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ON KINEMATICAL MINIMUM PRINCIPLES FOR RATES AND INCREMENTS IN PLASTICITY

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ABSTRACT

The optimization approach for elastoplastic analysis is discussed showing that some minimum principles related to numerical methods can be derived by means of duality and penalization procedures.

Three minimum principles for velocity and plastic multiplier rate fields are presented in the framework of perfect plasticity. The first one is the classical Greenberg formulation. The second one, due to Capurso, is developed here with different motivation, and modified by penalization of constraints so as to arrive at a third principle for rates.

The counterparts of these optimization formulations in terms of discrete increments of displacements and plastic multipliers are discussed. The third one of these minimum principles for finite increments is recognized to be closely related to Maier's formulation of holonomic plasticity.

RESUMO

~~Neste trabalho~~ Mostra-se que alguns princípios de mínimo relacionados a métodos numéricos para a análise elasto-plástica podem ser obtidos através de técnicas de penalização e dualidade.

Três princípios de mínimo para a velocidade e a taxa de deformação plástica são apresentados dentro do contexto da plasticidade ideal. O primeiro é o princípio clássico de Greenberg. O segundo, proposto por Capurso, é obtido a partir de uma formulação diferente que, posteriormente, via penalização é modificado de maneira a obter o terceiro princípio para velocidades e taxas. Apresenta-se também a formulação incremental destes princípios mostrando que o terceiro princípio variacional inclui como caso particular a formulação de Maier para plasticidade holonômica.

ON KINEMATICAL MINIMUM PRINCIPLES FOR RATES AND INCREMENTS IN PLASTICITY

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SUMMARY. The optimization approach for elastoplastic analysis is discussed showing that some minimum principles related to numerical methods can be derived by means of duality and penalization procedures.

Three minimum principles for velocity and plastic multiplier rate fields are presented in the framework of perfect plasticity. The first one is the classical Greenberg formulation. The second one, due to Capurso, is developed here with different motivation, and modified by penalization of constraints so as to arrive at a third principle for rates.

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1. INTRODUCTION

Optimization theory and mathematical programming techniques are increasingly used for the analysis of stress and deformation in elastic-plastic structures [1,2]. Both theoretical understanding and numerical methods are developed in such a way. A remarkable advantage of this approach is that the possibility of local elastic unloading and the consideration of plastic admissibility of final stresses for finite steps of load can be taken into account without a posteriori iterations.

The aim of this paper is to show that minimum principles directly related to numerical methods can be derived by means of basic concepts in optimization theory such as duality and penalization. The relation between optimum principles for rates and several formulations in terms of finite increments of kinematical unknowns is discussed. In particular, a formulation in finite increments of displacements and plastic multipliers is developed as the discrete version of a minimum principle in rates. These principles are shown to be closely related to Maier's formulation of holonomic incremental plasticity [4,1].

The framework of the discussion is the incremental (historical) analysis of structures under quasi-static loading process considering elastic-perfectly plastic homogeneous behaviour of the body. Workhardening and singular yielding functions are avoided for the only purpose of simplicity. Complete formulations taking account of these aspects of material behavior have been proposed [6,9,1] as extensions of some of the principles for the ideal case. Also for the special features focused in this paper workhardening and singularities in yield locus can be introduced without destroying the basic proposal.

The notations for kinematics and the variational deffinition of equilibrium are stablished first, and afterward a local constitutive equation is specified. This approach facilitates the interpretation of the characteristics of approximate incremental solutions.

Three kinematical minimum principles for velocity and plastic multiplier rate fields are presented next. The first one is the classical Greenberg formulation [3] in terms of velocity, and the second is the principle proposed by Capurso [5,6] involving both mentioned fields. The equivalence between them is shown using optimization concepts. This second principle appears as a natural consequence of the necessity to avoid the nondifferentiability of the functional in Greenberg's principle. The second principle is modified by introducing a penalty function so as to arrive to a third principle for rates in which some of the constraints are released.

The usefulness of this third principle is not justified in the context of the problem of rate determination. The motivation of the aforementioned rate principle is apparent when the discrete version of the three formulations are analyzed for finite increments of displacements and plastic multipliers.

