, we think that the present formalism will very often turn out to be more effective and transparent than standard methods[1].

3. Interaction energies in central field problems

We will now illustrate how useful our formalism is for calculating interaction energies in problems with spherical symmetry. As we will see, the reason for the usefulness is the manifest rotation symmetry of (6) which allows us to use (15) and the matrix algebra of Appendix C.

3.1. Currents and fields

We will give some examples on how to express currents and fields by means of MTM's. Consider a Dirac particle in some rotation symmetric potential - or in a spherical bag[8]. The wavefunction is generally given by [2,3]

$$\Psi_n(\hat{\underset{\sim}{r}},t) = N_n \begin{pmatrix} f_n(r) \\ i g_n(r)\, \underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}} \end{pmatrix} \varphi_n e^{-i\omega_n t} \quad , \qquad (24)$$

where $n=S_{1/2}, P_{1/2}, P_{3/2}, \cdots$. ϕ_n is some $\phi(s\ell j;m)$ given by (1) and/or (6).

f_n and g_n are radial functions. Phase ambiguities are contained in the normalization factors N_n. We will first show how the different currents involving particles in $S_{\frac{1}{2}}$, $P_{\frac{1}{2}}$ and $P_{3/2}$ states can be expressed using different MTM's. (It is trivial to go from the present results to the colour-electromagnetic case; just insert the Gell-Mann SU(3) colour matrices λ^a in front of every current and electromagnetic field expression.)

In the following, let us introduce the abbreviations

$$S_{1/2} \equiv S \quad , \quad P_{1/2} \equiv P \quad , \quad P_{3/2} \equiv A \quad . \tag{25}$$

The currents of the $s=\frac{1}{2}$ fermions can generally be written as

$$\underset{\sim}{j}_{XY}(\underset{\sim}{r},t) = \Psi_Y^{+} \underset{\sim}{\alpha} \Psi_X \quad . \tag{26}$$

Using Eq.(24) we find that $\underset{\sim}{j}_{XY}$ is given by

$$\underset{\sim}{j}_{XY}(\underset{\sim}{r},t)$$

$$= N_X N_Y^{*} \varphi_Y^{+} \left[(f_X g_Y + g_X f_Y)(\underset{\sim}{\sigma} \times \hat{\underset{\sim}{r}}) - i(f_X g_Y - g_X f_Y)\hat{\underset{\sim}{r}} \right] \varphi_X \, e^{-i(\omega_X - \omega_Y)t} \quad . \tag{27}$$

If X=Y this reduces to the time independent current

$$\underset{\sim}{j}_{XX}(\underset{\sim}{r}) = 2|N_X|^2 f_X g_X \, \varphi_X^{+}(\underset{\sim}{\sigma} \times \hat{\underset{\sim}{r}})\varphi_X \quad . \tag{28}$$

Writing the spherical harmonic spinors ϕ_X for X = S, P and A as in Eq.(3)-(5) and using the algebra of the MTM's (Appendix C) we find that the different currents can be written as a sum of terms involving different MTM's. We find :

$$\underset{\sim}{j}_{ss}(\underset{\sim}{r}) = \frac{|N_s|^2}{2\pi} f_s g_s \, \chi_m^{[\frac{1}{2}]^\dagger} (\underset{\sim}{\sigma} \times \hat{\underset{\sim}{r}}) \, \chi_{m'}^{[\frac{1}{2}]} \quad ,$$

$$\underset{\sim}{j}_{pp}(\underset{\sim}{r}) = -\frac{|N_p|^2}{2\pi} f_p g_p \, \chi_m^{[\frac{1}{2}]^\dagger} (\underset{\sim}{\sigma} \times \hat{\underset{\sim}{r}}) \, \chi_{m'}^{[\frac{1}{2}]} \quad ,$$

$$\underset{\sim}{j}_{AA}(\underset{\sim}{r}) = \frac{|N_A|^2}{\pi} f_A g_A \, \chi_m^{[\frac{3}{2}]^\dagger}$$

$$\cdot \left[\frac{2}{5} (\underset{\sim}{S}^{[\frac{3}{2}]} \times \hat{\underset{\sim}{r}}) - \underset{\sim}{e}_i \, \varepsilon_{ikt} \, \hat{r}_t \, \hat{r}_r \, \hat{r}_n \, \sigma_{rnk}^{[\frac{3}{2}]} \right] \chi_{m'}^{[\frac{3}{2}]} \quad ,$$

$$\underset{\sim}{j}_{PS}(\underset{\sim}{r}, t) = -\frac{i}{4\pi} N_p N_s^* \, \chi_m^{[\frac{1}{2}]^\dagger}$$

$$\cdot \left[(f_p g_s + g_p f_s) \underset{\sim}{\sigma} - 2 f_p g_s \, \hat{\underset{\sim}{r}} (\hat{\underset{\sim}{r}} \cdot \underset{\sim}{\sigma}) \right] \chi_{m'}^{[\frac{1}{2}]} \, e^{-i(\omega_p - \omega_s)t} \quad ,$$

$$\underset{\sim}{j}_{AS}(\underset{\sim}{r}, t) = -\frac{i\sqrt{3}}{8\pi} N_A N_s^* \, \chi_m^{[\frac{1}{2}]^\dagger}$$

$$\cdot \left[(f_A g_s + g_A f_s) \underset{\sim}{\sigma}^{[\frac{1}{2},\frac{3}{2}]} + (f_A g_s - 3 g_A f_s) \hat{\underset{\sim}{r}} \left(\hat{\underset{\sim}{r}} \cdot \underset{\sim}{\sigma}^{[\frac{1}{2},\frac{3}{2}]} \right) \right.$$

$$\left. - 2i \, (f_A g_s + g_A f_s) \, \underset{\sim}{e}_i \, \varepsilon_{ink} \, \hat{r}_n \, \hat{r}_t \, K_{kt}^{[\frac{1}{2},\frac{3}{2}]} \right] \chi_{m'}^{[\frac{3}{2}]} \, e^{-i(\omega_A - \omega_s)t} \quad ,$$

$$\underset{\sim}{j}_{AP}(\underset{\sim}{r},t) = -\frac{i\sqrt{3}}{8\pi} N_A N_P^{*} \chi_m^{[\frac{1}{2}]^+}$$

$$\cdot\Big[(f_A g_P + g_A f_P)\, i(\hat{\underset{\sim}{r}} \times \underset{\sim}{\sigma}^{[\frac{1}{2},\frac{1}{2}]}) - 2(f_A g_P + g_A f_P)\, \underset{\sim}{e}_i \hat{r}_j K_{ij}^{[\frac{1}{2},\frac{1}{2}]}$$

$$+ 4 g_A f_P\, \hat{\underset{\sim}{r}}\, \hat{r}_i \hat{r}_j K_{ij}^{[\frac{1}{2},\frac{1}{2}]}\Big] \chi_{m'}^{[\frac{1}{2}]} e^{-i(\omega_A - \omega_P)t} \quad . \qquad (29)$$

In the above expressions f_S, g_S, f_P, g_P, f_A and g_A are the radial functions coming from Eq.(24), which are assumed to be real. Furthermore, $\underset{\sim}{e}_i$ is the unit vector in the i-direction i.e. $\hat{\underset{\sim}{r}} = \hat{r}_i \underset{\sim}{e}_i$ for instance. Explicit *expressions for the MTM's appearing in* (Eq.(29) can be found in Appendix A. One should notice that the same radial function appears in front of different MTM's in some of the current expressions above.

The current expressions *(29) may now conveniently be used* to calculate the vector potentials and magnetic flux densities. The same kind of calculations will appear both in electromagnetic and in colour-electromagnetic calculations (Bag-models etc.). As an example we will now first show how it is possible to find the vector potential arising *because of* the current $\underset{\sim}{j}_{SS}(\underset{\sim}{r})$.