The third principle reduces the incremental elastoplastic analysis to a quadratic optimization problem. Whenever the structure is essentially discrete or has been approximated by a discrete model, this problem turns to be a quadratic programming (QP) problem which can be solved by any general QP algorithm or linear complementarity (LC) algorithm [1,2]. It is also possible to simplify this QP problem by elimination of displacements [1], taking advantage of usual techniques to solve elastic problems. This is a very attractive implementation of the method and it is certainly related to a minimum principle in terms of plastic multiplier rates only, in some way similar to the one stablished by Ceradini [7]. The latter topics are not developed in this paper.

2. KINEMATICS AND EQUILIBRIUM

The body occupies the reference region B in which position vector x distinguishes particles and vector u identifies displacements. This presentation is restricted to small deformations, then any configuration of the body is assumed to coincide with B whenever necessary.

We denote U the set of all displacement fields considered sufficiently regular. Let W be the space of strain fields E. The linear operator $\mathcal{D}$ establishes kinematical compatibility so that

$$E = \mathcal{D}(u) \tag{1}$$

The body is subjected to homogeneous displacement constraints in a part Γ_u of the boundary Γ of B. Displacements fields that are kinematically admissibles for the adopted supports constitute the subspace

$$V = \{u \in U \ ; \ u|_{\Gamma_u} = 0\} \tag{2}$$

Admissible velocity fields v are also elements of V. Consequently, the strain rate D is given by

$$D = \mathcal{D}(v) \tag{3}$$

External load L and stress field T respectively belong to U' and W' which are dual spaces of U and V. At any stage of the process the known loading is specified by means of the volumetric density b and surface tractions τ defined in Γ_T. The part Γ_T of Γ is the complement of Γ_u. The work done by the load L over the system, for a displacement u, is given by

$$L(u) = \int_B b \cdot u \, dB + \int_{\Gamma_T} \tau \cdot u \, d\Gamma \tag{4}$$

The external work rate is denoted

$$\dot{L}(u) = \int_B \dot{b} \cdot u \, dB + \int_{\Gamma_T} \dot{\tau} \cdot u \, d\Gamma \tag{5}$$

The internal work corresponding to a stress field T and a strain field E is written

$$-\int_B T \cdot E \, dB \tag{6}$$

The stress distribution T is in equilibrium with a load L if the virtual work principle is verified

$$\int_B T \cdot D(v) \, dB = L(v) \quad \forall \quad v \in V \tag{7}$$

Calling S the set of all stress fields in equilibrium with a fixed load the above condition reads $T \in S$.

Likewise, the stress rate field $\dot{T}$ equilibrates the external power $\dot{L}$ if

$$\int_B \dot{T} \cdot D(v) \, dB = \dot{L}(v) \quad \forall \quad v \in V \tag{8}$$

3. CONSTITUTIVE RELATIONS

It is assumed that the body is homogeneous and constituted by an elastic perfectly plastic material. Then, stress-strain equations are given by the following relations

i) plasticity criteria

$$f(T) \le 0 \tag{9}$$

where f is admitted convex and regular.

ii) associated flow rule $\dot{T} = \dot{T}(T,D)$, given by

$$\dot{T} = \mathbb{D}\,[D - \dot{\lambda}(T,D) f_T(T)] \tag{10}$$

where $\mathbb{D}$ is the symmetric positive definite elasticity tensor, f_T the gradient of f, and

$$\dot{\lambda}(T,D) = S(f(T)) \, \frac{(\mathbb{D} D \cdot f_T(T))^+}{\mathbb{D} f_T(T) \cdot f_T(T)} \tag{11}$$

with the notations S for the Heaviside function

$$S(c) = \begin{cases} 0 & \text{if} \quad c < 0 \\ 1 & \text{if} \quad c \ge 0 \end{cases} \tag{12}$$

and

$$(c)^+ = \max(c,0) \tag{13}$$

Equation (11) determines $\dot{\lambda}$ as a function of T and D and is equivalent to the following set of relations

$$\text{For} \quad f(T) = 0 : \quad \dot{\lambda} \geq 0 \quad \dot{f} \leq 0 \quad \dot{\lambda}\,\dot{f} = 0 \tag{14}$$

$$\text{For} \quad f(T) < 0 : \quad \dot{\lambda} = 0 \tag{15}$$

where the notation

$$\dot{f} = \mathbb{D}\,D\cdot f_T(T) - \dot{\lambda}\;\mathbb{D}\,f_T(T)\cdot f_T(T) \tag{16}$$

holds for the time derivative of f evaluated as a function of T and D by substituting equation (10) in $\dot{f} = f_T\cdot\dot{T}$.