The vector potential will get a contribution from the volume integral

$$\underset{\sim}{A}_{SS}^{vol}(\underset{\sim}{r}) = \frac{1}{4\pi} \int \frac{\underset{\sim}{j}_{SS}(\underset{\sim}{r}')}{|\underset{\sim}{r} - \underset{\sim}{r}'|}\, d^3\underset{\sim}{r}' \qquad (30)$$

and, depending on the surface conditions, we will get a contribution *from an* integral over the surface. *In performing the* integral (32) it is customary to write

$$\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|} = \sum_{\ell,m} \frac{4\pi}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_\ell^{m^*}(\hat{\underset{\sim}{r}})\, Y_\ell^{m}(\hat{\underset{\sim}{r}}') \quad . \qquad (31)$$

By putting the current $\underset{\sim}{j}_{S}(\underset{\sim}{r})$ Eq.(29) and the expression (31) into (30) one finally arrives at the following angular integral:

$$I_1 = \sum_{\ell,m} Y_\ell^{m*}(\hat{\underset{\sim}{r}}) \int d^2\hat{\underset{\sim}{r}}' (\underset{\sim}{\sigma} \times \hat{\underset{\sim}{r}}') Y_\ell^m(\hat{\underset{\sim}{r}}') \quad , \tag{32}$$

which is easily calculated using the methods of chapter 2. One finds $I_1 = \underset{\sim}{\sigma} \times \hat{\underset{\sim}{r}}$. The contribution from the surface integral to $\underset{\sim}{A}_{SS}(\underset{\sim}{r})$ has the same angular dependence as (32), and one may generally write

$$\underset{\sim}{A}_{SS}(\underset{\sim}{r}) = (\hat{\underset{\sim}{r}} \times \underset{\sim}{\sigma}) A(r) \quad , \tag{33}$$

where $A(r)$ is generally a radial function expressed by integrals over r involving $f_S(r)$, $g_S(r)$ and powers of r.

Similar calculations can be done for the other currents. A little complication arises when the time dependent currents are used. In Eq.(30) one must then also include the time dependence in the current writing

$$\underset{\sim}{A}_{XY}^{vol}(\underset{\sim}{r}, t) = \frac{1}{4\pi} \int \frac{\underset{\sim}{j}_{XY}(\underset{\sim}{r}', t')}{|\underset{\sim}{r} - \underset{\sim}{r}'|} d^3\underset{\sim}{r}' \quad , \tag{34}$$

where $t' = t - |\underset{\sim}{r} - \underset{\sim}{r}'|$. To perform the calculation in (34) we need the formula[9)]

$$\frac{e^{i\omega|\underset{\sim}{r} - \underset{\sim}{r}'|}}{4\pi|\underset{\sim}{r} - \underset{\sim}{r}'|} = i\omega \sum_{\lambda,m} j_\lambda(\omega r_<) h_\lambda^{(1)}(\omega r_>) Y_\lambda^{m*}(\hat{\underset{\sim}{r}}) Y_\lambda^m(\hat{\underset{\sim}{r}}') \quad , \tag{35}$$

which will be used instead of (31) when we have time dependent currents. We see that the angular integrals will be of the same type. The only difference is introduced in the radial integrals, where we now have to integrate over combinations of $f_i(r)$, $g_i(r)$, $j_\ell(\omega r)$ and $n_\ell^{(1)}(\omega r)$ to find the radial functions $A_{XY}^{(i)}(r)$ appearing in the vector potential.

In the present case we see that the vector potential $\underset{\sim}{A}_{SS}(r)$ has the same angular dependence, $(\underset{\sim}{\sigma}\times\hat{\underset{\sim}{r}})$, as the current $\underset{\sim}{j}_{SS}$ (see Eq.(29)). This can be shown to be quite general: The vector potential $\underset{\sim}{A}_{XY}(\underset{\sim}{r},t)$ will have same type of angular dependence as $\underset{\sim}{j}_{XY}(\underset{\sim}{r},t)$. For some of the $\underset{\sim}{j}_{XY}$'s in Eqs.(29), the same radial function is multiplying different MTM's. This is not the case for the corresponding $\underset{\sim}{A}_{XY}$'s.

The electric charge density can of course also be expressed as a sum of terms involving MTM's and calculations of the electric potential can be performed using the same techniques as used when calculating the magnetic vector potential.

3.2 Interaction energies from currents and fields

Let us now, as an example, consider the magnetic interaction between a $P_{3/2}$ fermion and a $S_{\frac{1}{2}}$ fermion in a spherical cavity. One might then calculate

$$H_M = -\frac{1}{2}\int d^3\underset{\sim}{r}\ \underset{\sim}{j}_{AA} \otimes \underset{\sim}{A}_{SS} \quad . \tag{36}$$

Using the expressions (29) and (33) for $\underset{\sim}{j}_{AA}$ and $\underset{\sim}{A}_{SS}$ we arrive at the following expression :

$$H_M = \frac{|N_A|^2}{2\pi}\int dr\, r^2 f_A(r) g_A(r)\, A(r)\ I_{\omega} \quad , \tag{37}$$

where I_Ω is the angular integral (spinors are now suppressed)

$$I_\Omega = \int d\hat{\underset{\sim}{r}} \left[\frac{2}{5} \hat{\underset{\sim}{r}} \times \underset{\sim}{S}^{[\frac{3}{2}]} + \underset{\sim}{e}_i \varepsilon_{ikt} \hat{r}_t \hat{r}_r \hat{r}_n \sigma_{rnk}^{[\frac{3}{2}]} \right] \otimes \left[\hat{\underset{\sim}{r}} \times \underset{\sim}{\sigma} \right] \quad , \tag{38}$$

which can be calculated by using Eq.(15). We find

$$I_\Omega = \frac{16\pi}{15} \underset{\sim}{S}^{[\frac{3}{2}]} \otimes \underset{\sim}{\sigma} \quad , \tag{39}$$

thus giving rise to a pure dipole × dipole interaction. The octupole contribution vanish when the integral is calculated because the MTM's are completely symmetric in the vector indices and Eq.(8).

In the literature[10)] the following expression for the $P_{3/2}$ spinor have been used:

$$\phi(\tfrac{1}{2} 1 \tfrac{3}{2}; \pm\tfrac{1}{2}) \sim \left[\hat{r}_3 - \tfrac{1}{3}(\underset{\sim}{\sigma} \cdot \hat{\underset{\sim}{r}}) \sigma_3 \right] \chi_{\pm\frac{1}{2}}^{[\frac{1}{2}]} \quad . \tag{40}$$

One should however, notice that (40) only gives the subspace $j=3/2$, $m=\pm\frac{1}{2}$. Using (40) instead of the full $\phi(\frac{1}{2},1,3/2;m)$ of Eq.(5) in the calculations above will therefore only give a part of the answer, and one arrives in the end at the magnetic operator working in the subspace $j=3/2$, $m=\pm\frac{1}{2}$ given by

$$H_M \sim \underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma} - \tfrac{1}{2}\, \sigma_3 \otimes \sigma_3 \quad . \tag{41}$$

Since Eq.(41) is valid only for $j=3/2$, $m=\pm\frac{1}{2}$ one has to do the calculation once more for $j=3/2$, $m=\pm 3/2$ using for instance

$$\phi(\tfrac{1}{2} 1 \tfrac{3}{2}; \pm\tfrac{3}{2}) \sim \left[\hat{r}_1 \sigma_3 + i \hat{r}_2 \right] \chi_{\pm\frac{1}{2}}^{[\frac{1}{2}]} \quad . \tag{42}$$

In the end one has to perform a similar calculation even once more, now with a $j=3/2$, $m=\pm\frac{1}{2}$ fermion coming in, and a $j=3/2$, $m=\pm 3/2$ fermion going out (or vice versa).

We assert that the new method proposed here using MTM's leads to calculations far less time consuming and more transparent than the method using expressions (40) and (42) for the spinors.

Let us end this section by writing down the operators (in spin-space) of the magnetic interaction Hamiltonian for some more diagrams. Generally let

$$H_M\begin{pmatrix} ab \\ cd \end{pmatrix} \sim \int d^3\underset{\sim}{r}\; \underset{\sim ab}{j} \otimes \underset{\sim cd}{A} \tag{43}$$

correspond to the diagram of Fig. 1. This will give, using Eqs.(15), (29) and the fact that $\underset{\sim}{A}_{cd}$ has the same angular structure as $\underset{\sim}{j}_{cd}$, the following operators:

$$H_M\begin{pmatrix} SS \\ SS \end{pmatrix} \sim H_M\begin{pmatrix} SS \\ PP \end{pmatrix} \sim H_M\begin{pmatrix} PP \\ PP \end{pmatrix} \sim H_M\begin{pmatrix} SP \\ PS \end{pmatrix} \sim H_M\begin{pmatrix} SP \\ SP \end{pmatrix} \sim \underset{\sim}{\sigma} \otimes \underset{\sim}{\sigma} \quad ,$$

$$H_M\begin{pmatrix} AA \\ SS \end{pmatrix} \sim H_M\begin{pmatrix} AA \\ PP \end{pmatrix} \sim \underset{\sim}{S}^{[\frac{3}{2}]} \otimes \underset{\sim}{\sigma} \quad ,$$

$$H_M\begin{pmatrix} AA \\ AA \end{pmatrix} \sim \underset{\sim}{S}^{[\frac{3}{2}]} \otimes \underset{\sim}{S}^{[\frac{3}{2}]} + \beta\, \sigma_{tnk}^{[\frac{3}{2}]} \otimes \sigma_{tnk}^{[\frac{3}{2}]} \quad ,$$

$$H_M\begin{pmatrix} AS \\ SP \end{pmatrix} \sim H_M\begin{pmatrix} AP \\ SS \end{pmatrix} \sim \underset{\sim}{\sigma}^{[\frac{1}{2},\frac{3}{2}]} \otimes \underset{\sim}{\sigma} \quad ,$$

$$H_M\begin{pmatrix} AS \\ SA \end{pmatrix} \sim H_M\begin{pmatrix} AP \\ PA \end{pmatrix} \sim \underset{\sim}{\sigma}^{[\frac{1}{2},\frac{3}{2}]} \otimes \underset{\sim}{\sigma}^{[\frac{3}{2},\frac{1}{2}]} + \gamma\, K_{tn}^{[\frac{1}{2},\frac{3}{2}]} \otimes K_{tn}^{[\frac{3}{2},\frac{1}{2}]} \quad . \tag{44}$$

where the coefficients in front of the different MTM's involve radial integrals depending on the details of the model. The above structures are however, quite general. It will for instance also appear[11] in weak decay calculations, (see Fig.2)

3.3. Interaction Hamiltonian terms in atomic physics

In atomic physics one is often faced with the problem of calculating matrix elements of interaction Hamiltonian terms involving spin-orbit and spin-spin forces.

Let us consider a spin s particle with orbital angular momentum ℓ and total angular momentum j in the field of another particle (- which may be thought of as the nucleus) with spin operator $\underset{\sim}{N}$. The spin-spin interaction has the form[1,2]:

$$H_{ss} = \frac{h_{ss}}{r^3} S_i^{[s]} \left(\hat{r}_i \hat{r}_j - \frac{1}{3} \delta_{ij} \right) N_j \quad . \tag{45}$$

We will then need

$$B_k(s;\ell'\ell;j'j) \equiv \int d^2\hat{\underset{\sim}{r}}\, \varphi(s\ell'j')^{\dagger} S_k^{[s]} \varphi(s\ell j) \tag{46a}$$

and

$$D_k(s;\ell'\ell;j'j) \equiv \int d^2\hat{\underset{\sim}{r}}\, \hat{r}_k \varphi(s\ell'j')^{\dagger} \hat{\underset{\sim}{r}} \cdot \underset{\sim}{S}^{[s]} \varphi(s\ell j) \quad . \tag{46b}$$

Inserting the expression (6) for the ϕ's and using Eq.(15) for the angular integration we find that B_k is only nonzero for $\ell'=\ell$. Moreover, for any s, ℓ, j, j' B_k is proportional to:

$$T^{[j',s]}_{n_1\cdots n_\ell}\, S^{[s]}_k\, T^{[s,j]}_{n_1\cdots n_\ell} \sim \sigma^{[j',j]}_k \qquad (47)$$

Other terms cancel due to (8). Note that[5] $\sigma_k^{[j',j]}=0$ if $|j'-j|\neq 0,1$

For D_k we get nonzero contributions if $|\ell'-\ell|=0,2$; i.e. in addition to the expression in (47), D_k also gets contributions from two other matrix products:

$$T^{[j',s]}_{n_1\cdots n_\ell k t}\, S^{[s]}_t\, T^{[s,j]}_{n_1\cdots n_\ell} \;;\; T^{[j',s]}_{n_1\cdots n_\ell}\, S^{[s]}_t\, T^{[s,j]}_{t k n_1\cdots n_\ell} \;, \qquad (48)$$

which must also be proportional to $\sigma_k^{[j',j]}$ if they are non-zero.

Some examples (spinors $\chi^{[j']^\dagger}$ and $\chi^{[j]}$ are suppressed):

$$B_k(\tfrac{1}{2};11;\tfrac{1}{2}\tfrac{1}{2}) = -\tfrac{1}{6}\,\sigma_k \;,\; B_k(\tfrac{1}{2};11;\tfrac{3}{2}\tfrac{1}{2}) = -\tfrac{1}{\sqrt{3}}\,\sigma_k^{[\frac{3}{2},\frac{1}{2}]} \;,$$

$$B_k(\tfrac{1}{2};11;\tfrac{3}{2}\tfrac{3}{2}) = \tfrac{2}{3}\,S_k^{[\frac{3}{2}]} \;,\; D_k(\tfrac{1}{2};11;j'j) = \tfrac{1}{5}\,B_k(\tfrac{1}{2};11;j'j) \;,$$

$$B_k(1;11;11) = \tfrac{1}{2}\,S_k^{[1]} \;,\; B_k(1;11;21) = \tfrac{1}{2\sqrt{2}}\,\sigma_k^{[2,1]} \;,$$

$$B_k(1;11;22) = \tfrac{3}{4}\,S_k^{[2]} \;,\; D_k(1;11;j'j) = \tfrac{1}{5}\,B(1;11;j'j) \;. \qquad (49)$$

Using (46)-(49) matrix elements of (45) can be found. Generally, we have shown that the "effective interaction" is proportional to

$$\underset{\sim}{\sigma}^{[j',j]}\cdot\underset{\sim}{N} \;. \qquad (50)$$

The ordinary spin-orbit coupling is trivial because $2\underset{\sim}{L}\cdot\underset{\sim}{S} = \underset{\sim}{J}^2-\underset{\sim}{L}^2-\underset{\sim}{S}^2$. Consider instead the coupling of the orbital angular momentum to the spin $\underset{\sim}{N}$ of the nucleus. Then we need

$$F_k(s;\ell'\ell;j'j) = \int d\hat{\underset{\sim}{r}}\, \varphi(s\ell'j')^{\dagger}\, L_k\, \varphi(s\ell j) \quad . \tag{51}$$

Using (6) and

$$L_k = -i\varepsilon_{knt}\, r_n \frac{\partial}{\partial r_t} \;,\; \frac{\partial f}{\partial r_t} = \frac{\partial f}{\partial r}\hat{r}_t \;,\; \frac{\partial r_k}{\partial r_t} = \delta_{kt} \quad , \tag{52}$$

we find the following expression for $\underset{\sim}{L}\phi$:

$$L_k\, \varphi(s\ell j) = -i\ell\, c(s\ell j)\, \varepsilon_{knt}\, \hat{r}_n \hat{r}_{k_2} \cdots \hat{r}_{k_\ell}\, T^{[s,j]}_{tk_2\cdots k_\ell}\, \chi^{[j]} \quad . \tag{53}$$

Inserting (53) and (6) in (51) we see that F_k is nonzero only for $\ell'=\ell$. Moreover, F_k is proportional to

$$-i\varepsilon_{knt}\, T^{[j',s]}_{nk_2\cdots k_\ell}\, T^{[s,j]}_{tk_2\cdots k_\ell} \sim \sigma_k^{[j',j]} \quad . \tag{54}$$

Some examples (spinors are suppressed):

$$F_k(\tfrac{1}{2};11;\tfrac{1}{2}\tfrac{1}{2}) = \tfrac{2}{3}\sigma_k \quad , \quad F_k(\tfrac{1}{2};11;\tfrac{3}{2}\tfrac{1}{2}) = \tfrac{1}{\sqrt{3}}\sigma_k^{[\frac{3}{2},\frac{1}{2}]} \quad ,$$

$$F_k(\tfrac{1}{2};11;\tfrac{3}{2}\tfrac{3}{2}) = \tfrac{2}{3} S_k^{[\frac{3}{2}]} \quad , \quad F_k(\tfrac{1}{2};22;\tfrac{1}{2}\tfrac{1}{2}) = 0 \quad ,$$

$$F_k(\tfrac{1}{2};22;\tfrac{3}{2}\tfrac{1}{2}) = 0 \quad , \quad F_k(\tfrac{1}{2};22;\tfrac{3}{2}\tfrac{3}{2}) = \tfrac{3}{5} S_k^{[\frac{3}{2}]} \quad . \tag{55}$$

Also in this case (50) gives the form of the "effective interaction".