The set P of stress fields satisfying plastic admissibility is defined next to simplify the notation

$$P = \{T \in W' \;;\; f(T(x)) \leq 0 \quad \forall \quad x \in B\} \tag{17}$$

4. ELASTOPLASTIC ANALYSIS FOR RATES

The elastoplastic problem in terms of rates is stated at an instant t when the stress distribution $T \in K \equiv P \cap S$ and displacement field $u \in V$ are assumed to be know. The kinematical formulation for rates is the problem of finding a velocity field $\dot{u} \in V$ such that

$$\int_B \dot{T}\cdot D(v)\,dB = \dot{L}(v) \quad \forall \quad v \in V \tag{18}$$

where $\dot{T}$ is related with $\dot{u}$ by the constitutive equation

$$\dot{T} = \mathbb{D}\,[D(\dot{u}) - \dot{\lambda}(T, D(\dot{u}))\; f_T(T)] \tag{19}$$

and $\dot{\lambda}(T, D(\dot{u}))$ is determined by equation (11).

The solution of this problem in ideal plasticity is not unique in general for admissible loads, even when rigid movements are ruled out. For the collapse load the problem in velocities certainly has no unique solution. A characterization of the solution of this problem by means of a minimum principle is useful when numerical methods are considered.

In what follows several functionals are stated, using the known fields $f = f(T)$ and $f_T = f_T(T)$, and used to establish minimum principles for the elastic plastic problem. From now on it is not explicity denoted the dependence of any field with respect to the fixed stress field T.

PRINCIPLE I

The solutions $\dot{u}$ of the elastoplastic problem coincide with the solutions of the following optimization problem

$$\min_{v \in V} \pi_1(v) \tag{20}$$

where

$$\pi_1(v) = \frac{1}{2}\int_B [\mathbb{D}\mathcal{D}(v)\cdot\mathcal{D}(v) - \mathbb{D}\mathcal{D}(v)\cdot f_T\ \dot{\lambda}(v)]dB - \dot{L}(v) \tag{21}$$

and

$$\dot{\lambda}(v) = S(f)\ \frac{(\mathbb{D}\mathcal{D}(v)\cdot f_T)^+}{\mathbb{D}f_T\cdot f_T} \tag{22}$$

The equivalence between this principle, proposed by Greenberg [3], and the previous statement of the problem in terms of velocity is a well known relation [8] that will not be proven here.

The lack of differentiability of π_1 is a remarkable characteristic of this formulation that should be considered when this principle is invoqued to generate approximations in discrete intervals of t. Let us call π_1^d and π_1^{nd} the respective differentiable and non-differentiable terms in π_1

$$\pi_1(v) = \pi_1^d(v) + \pi_1^{nd}(v) \tag{23}$$

$$\pi_1^d(v) = \frac{1}{2}\int_B \mathbb{D}\mathcal{D}(v)\cdot\mathcal{D}(v)dB - \dot{L}(v) \tag{24}$$

$$\pi_1^{nd}(v) = -\frac{1}{2}\int_B \mathbb{D}\mathcal{D}(v)\cdot f_T\dot{\lambda}(v)dB \tag{25}$$

The inconvenient presence of a non-differentiable functional in the formulation can be avoided by the introduction of a new functional, depending on both velocity $\dot{u}$ and plastic multipliers $\dot{\lambda}$, that must be appropriate to take the place of π_1^{nd}. We define for this purpose

$$\phi(D,\hat{\lambda}) = \int_B \left(\frac{1}{2}\, \mathbb{D} f_T \cdot f_T \hat{\lambda}^2 - \mathbb{D} D \cdot f_T \hat{\lambda}\right) dB \tag{26}$$

The following properties show the relation between ϕ and π_2^{nd}

i) For any $T \in P$ and any $D^* \in W$ there exists a single solution $\hat{\lambda}^*$ of the problem

$$\min_{\hat{\lambda} \in \Lambda_f} \phi(D^*,\hat{\lambda}) \tag{27}$$

where Λ_f is the cone defined in the space of scalar fields in B by means of

$$\Lambda_f = \{\hat{\lambda} : \ \hat{\lambda}(x) \geq 0 \quad \forall \quad x \in B \quad ; \quad \hat{\lambda}(x) = 0 \quad \text{if} \quad f(T(x)) < 0\} \tag{28}$$

ii) The minimization of ϕ with respect to $\hat{\lambda}$ imposes the constitutive relation (10). In other words, the solution $\hat{\lambda}^*$ of (27) satisfies

$$\hat{\lambda}^* = \hat{\lambda}(T,D^*) \qquad \forall \qquad x \in B \tag{29}$$

iii) The minimum value of ϕ for $\hat{\lambda}$ in Λ_f verifies

$$\phi(D(v^*),\hat{\lambda}^*) = \pi_2^{nd}(v^*) \tag{30}$$

To prove assertion (i) we apply the property that any functional of the form

$$\int_B (\alpha\, \hat{\lambda}^2 + \beta\, \hat{\lambda})\, dB \tag{31}$$

is strictly convex and coercive if $\alpha(x)$ is strictly positive for any x in B. This condition is valid for the functional ϕ because $\mathbb{D}$ is positive definite and f_T is nonzero due to the assumed regularity of f. The domain Λ_f where problem (27) is defined is a convex set. Thus, the problem stated in (27) is a strictly convex optimization problem, and since the domain is not empty, we conclude that there exists an unique solution $\hat{\lambda}^*$ for this problem.

In order to prove property (ii), the variational optimality condition for the problem (27) is considered now

$$\int_B (\mathbb{D} f_T \cdot f_T \hat{\lambda}^* - \mathbb{D} D^* \cdot f_T)(\hat{\lambda} - \hat{\lambda}^*)\, dB \geq 0 \qquad \forall \qquad \hat{\lambda} \in \Lambda_f \tag{32}$$

The particularization of this inequality for $\hat{\lambda} = 0$ and also for $\hat{\lambda} = 2\hat{\lambda}^*$ leads to

$$\int_B (\mathbb{D}\, f_T \cdot f_T \hat{\lambda}^* - \mathbb{D}\, D^* \cdot f_T)\hat{\lambda}^* \, dB = 0 \tag{33}$$

This equality is substituted in the previous inequality to obtain

$$\int_B (\mathbb{D}\, f_T \cdot f_T \hat{\lambda}^* - \mathbb{D}\, D^* \cdot f_T)\hat{\lambda} \, dB \geq 0 \quad \forall \quad \hat{\lambda} \in \Lambda_f \tag{34}$$

According to the fundamental theorem of variational calculus this condition implies the following local relation

$$\mathbb{D}\, f_T \cdot f_T \hat{\lambda}^* - \mathbb{D}\, D^* \cdot f_T \geq 0 \quad \forall \quad x \in B \tag{35}$$

Combining this inequality with (33) and also with

$$\hat{\lambda}^* \geq 0 \quad \forall \quad x \in B \tag{36}$$

it follows an additional local property

$$(\mathbb{D}\, f_T \cdot f_T \hat{\lambda}^* - \mathbb{D}\, D^* \cdot f_T)\hat{\lambda}^* = 0 \quad \forall \quad x \in B \tag{37}$$

Relations (35), (36) and (37) are the constitutive equations (14) corresponding to points of the body where $f = 0$. Any other point has $\hat{\lambda}^* = 0$, that is, the second part of the constitutive relation, equation (15) is imposed by the definition (28) of the domain Λ_f. This completes the proof of the property (ii).

Property (iii) can be proven now. From equation (33) substituted in (26) it follows that

$$\phi(D^*, \lambda^*) = -\frac{1}{2} \int_B \mathbb{D}\, D^* \cdot f_T \hat{\lambda}^* dB \tag{38}$$

Using this equation for $D^* = \mathcal{D}(v^*)$ and the result stated in (ii) we get

$$\phi(\mathcal{D}(v^*), \hat{\lambda}^*) = \pi_1^{nd}(v^*) \tag{39}$$

So the last property, namely (iii), is shown to be valid.