Consider now <u>two</u> particles with spin $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ respectively in the field of a nucleus - i.e. some Helium-like system. The spin-spin interaction between the two particles have the same form as (45):

$$H_{12} = \frac{h_{12}}{\rho^5} \; S_i^{(1)} \left[\rho_i \rho_j - \frac{1}{3} \rho^2 \delta_{ij} \right] S_j^{(2)} \quad , \tag{56}$$

where $\rho \equiv \underset{\sim}{r}^{(1)} - \underset{\sim}{r}^{(2)}$. To perform angular integrations we use (- similar to (31) and (35)):

$$\frac{1}{\rho^N} = \sum_{LM} f_L^{(N)}(r^{(1)}, r^{(2)}) \, Y_L^M(\hat{\underset{\sim}{r}}^{(1)}) \, Y_L^M(\hat{\underset{\sim}{r}}^{(2)})^* \quad . \tag{57}$$

Let particle 1 and 2 be in states $\phi(s_1 \ell_1 j_1)$ and $\phi(s_2 \ell_2 j_2)$ respectively. Then we obtain for the term $\sim\delta_{ij}$ in (56) :

$$\left\langle \frac{1}{\rho^N} \underset{\sim}{S}^{(1)} \cdot \underset{\sim}{S}^{(2)} \right\rangle = \sum_{LM} f_L^{(N)}(r^{(1)}, r^{(2)}) \, G_k(s_1 \ell_1 j_1; LM) \, G_k^{(*)}(s_2 \ell_2 j_2; LM) \quad , \tag{58}$$

where

$$G_k(s\ell j; LM) \equiv \int d^2\hat{\underset{\sim}{r}} \; \varphi(s\ell j)^\dagger \, S_k^{[s]} \, \varphi(s\ell j) \, Y_L^M(\hat{\underset{\sim}{r}}) \quad . \tag{59}$$

$G_k^{(*)}$ has Y_L^M replaced by $(Y_L^M)^*$. We see that $G_k=0$ for $\ell=0$ unless $L=0$. For $\ell=1$, G_k is nonzero only for $L=0$ and $L = 2$:

$$\begin{aligned} G_k(s1j;00) &\sim \sigma_n^{[j,s]} S_k^{[s]} \sigma_n^{[s,j]} \sim S_k^{[j]} \quad , \\ G_k(s1j;2M) &\sim \sigma_n^{[j,s]} S_k^{[s]} \sigma_t^{[s,j]} \left(K_{nt}^{[0,2]} \chi_M^{[2]} \right) . \end{aligned} \tag{60}$$

Using (see (C.13))

$$\sum_M K_{n_1 t_1}^{[0,2]} \chi_M^{[2]} \chi_M^{[2]\dagger} K_{n_2 t_2}^{[2,0]} = -\frac{1}{3} \delta_{n_1 t_1} \delta_{n_2 t_2} + \frac{1}{2} \left[\delta_{n_1 n_2} \delta_{t_1 t_2} + \delta_{n_1 t_2} \delta_{n_2 t_1} \right] \quad , \tag{61}$$

one can see from the matrix algebra of Appendix C that

$$\sum_M G_k(s_1 1 j_1; 2M)\, G_k^{(M)}(s_2 1 j_2; 2M)$$

$$= a_1 \underset{\sim}{S}^{[j_1]} \otimes \underset{\sim}{S}^{[j_2]} + a_2 K_{kn}^{[j_1]} \otimes K_{kn}^{[j_2]} + a_3 \sigma_{knt}^{[j_1]} \otimes \sigma_{knt}^{[j_2]} \quad . \tag{62}$$

Here $K_{kn}^{[j]}=0$ for $j=\frac{1}{2}$ and $\sigma_{knt}^{[j]}=0$ for $j\leq 1$. Eq.(62) illustrates the general property of the interaction Hamiltonian (56): The "effective interaction" is always of the form dipole × dipole + quadrupole × quadrupole + octupole × octupole + ... The highest contributing multipole moment operators (matrices) depend on s_1, s_2, ℓ_1, ℓ_2, j_1 and j_2.

4. Conclusions

We have given expressions for angular momentum eigenfunctions $\phi(s\ell j)$ in terms of multipole transition matrices (MTM). We have shown by means of some typical examples that the manifest rotation symmetric character of the expressions and the knowledge of the explicite matrix algebra of the MTM's enable us to simplify angular integration calculations considerably. We have been motivated by calculations involving $P_{3/2}$ quarks[10,12] in the bag model. But as we show in section 3.3, our formalism could also be useful in other central field problems as for instance atomic physics. Of course our formalism will - at the present stage - only give explicit expressions for angular integrations involving ϕ's , $\hat{\underset{\sim}{r}}\cdot\underset{\sim}{s}$, ... for the lowest spin/angular momentum values (- for which we know explicit expressions in Appendix A, B and C). But also some general properties like (50) can be shown. In our opinion our formalism has a very transparent and "compact" structure and will often be easier to deal with than the sums of products of n-j symbols which are generally obtained by standard methods.

* * *

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Appendix A. $T^{[s,j]}$-matrices

In this section we will give the explicite expressions for the MTM's. The matrices are given in the representation where $S_3^{[s]}$ is diagonal.

For s=1, j=0 one finds the dipole transition matrices[5,7]:

$$\sigma_1^{[1,0]} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix} \quad , \quad \sigma_2^{[1,0]} = \frac{i}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} \quad , \quad \sigma_3^{[1,0]} = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad . \tag{A.1}$$

For s=3/2, j=½ the dipole transition matrices are[5]:

$$\sigma_1^{[\frac{3}{2},\frac{1}{2}]} = \frac{1}{\sqrt{6}}\begin{bmatrix}-\sqrt{3} & 0\\0 & -1\\1 & 0\\0 & \sqrt{3}\end{bmatrix} \quad , \quad \sigma_2^{[\frac{3}{2},\frac{1}{2}]} = \frac{i}{\sqrt{6}}\begin{bmatrix}\sqrt{3} & 0\\0 & 1\\1 & 0\\0 & \sqrt{3}\end{bmatrix} \quad , \quad \sigma_3^{[\frac{3}{2},\frac{1}{2}]} = \sqrt{\frac{2}{3}}\begin{bmatrix}0 & 0\\1 & 0\\0 & 1\\0 & 0\end{bmatrix} \quad . \tag{A.2}$$

The quadrupole transition matrices for this case are already defined in Eq.(7c).

For s=0, j=2 the quadrupole transition matrices are[5]:

$$K_{11}^{[0,2]} = \frac{1}{2\sqrt{6}}\left(\sqrt{6} \quad 0 \quad -2 \quad 0 \quad \sqrt{6}\right) \quad , \quad K_{22}^{[0,2]} = \frac{-1}{2\sqrt{6}}\left(\sqrt{6} \quad 0 \quad 2 \quad 0 \quad \sqrt{6}\right) \quad ,$$

$$K_{33}^{[0,2]} = \sqrt{\frac{2}{3}}\left(0 \quad 0 \quad 1 \quad 0 \quad 0\right) \quad , \quad K_{12}^{[0,2]} = \frac{-i}{2}\left(-1 \quad 0 \quad 0 \quad 0 \quad 1\right) \quad ,$$

$$K_{13}^{[0,2]} = \frac{1}{2}\left(0 \quad -1 \quad 0 \quad 1 \quad 0\right) \quad , \quad K_{23}^{[0,2]} = \frac{-i}{2}\left(0 \quad 1 \quad 0 \quad 1 \quad 0\right) \quad . \tag{A.3}$$

For $s=\frac{1}{2}$, $j=5/2$ the quadrupole transition matrices are[5]:

$$K_{11}^{[\frac{1}{2},\frac{5}{2}]}=\frac{1}{2\sqrt{5}}\begin{bmatrix}\sqrt{5} & 0 & -\sqrt{2} & 0 & 1 & 0\\ 0 & 1 & 0 & -\sqrt{2} & 0 & \sqrt{5}\end{bmatrix},\quad K_{22}^{[\frac{1}{2},\frac{5}{2}]}=\frac{-1}{2\sqrt{5}}\begin{bmatrix}\sqrt{5} & 0 & \sqrt{2} & 0 & 1 & 0\\ 0 & 1 & 0 & \sqrt{2} & 0 & \sqrt{5}\end{bmatrix},$$

$$K_{33}^{[\frac{1}{2},\frac{5}{2}]}=\sqrt{\frac{2}{5}}\begin{bmatrix}0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\end{bmatrix},\quad K_{12}^{[\frac{1}{2},\frac{5}{2}]}=\frac{-i}{2\sqrt{5}}\begin{bmatrix}-\sqrt{5} & 0 & 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0 & 0 & \sqrt{5}\end{bmatrix},$$

$$K_{13}^{[\frac{1}{2},\frac{5}{2}]}=\frac{1}{\sqrt{10}}\begin{bmatrix}0 & -\sqrt{2} & 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & \sqrt{2} & 0\end{bmatrix},\quad K_{23}^{[\frac{1}{2},\frac{5}{2}]}=\frac{-i}{\sqrt{10}}\begin{bmatrix}0 & \sqrt{2} & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & \sqrt{2} & 0\end{bmatrix}. \tag{A.4}$$

In terms of these, the octupole transition matrices are[5]:

$$\sigma_{k\ell n}^{[\frac{1}{2},\frac{5}{2}]}\equiv\frac{1}{3}\left(\sigma_k K_{\ell n}^{[\frac{1}{2},\frac{5}{2}]}+\sigma_\ell K_{kn}^{[\frac{1}{2},\frac{5}{2}]}+\sigma_n K_{k\ell}^{[\frac{1}{2},\frac{5}{2}]}\right). \tag{A.5}$$

For $s=0$, $j=3$ we find the following octupole transition matrices

$$\sigma_{111}^{[0,3]}=\frac{1}{2\sqrt{10}}\begin{pmatrix}-\sqrt{5} & 0 & \sqrt{3} & 0 & -\sqrt{3} & 0 & \sqrt{5}\end{pmatrix},$$

$$\sigma_{112}^{[0,3]}=\frac{-i}{2\sqrt{10}}\begin{pmatrix}\sqrt{5} & 0 & \frac{-1}{\sqrt{3}} & 0 & \frac{-1}{\sqrt{3}} & 0 & \sqrt{5}\end{pmatrix},$$

$$\sigma_{113}^{[0,3]}=\begin{pmatrix}0 & \frac{1}{\sqrt{12}} & 0 & \frac{-1}{\sqrt{10}} & 0 & \frac{1}{\sqrt{12}} & 0\end{pmatrix},$$

$$\sigma_{122}^{[0,3]}=\frac{1}{2\sqrt{30}}\begin{pmatrix}\sqrt{15} & 0 & 1 & 0 & -1 & 0 & -\sqrt{15}\end{pmatrix},$$

$$\sigma_{123}^{[0,3]}=\frac{i}{\sqrt{12}}\begin{pmatrix}0 & 1 & 0 & 0 & 0 & -1 & 0\end{pmatrix},$$

$$\sigma^{[0,3]}_{222} = \frac{i}{2\sqrt{10}} \left(\sqrt{5} \quad 0 \quad \sqrt{3} \quad 0 \quad \sqrt{3} \quad 0 \quad \sqrt{5} \right) \qquad ,$$

$$\sigma^{[0,3]}_{333} = \sqrt{\tfrac{2}{5}} \left(0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \right) \qquad ,$$

$$\sigma^{[0,3]}_{133} = -\sqrt{\tfrac{2}{15}} \left(0 \quad 0 \quad 1 \quad 0 \quad -1 \quad 0 \quad 0 \right) \qquad ,$$

$$\sigma^{[0,3]}_{223} = -\left(0 \quad \tfrac{1}{\sqrt{12}} \quad 0 \quad \tfrac{1}{\sqrt{10}} \quad 0 \quad \tfrac{1}{\sqrt{12}} \quad 0 \right) \qquad ,$$

$$\sigma^{[0,3]}_{233} = -i\sqrt{\tfrac{2}{15}} \left(0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \right) \qquad . \tag{A.6}$$

For $s=2$, $j=1$ the dipole transition matrices are[5]:

$$\sigma^{[2,1]}_1 = \frac{1}{2\sqrt{3}} \begin{bmatrix} -\sqrt{6} & 0 & 0 \\ 0 & -\sqrt{3} & 0 \\ 1 & 0 & -1 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix}, \quad \sigma^{[2,1]}_2 = \frac{i}{2\sqrt{3}} \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 1 & 0 & 1 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix}, \quad \sigma^{[2,1]}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}, \tag{A.7}$$

and the *quadrupole* and *octupole* transition matrices are defined by[5]:

$$K^{[2,1]}_{kn} \equiv \frac{1}{2} \left(\sigma^{[2,1]}_k S^{[1]}_n + \sigma^{[2,1]}_n S^{[1]}_k \right) \qquad , \tag{A.8}$$

and

$$\sigma^{[2,1]}_{knt} \equiv \frac{1}{3} \left(K^{[2,1]}_{kn} S^{[1]}_t + K^{[2,1]}_{kt} S^{[1]}_n + K^{[2,1]}_{nt} S^{[1]}_k \right) - \frac{1}{5} \left(\delta_{kn} \sigma^{[2,1]}_t + \delta_{kt} \sigma^{[2,1]}_n + \delta_{nt} \sigma^{[2,1]}_k \right) \tag{A.9}$$

respectively.

Appendix B. Explicit expressions for $\phi(s\ell j;m)$

In addition to Eqs. (3), (4) and (5), we find the following expressions for $s=\frac{1}{2}$:

$$\varphi(\tfrac{1}{2}\,2\,\tfrac{3}{2};m) = -\sqrt{\tfrac{3}{4\pi}}\;\hat{r}_k\hat{r}_n\;K^{[\frac{1}{2},\frac{3}{2}]}_{kn}\;\chi^{[\frac{3}{2}]}_m \quad , \tag{B.1}$$

$$\varphi(\tfrac{1}{2}\,2\,\tfrac{5}{2};m) = \sqrt{\tfrac{15}{8\pi}}\;\hat{r}_k\hat{r}_n\;K^{[\frac{1}{2},\frac{5}{2}]}_{kn}\;\chi^{[\frac{5}{2}]}_m \quad . \tag{B.2}$$

For $s=0$, $\phi(0\ell\ell;m) = Y^m_\ell$, and we find:

$$Y^m_{\ell=1}(\hat{\underset{\sim}{r}}) = \sqrt{\tfrac{3}{4\pi}}\;\hat{\underset{\sim}{r}}\cdot\underset{\sim}{\sigma}^{[0,1]}\,\chi^{[1]}_m \quad , \tag{B.3}$$

$$Y^m_{\ell=2}(\hat{\underset{\sim}{r}}) = \sqrt{\tfrac{15}{8\pi}}\;\hat{r}_k\hat{r}_n\;K^{[0,2]}_{kn}\,\chi^{[2]}_m \quad , \tag{B.4}$$

$$Y^m_{\ell=3}(\hat{\underset{\sim}{r}}) = \sqrt{\tfrac{35}{8\pi}}\;\hat{r}_k\hat{r}_n\hat{r}_t\;\sigma^{[0,3]}_{knt}\,\chi^{[3]}_m \quad . \tag{B.5}$$