As a consequence of this third property, equation (30), and using also equation (23), the functional π_1 may be written in the following form

$$\pi_1(v) = \min_{\mathring{\lambda} \in \Lambda_f} [\pi_1^d(v) + \phi(\mathcal{D}(v),\mathring{\lambda})] \tag{40}$$

This expression is used next to deduce from the classical formulation (20) the Capurso principle [5] stated below

PRINCIPLE II

All the pairs $(\mathring{u},\mathring{\lambda}(\mathring{u}))$, containing a solution field $\mathring{u}$ for the elastoplastic problem (20) and its corresponding plastic multiplier via the material relation (22), constitute the set of solutions of

$$\min_{\substack{v \in V \\ \mathring{\lambda} \in \Lambda_f}} \pi_2(v,\mathring{\lambda}) \tag{41}$$

where

$$\pi_2(v,\mathring{\lambda}) = \pi_1^d(v) + \phi(\mathcal{D}(v),\mathring{\lambda}) =$$

$$= \int_B [\frac{1}{2} \mathbb{D}\mathcal{D}(v)\cdot\mathcal{D}(v) + \frac{1}{2} \mathbb{D} f_T\cdot f_T \mathring{\lambda}^2 - \mathbb{D}\mathcal{D}(v)\cdot f_T\mathring{\lambda}]dB - \mathring{L}(v) \tag{42}$$

To prove this proposition we note that the convexity of π_1^d and ϕ guarantees the existence of a pair of fields $(\bar{v},\bar{\mathring{\lambda}})$ for which π_2 attains its minimum value, hence

$$\pi_2(\bar{v},\bar{\mathring{\lambda}}) \leq \pi_2(\mathring{u},\mathring{\lambda}(\mathring{u})) \tag{43}$$

On the other hand, $\bar{\mathring{\lambda}}$ minimizes ϕ for $D = \mathcal{D}(\bar{v})$, then equations (30), (23) and (42) leads to

$$\pi_2(\bar{v},\bar{\mathring{\lambda}}) = \pi_1(\bar{v}) \tag{44}$$

when the minimization of ϕ with $D = \mathcal{D}(\mathring{u})$ is considered the property (29) assures that the solution is $\mathring{\lambda}(\mathring{u})$. Applying again equations (30), (23) and (42) we get

$$\pi_2(\mathring{u},\mathring{\lambda}(\mathring{u})) = \pi_1(\mathring{u}) \tag{45}$$

Any field $\dot{u}$ minimizes π_1 therefore

$$\pi_1(\dot{u}) \leq \pi_1(\tilde{v}) \tag{46}$$

Combining equations (43) to (46) it follows that

$$\pi_1(\dot{u}) = \pi_1(\tilde{v}) = \pi_2(\dot{u},\hat{\lambda}(\dot{u})) = \pi_2(\tilde{v},\tilde{\hat{\lambda}}) \tag{47}$$

These equalities establish the equivalence of both optimization problems (20) and (41), so the proof of Principle II is completed.

Principle II states the elastoplastic analysis for rates as the problem of minimizing a differentiable quadratic functional depending on two fields.

Nevertheless this formulation (Principle II) is completely adequate to solve the problem in terms of velocity, we introduce in what follows a new formulation that will be justified later by its relation with the process of approximation in finite time increments.

This new formulation is obtained modifying the problem stated in Principle II so as to release some of the constraints, more precisely those restrictions imposed by means of the definition of its domain Λ_f. A broader domain for the field $\hat{\lambda}$ is the set

$$\Lambda = \{\hat{\lambda} : \ \hat{\lambda}(x) \geq 0 \quad \forall \quad x \in B\} \tag{48}$$

Obviously $\Lambda_f \subset \Lambda$, so $\hat{\lambda} \in \Lambda$ is less restrictive than $\hat{\lambda} \in \Lambda_f$. In the framework of the rate problem there is no need to release constraints of this kind, that force the plastic strain rate to be zero in points where the stress is elastic, since Λ_f is known and the representation of its elements is a simple task. When finite time increments are performed, other reasons suggest that it is more convenient to let $\hat{\lambda}$ vary in Λ and to impose released constraints through penalization of the objective functional.