For the case $s=1$, we find:

$$\varphi(1\,1\,0) = \frac{-1}{\sqrt{4\pi}}\;\hat{\underset{\sim}{r}}\cdot\underset{\sim}{\sigma}^{[1,0]} \quad ,$$

$$\varphi(101;m) = \frac{1}{\sqrt{4\pi}}\;\chi^{[1]}_m \quad ,$$

$$\varphi(111;m) = -\sqrt{\tfrac{3}{8\pi}}\;\hat{\underset{\sim}{r}}\cdot\underset{\sim}{S}^{[1]}\,\chi^{[1]}_m \quad ,$$

$$\varphi(121;m) = \frac{3}{\sqrt{8\pi}}\;\hat{r}_k\hat{r}_n\;K^{[1]}_{kn}\,\chi^{[1]}_m \quad ,$$

$$\varphi(112;m) = \sqrt{\tfrac{3}{4\pi}}\ \hat{\underset{\sim}{r}}\cdot\underset{\sim}{\sigma}^{[1,2]}\chi_m^{[2]} \quad ,$$

$$\varphi(122;m) = -\sqrt{\tfrac{5}{4\pi}}\ \hat{r}_k\hat{r}_n\ K_{kn}^{[1,2]}\chi_m^{[2]} \quad ,$$

$$\varphi(132;m) = \frac{5}{\sqrt{8\pi}}\ \hat{r}_k\hat{r}_n\hat{r}_t\ \sigma_{knt}^{[1,2]}\ \chi_m^{[2]} \quad . \tag{B.6}$$

For $s=\frac{1}{2}$ the simple relation (2) is valid. Using the matrix algebra (see Appendix C) we find the following generalization of (2) to $s=1$ for the lowest ℓ values:

$$\hat{\underset{\sim}{r}}\cdot\underset{\sim}{S}^{[1]}\varphi(101) = -\sqrt{\tfrac{2}{3}}\ \varphi(111) \quad ,$$

$$\hat{\underset{\sim}{r}}\cdot\underset{\sim}{S}^{[1]}\varphi(111) = \frac{-1}{\sqrt{3}}\varphi(121) - \sqrt{\tfrac{2}{3}}\ \varphi(101) \quad ,$$

$$\hat{\underset{\sim}{r}}\cdot\underset{\sim}{S}^{[1]}\varphi(121) = -\frac{1}{\sqrt{3}}\ \varphi(111) \quad ,$$

$$\hat{\underset{\sim}{r}}\cdot\underset{\sim}{S}^{[1]}\varphi(110) = 0 \quad ,$$

$$\hat{\underset{\sim}{r}}\cdot\underset{\sim}{S}^{[1]}\varphi(112) = -\sqrt{\tfrac{2}{5}}\ \varphi(132) - \sqrt{\tfrac{3}{5}}\ \varphi(112) \ ,$$

$$\hat{\underset{\sim}{r}}\cdot\underset{\sim}{S}^{[1]}\varphi(132) = -\sqrt{\tfrac{2}{5}}\ \varphi(122) \quad . \tag{B.7}$$

Appendix C. Matrix algebra

C.1. Non-quadratic matrices

In addition to the algebra in (7a) we obtain for $s=0,\ j=1$:

$$\sigma_k^{[0,1]}\,\sigma_n^{[1,0]} = \delta_{kn} \quad , \tag{C.1}$$

$$\sigma_k^{[1,0]}\,\sigma_n^{[0,1]} = \frac{1}{3}\delta_{kn} + \frac{1}{2}i\,\varepsilon_{knt}S_t^{[1]} - K_{kn}^{[1]} \quad . \tag{C.2}$$

In addition to (7b) we have for $s=\frac{1}{2},\ j=3/2$[5]:

$$\sigma_n K_{kt}^{[\frac{1}{2},\frac{3}{2}]} = \frac{1}{2}i\,\varepsilon_{nkr}K_{rt}^{[\frac{1}{2},\frac{3}{2}]} + \frac{1}{2}i\,\varepsilon_{ntr}K_{rk}^{[\frac{1}{2},\frac{3}{2}]} + \frac{3}{4}\delta_{nt}\sigma_k^{[\frac{1}{2},\frac{3}{2}]} + \frac{3}{4}\delta_{nk}\sigma_t^{[\frac{1}{2},\frac{3}{2}]} - \frac{1}{2}\delta_{kt}\sigma_n^{[\frac{1}{2},\frac{3}{2}]} \quad , \tag{C.3}$$

$$\sigma_k^{[\frac{1}{2},\frac{3}{2}]}S_n^{[\frac{3}{2}]} = \frac{1}{2}K_{kn}^{[\frac{1}{2},\frac{3}{2}]} + \frac{5}{4}i\,\varepsilon_{knt}\sigma_t^{[\frac{1}{2},\frac{3}{2}]} \quad , \tag{C.4}$$

$$K_{kn}^{[\frac{1}{2},\frac{3}{2}]}S_t^{[\frac{3}{2}]} = \frac{3}{4}i\,\varepsilon_{ktr}K_{rn}^{[\frac{1}{2},\frac{3}{2}]} + \frac{3}{4}i\,\varepsilon_{ntr}K_{rk}^{[\frac{1}{2},\frac{3}{2}]} + \frac{3}{8}\delta_{kt}\sigma_n^{[\frac{1}{2},\frac{3}{2}]} + \frac{3}{8}\delta_{nt}\sigma_k^{[\frac{1}{2},\frac{3}{2}]} - \frac{1}{4}\delta_{kn}\sigma_t^{[\frac{1}{2},\frac{3}{2}]} \quad , \tag{C.5}$$

$$\sigma_k^{[\frac{1}{2},\frac{3}{2}]}\,\sigma_n^{[\frac{3}{2},\frac{1}{2}]} = \frac{2}{3}\delta_{kn} - \frac{1}{3}i\,\varepsilon_{knt}\sigma_t \quad , \tag{C.6}$$

$$\sigma_k^{[\frac{1}{2},\frac{3}{2}]}K_{nt}^{[\frac{3}{2},\frac{1}{2}]} = \frac{1}{2}\delta_{kn}\sigma_t + \frac{1}{2}\delta_{kt}\sigma_n - \frac{1}{3}\delta_{nt}\sigma_k \quad , \tag{C.7}$$

$$K_{k\ell}^{[\frac{1}{2},\frac{3}{2}]} K_{mn}^{[\frac{3}{2},\frac{1}{2}]} = \frac{1}{2}\left[\delta_{km}\delta_{\ell n}+\delta_{kn}\delta_{\ell m}\right] - \frac{1}{3}\delta_{k\ell}\delta_{mn}$$

$$+\frac{1}{4}i\left[\delta_{km}\varepsilon_{\ell nr}+\delta_{kn}\varepsilon_{\ell mr}+\delta_{\ell m}\varepsilon_{knr}+\delta_{\ell n}\varepsilon_{kmr}\right]\sigma_r \quad , \qquad \text{(C.8)}$$

$$\sigma_k^{[\frac{3}{2},\frac{1}{2}]}\sigma_n^{[\frac{1}{2},\frac{3}{2}]} = \frac{1}{3}\delta_{kn} + \frac{1}{3}i\,\varepsilon_{knt}S_t^{[\frac{3}{2}]} - \frac{1}{3}K_{kn}^{[\frac{3}{2}]} \quad , \qquad \text{(C.9)}$$

$$K_{kn}^{[\frac{3}{2},\frac{1}{2}]}\sigma_t^{[\frac{1}{2},\frac{3}{2}]} = \frac{1}{10}\delta_{kt}S_n^{[\frac{3}{2}]} + \frac{1}{10}\delta_{tn}S_k^{[\frac{3}{2}]} - \frac{1}{15}\delta_{kn}S_t^{[\frac{3}{2}]}$$

$$+\frac{1}{6}i\,\varepsilon_{ktr}K_{rn}^{[\frac{3}{2}]} + \frac{1}{6}i\,\varepsilon_{ntr}K_{rk}^{[\frac{3}{2}]} - \frac{2}{3}\sigma_{knt}^{[\frac{3}{2}]} \quad , \qquad \text{(C.10)}$$

$$K_{k\ell}^{[\frac{3}{2},\frac{1}{2}]}K_{mn}^{[\frac{1}{2},\frac{3}{2}]} = -\frac{1}{6}\delta_{k\ell}\delta_{mn} + \frac{1}{4}(\delta_{km}\delta_{\ell n}+\delta_{kn}\delta_{\ell m})$$

$$+\frac{3}{20}i\left[\delta_{km}\varepsilon_{\ell nr}+\delta_{kn}\varepsilon_{\ell mr}+\delta_{\ell m}\varepsilon_{knr}+\delta_{\ell n}\varepsilon_{kmr}\right]S_r$$

$$+\frac{1}{3}\left[\delta_{k\ell}K_{mn}+\delta_{mn}K_{k\ell}\right] - \frac{1}{4}\left[\delta_{km}K_{\ell n}+\delta_{kn}K_{\ell m}+\delta_{\ell m}K_{kn}+\delta_{\ell n}K_{km}\right]$$

$$-\frac{1}{6}i\left[\varepsilon_{kmr}\sigma_{r\ell n}+\varepsilon_{knr}\sigma_{r\ell m}+\varepsilon_{\ell mr}\sigma_{rkn}+\varepsilon_{\ell nr}\sigma_{rkm}\right] \quad . \qquad \text{(C.11)}$$

In the last expression (C.11), $\underset{\sim}{S}\equiv\underset{\sim}{S}^{[3/2]}$, $K_{k\ell}\equiv K_{k\ell}^{[3/2]}$ and $\sigma_{k\ell n}\equiv\sigma_{k\ell n}^{[3/2]}$.

For the case $s=0$, $j=2$, we obtain:

$$K_{kn}^{[0,2]}S_t^{[2]} = i\left(\varepsilon_{ktr}K_{rn}^{[0,2]} + \varepsilon_{ntr}K_{rk}^{[0,2]}\right) \qquad \text{(C.12)}$$

$$K_{k\ell}^{[0,2]}K_{mn}^{[2,0]} = -\frac{1}{3}\delta_{k\ell}\delta_{mn} + \frac{1}{2}\left[\delta_{km}\delta_{\ell n}+\delta_{kn}\delta_{\ell m}\right] \qquad \text{(C.13)}$$

For the case $s=2$, $j=1$ one obtains[5]:

$$\sigma_k^{[2,1]}S_n^{[1]} = K_{kn}^{[2,1]} - i\,\varepsilon_{knt}\sigma_t^{[2,1]} \quad , \qquad \text{(C.14)}$$

$$S_k^{[1]}\sigma_n^{[2,1]} = K_{kn}^{[2,1]} + \frac{3}{2}i\,\varepsilon_{knt}\sigma_t^{[2,1]} \quad . \qquad \text{(C.15)}$$

We also find:

$$S_k^{[1]} K_{ij}^{[1,2]} = -\frac{3}{10}\delta_{ij}\sigma_k^{[1,2]} + \frac{9}{20}\left[\delta_{ki}\sigma_j^{[1,2]} + \delta_{kj}\sigma_i^{[1,2]}\right] + \frac{1}{6}i\left(\varepsilon_{kin}K_{nj}^{[1,2]} + \varepsilon_{kjn}K_{ni}^{[1,2]}\right) + \sigma_{kij}^{[1,2]} \quad , \tag{C.16}$$

$$K_{ij}^{[1,2]} S_k^{[2]} = -\frac{3}{10}\delta_{ij}\sigma_k^{[1,2]} + \frac{9}{20}\left(\delta_{ki}\sigma_j^{[1,2]} + \delta_{kj}\sigma_i^{[1,2]}\right) - \frac{5}{6}i\left(\varepsilon_{kin}K_{nj}^{[1,2]} + \varepsilon_{kjn}K_{ni}^{[1,2]}\right) + \sigma_{kij}^{[1,2]} \quad , \tag{C.17}$$

$$\sigma_k^{[1,0]} K_{ij}^{[0,2]} = -\frac{1}{5}\delta_{ij}\sigma_k^{[1,2]} + \frac{3}{10}\left[\delta_{ki}\sigma_j^{[1,2]} + \delta_{kj}\sigma_i^{[1,2]}\right] + \frac{1}{3}i\left(\varepsilon_{kin}K_{nj}^{[1,2]} + \varepsilon_{kjn}K_{ni}^{[1,2]}\right) - \sigma_{kij}^{[1,2]} \quad , \tag{C.18}$$

$$\sigma_k^{[1,2]}\sigma_n^{[2,1]} = \frac{1}{3}\delta_{kn} + \frac{1}{2}i\,\varepsilon_{knt}S_t^{[1]} - K_{kn}^{[1]} \quad , \tag{C.19}$$

$$\sigma_k^{[2,1]}\sigma_n^{[1,2]} = \frac{1}{3}\delta_{kn} + \frac{1}{4}i\,\varepsilon_{knt}S_t^{[2]} - \frac{1}{6}K_{kn}^{[2]} \quad , \tag{C.20}$$

$$K_{k\ell}^{[2,1]}\sigma_n^{[1,2]} = -\frac{1}{20}\delta_{k\ell}S_n^{[2]} + \frac{3}{40}\left(\delta_{kn}S_\ell^{[2]} + \delta_{\ell n}S_k^{[2]}\right) + \frac{1}{12}i\left(\varepsilon_{knt}K_{t\ell}^{[2]} + \varepsilon_{\ell nt}K_{tk}^{[2]}\right) - \frac{1}{6}\sigma_{k\ell n}^{[2]} \quad , \tag{C.21}$$

For the case $s=\frac{1}{2}$, $j=5/2$ one obtains[5]:

$$\sigma_t K_{kn}^{[\frac{1}{2},\frac{5}{2}]} = \frac{1}{3}i\left(\varepsilon_{ktr}K_{rn}^{[\frac{1}{2},\frac{5}{2}]} + \varepsilon_{ntr}K_{rk}^{[\frac{1}{2},\frac{5}{2}]}\right) + \sigma_{knt}^{[\frac{1}{2},\frac{5}{2}]} \quad , \tag{C.22}$$

$$K_{kn}^{[\frac{1}{2},\frac{5}{2}]} S_t^{[\frac{5}{2}]} = \frac{7}{6}i\left(\varepsilon_{ktr}K_{rn}^{[\frac{1}{2},\frac{5}{2}]} + \varepsilon_{ntr}K_{rk}^{[\frac{1}{2},\frac{5}{2}]}\right) + \frac{1}{2}\sigma_{knt}^{[\frac{1}{2},\frac{5}{2}]} \quad , \tag{C.23}$$

$$\sigma_n \sigma_{klm}^{[\frac{1}{2},\frac{5}{2}]} = -\tfrac{1}{3} i \left(\varepsilon_{knr} \sigma_{rlm} + \varepsilon_{lnr} \sigma_{rkm} + \varepsilon_{mnr} \sigma_{rkl} \right)^{[\frac{1}{2},\frac{5}{2}]}$$

$$+ \tfrac{5}{9} \left(\delta_{kn} K_{lm} + \delta_{ln} K_{km} + \delta_{mn} K_{kl} \right)^{[\frac{1}{2},\frac{5}{2}]}$$

$$- \tfrac{2}{9} \left(\delta_{kl} K_{mn} + \delta_{km} K_{ln} + \delta_{ml} K_{kn} \right)^{[\frac{1}{2},\frac{5}{2}]} , \qquad \text{(C.24)}$$

$$\sigma_{klm}^{[\frac{1}{2},\frac{5}{2}]} S_n^{[\frac{5}{2}]} = \tfrac{5}{6} i \left(\varepsilon_{knr} \sigma_{rlm} + \varepsilon_{lnr} \sigma_{rkm} + \varepsilon_{mnr} \sigma_{rkl} \right)^{[\frac{1}{2},\frac{5}{2}]}$$