A proper functional for the purpose of generating a penalty function in Λ to force the constraint $\hat{\lambda} \in \Lambda_f$ is

$$\psi(\hat{\lambda}) = - \int_B f \hat{\lambda} \, dB \tag{49}$$

because it verifies the properties stated below .

i) ψ is linear, hence differentiable and (nonstrictly) convex; Λ_f and Λ are closed convex sets; and $\Lambda_f \subset \Lambda$.

$$\text{ii)}\quad \psi(\dot{\lambda}) = 0 \quad \forall \quad \dot{\lambda} \in \Lambda_f \tag{50}$$

$$\text{iii)}\quad \psi(\dot{\lambda}) > 0 \quad \forall \quad \dot{\lambda} \in \Lambda - \Lambda_f \tag{51}$$

Consequently, we search for a field v_a depending on a positive number a, to approximate when $a \to 0$ the solution u of the elastoplastic problem, solving the problem stated in the following principle.

PRINCIPLE III

The solution $(v_a, \dot{\lambda}_a)$ of the problem

$$\min_{\substack{v \in V \\ \dot{\lambda} \in \Lambda}} \pi_a(v, \dot{\lambda}) \tag{52}$$

where $a > 0$ and

$$\pi_a(v,\dot{\lambda}) = \pi_2(v,\dot{\lambda}) + \frac{1}{a}\,\psi(\dot{\lambda}) =$$

$$= \int_B [\frac{1}{2}\,\mathbb{D}\,\mathcal{D}(v)\cdot\mathcal{D}(v) + \frac{1}{2}\,\mathbb{D}\,f_T\cdot f_T\dot{\lambda}^2 - \mathbb{D}\,\mathcal{D}(v)\cdot f_T\dot{\lambda} - \frac{1}{a}\,f\dot{\lambda}]\,dB - \dot{L}(v) \tag{53}$$

approximates the solution $\dot{u}$ of the elastoplastic problem, that is

$$v_a \xrightarrow[a \to 0^+]{} \dot{u} \qquad\qquad \dot{\lambda}_a \xrightarrow[a \to 0^+]{} \dot{\lambda}(\dot{u}) \tag{54}$$

This notation means weak convergence. Indeed, since the solution $\dot{u}$ is not unique we can only expect that any convergent subsequence included in v_a has limit $\dot{u}$.

5. ELASTOPLASTIC ANALYSIS FOR INCREMENTS

To approximate the values of the displacement field in a discrete number of values of the parameter t which describes process evolution, rate formulations are used in generating the displacement sequence for the selected instants.

The following notation will be used

$$\Delta u = u_{t+\Delta t} - u_t \tag{55}$$

$$\Delta\lambda = \lambda_{t+\Delta t} - \lambda_t \tag{56}$$

$$\Delta T = T_{t+\Delta t} - T_t \tag{57}$$

$$\Delta L = L_{t+\Delta t} - L_t \tag{58}$$

where subindex t identifies values corresponding to this instant.

An Euler type approximation is used next. Accordingly, we consider that any field is known in t (in particular $f = f(T_t)$ and $f_T = f_T(T_t)$) and we substitute rates by means of

$$\dot{u} = \frac{1}{\Delta t}\Delta u \qquad \dot{\lambda} = \frac{1}{\Delta t}\Delta\lambda \qquad \dot{L} = \frac{1}{\Delta t}\Delta L \tag{59}$$

The classical principle for velocities, equation (20), is invoqued when pseudoelastic methods are used with assumed initial stress or strains. An iterative strategy is needed for the determination of a tangent operator because it depends on the solution itself due to the fact that the functional involved is nondifferentiable.

A second approach follows from the substitution of the approximation proposed in (59) into Principle II. This procedure leads to the incremental formulation

$$\min_{\substack{\Delta u \in V \\ \Delta\lambda \in \Lambda_f}} \overline{\pi}_2(\Delta u, \Delta\lambda) \tag{60}$$

where

$$\overline{\pi}_2(\Delta u,\Delta\lambda) = \int_B [\frac{1}{2}\,\mathbb{D}\mathcal{D}(\Delta u)\cdot\mathcal{D}(\Delta u) + \frac{1}{2}\,\mathbb{D}f_T\cdot f_T(\Delta\lambda)^2 - \mathbb{D}\mathcal{D}(\Delta u)\cdot f_T\Delta\lambda]\,dB - \Delta L(\Delta u) \tag{61}$$