$$+ \tfrac{5}{18} \left(\delta_{kn} K_{lm} + \delta_{ln} K_{km} + \delta_{mn} K_{kl} \right)^{[\frac{1}{2},\frac{5}{2}]}$$

$$- \tfrac{1}{9} \left(\delta_{kl} K_{mn} + \delta_{ml} K_{nk} + \delta_{km} K_{ln} \right)^{[\frac{1}{2},\frac{5}{2}]} , \qquad \text{(C.25)}$$

$$K_{kl}^{[\frac{3}{2},\frac{5}{2}]} K_{mn}^{[\frac{5}{2},\frac{1}{2}]} = \tfrac{3}{10} \left[\delta_{km} \delta_{ln} \quad \delta_{lm} \right] - \tfrac{1}{5} \delta_{kl} \delta_{mn}$$

$$- \tfrac{1}{10} i \left[\delta_{km} \varepsilon_{lnr} + \delta_{kn} \varepsilon_{lmr} + \delta_{lm} \varepsilon_{knr} + \delta_{ln} \varepsilon_{kmr} \right] \sigma_r \quad . \qquad \text{(C.26)}$$

For $s=0$, $\ell=j=3$ we obtain:

$$\sigma_{klm}^{[0,3]} S_n^{[3]} = i \left(\varepsilon_{knr} \sigma_{rlm}^{[0,3]} + \varepsilon_{lnr} \sigma_{rkm}^{[0,3]} + \varepsilon_{mnr} \sigma_{r}^{[0,3]} \right) , \qquad \text{(C.27)}$$

$$
\begin{aligned}
\sigma^{[0,3]}_{k\ell m}\,\sigma^{[3,0]}_{nrt} = -\frac{1}{15}\big[\,&\delta_{k\ell}\delta_{mn}\delta_{rt}+\delta_{k\ell}\delta_{mr}\delta_{nt}+\delta_{k\ell}\delta_{mt}\delta_{rn}\\
&+\delta_{km}\delta_{\ell n}\delta_{rt}+\delta_{km}\delta_{\ell r}\delta_{nt}+\delta_{km}\delta_{\ell t}\delta_{rn}\\
&+\delta_{\ell m}\varepsilon_{kn}\delta_{rt}+\delta_{\ell m}\delta_{kr}\delta_{nt}+\delta_{\ell m}\delta_{kt}\delta_{rn}\,\big]\\
+\frac{1}{6}\big[\,&\delta_{kn}\delta_{\ell r}\delta_{mt}+\delta_{kn}\delta_{\ell t}\delta_{mr}+\delta_{kr}\delta_{\ell n}\delta_{mt}\\
&+\delta_{kr}\delta_{\ell t}\delta_{mn}+\delta_{kt}\delta_{\ell n}\delta_{mr}+\delta_{kt}\delta_{\ell r}\delta_{mn}\,\big]\quad, \qquad (C.28)
\end{aligned}
$$

Of course we could continue to construct algebraic relations for MTM's for higher values of s, ℓ and j. However, we think that the general pattern of such algebras are clarified through the examples (C.1)-(C.29): The left hand side will generally be the product of *a* $T^{[j_1,s]}_{k_1--k_{\ell_1}}$ *and a* $T^{[s,j_2]}_{n_1--n_{\ell_2}}$ MTM. This will generally be a linear combination of different $T^{[j_1,j_2]}_{i_1--i_\ell}$ MTM's. The number of such matrices - which is characterized by the maximal value of ℓ - is determined by j_1 and j_2. For given ℓ_1 and ℓ_2, the MTM at the right hand side with the highest ℓ-value has (at most) $\ell=\ell_1+\ell_2$ (- as an illustrative example (C.20) does not contain the matrices $\sigma^{[2]}_{k\ell n}$ even if these exist.) In addition to the different $T^{[j_1,j_2]}_{i_1--i_\ell}$'s, the right hand side also contains products of Kroenecker δ's and the Levi-Civita tensor ε_{ijk}. In constructing the algebra for given j_1, j_2, ℓ_1, ℓ_2 and s, one can write down the most general expression for the right hand side (- with undetermined coefficients in front of the MTM's), *remembering* that

δ_{ij}, $K_{ij}^{[j_1,j_2]}$, $K_{ijnk}^{[j_1,j_2]}$ ---- are tensors and ε_{ijk}, $\sigma_k^{[j_1,j_2]}$, $\sigma_{kij}^{[j_1,j_2]}$ are pseudotensors. The different numerical coefficients appearing on the right hand side of (C.1)-(C.28) can then be determined by calculating the product of matrices at the left hand side for some explicit cases.

C.2. Quadratic matrices

In addition to (C.1)-(C.28) for non-quadratic matrices, we will sometimes need some algebra for ordinary (i.e. quadratic) spin, quadrupole, octupole --- matrices. The quadrupole matrices are defined in (11) for arbitrary spin s. Similarly the octupole matrices are[13)]

$$T_{kij}^{[s,s]} \equiv \sigma_{kij}^{[s]} \equiv \frac{1}{3!}\left[S_k S_i S_j + \text{-----}\right]^{[s]} - \frac{1}{15}\left[3s^2+3s-1\right]\left[\delta_{ki} S_j + \text{--}\right]^{[s]} \quad , \qquad \text{(C.29)}$$

which in addition to (8) and (9) satisfy

$$\mathrm{Tr}\left\{ S_n^{[s]} \sigma_{kij}^{[s]} \right\} = 0 \quad . \qquad \text{(C.30)}$$

For arbitrary s we obtain:

$$S_k^{[s]} S_n^{[s]} = \frac{1}{3} s(s+1)\,\delta_{kn} + \frac{1}{2} i\,\varepsilon_{knt} S_t^{[s]} + K_{kn}^{[s]} \quad , \qquad \text{(C.31)}$$

$$S_k^{[s]} K_{ij}^{[s]} = -\frac{1}{30}\left[4s(s+1)-3\right]\delta_{ij} S_k^{[s]} + \frac{1}{20}\left[4s(s+1)-3\right]\left[\delta_{ik} S_j^{[s]} + \delta_{jk} S_i^{[s]}\right]$$
$$+ \frac{1}{2} i \left[\varepsilon_{kit} K_{tj}^{[s]} + \varepsilon_{kjt} K_{ti}^{[s]}\right] + \sigma_{kij}^{[s]} \quad , \qquad \text{(C.32)}$$

and so on.

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Figure captions

Fig. 1: Quark diagram corresponding to colour magnetic interaction.

Fig. 2: Quark diagram contributing to a weak parity violating transition.

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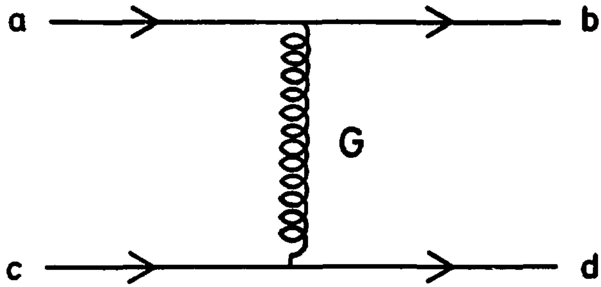

Fig. 1

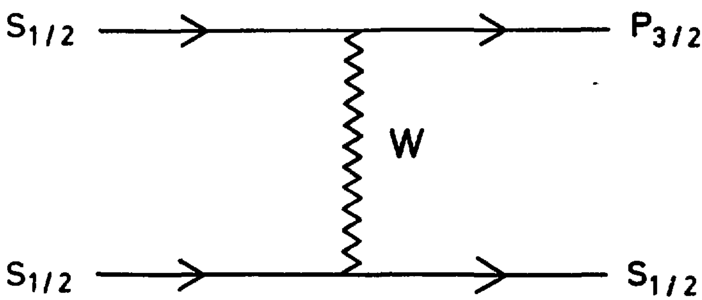

Fig.2