On the other hand, the same procedure applied on Principle III gives

$$\min_{\substack{\Delta u \in V \\ \Delta\lambda \in \Lambda}} \overline{\pi}_e(\Delta u, \Delta\lambda) \tag{62}$$

where

$$\bar{\pi}_a(\Delta u,\Delta\lambda) = \int_B [\frac{1}{2} \mathbb{D}\mathcal{D}(\Delta u)\cdot\mathcal{D}(\Delta u) + \frac{1}{2} \mathbb{D} f_T\cdot f_T(\Delta\lambda)^2 -$$

$$- \mathbb{D}\mathcal{D}(\Delta u)\cdot f_T\Delta\lambda - \frac{\Delta t}{a} f \;\Delta\lambda]dB - \Delta L(\Delta u) \qquad (63)$$

Optimality conditions for the solution $(\Delta u,\Delta\lambda)$ of this problem are obtained by operations similar to those described in equation (32) to (37). In this way, the global conditions that identify the solution of (62) are

$$\int_B [\mathbb{D}\mathcal{D}(\Delta u)\cdot\mathcal{D}(v) - \mathbb{D}\mathcal{D}(v)\cdot f_T \;\Delta\lambda]dB = \Delta L(v) \qquad \forall \quad v \in V \qquad (64)$$

$$\int_B [\mathbb{D} f_T\cdot f_T \;\Delta\lambda - \mathbb{D}\mathcal{D}(\Delta u)\cdot f_T - \frac{\Delta t}{a} f]\hat{\lambda} \; dB \geq 0 \qquad \forall \quad \hat{\lambda} \in \Lambda \qquad (65)$$

$$\int_B [\mathbb{D} f_T\cdot f_T \;\Delta\lambda - \mathbb{D}\mathcal{D}(\Delta u)\cdot f_T - \frac{\Delta t}{a} f]\Delta\lambda \; dB = 0 \qquad (66)$$

The corresponding local conditions are written below excepting the one associated to equilibrium equation (64).

$$\mathbb{D} f_T\cdot f_T \;\Delta\lambda - \mathbb{D}\mathcal{D}(\Delta u)\cdot f_T - \frac{\Delta t}{a} f \geq 0 \qquad \forall \quad x \in B \qquad (67)$$

$$[\mathbb{D} f_T\cdot f_T \;\Delta\lambda - \mathbb{D}\mathcal{D}(\Delta u)\cdot f_T - \frac{\Delta t}{a} f]\Delta\lambda = 0 \qquad \forall \quad x \in B \qquad (68)$$

$$\Delta\lambda \geq 0 \qquad \forall \quad x \in B \qquad (69)$$

Optimality conditions related to formulation (60) can be read from equations (64) to (69) cancelling the penalty term distinguished by the factor $1/a$. It is noted that equilibrium condition (64) remains unchanged.

We analyze now the characteristics of the solution of the approximate problem (62) assuming by the moment that the plastic potential f is a linear function. Actually the case of practical interest is when this function is piecewise linear but this situation cannot be properly treated in the framework of regularity assumed for the yield function. Anyway, all the conclusions are generalized to piecewise linear functions by using the appropriate notation [1].

For a linear yielding function f_T is constant and then the exact increment in tension is computed by the equation

$$\Delta T = \mathbb{D}\,(\mathcal{D}(\Delta u) - f_T \;\Delta\lambda) \qquad (70)$$

Substitution of this equation in (64) gives

$$\int_B \Delta T \cdot D(v) dB = \Delta L(v) \quad \forall \quad v \in V \tag{71}$$

Since at the beginning of the step the stresses equilibrate the external load, that is to say $T_t \in S_t$, we conclude from the previous equation that $T_{t+\Delta t} \in S_{t+\Delta t}$. In other words, the incremental approximation method derived from the functional π_a (equation (62)) leads to a final stress, for the load step, that is exactly equilibrated with the final load. This conclusion is valid under the hipotesis of linearity in f, and for any value of the parameter a.

Exact equilibrium at the end of a load increment is also reached with the algorithm (60) derived from Capurso's principle. Again, the linearity of the plasticity limit is needed to guarantee this property.

Another aspect to be observed in the approximate solution is the plastic admissibility of the stress distribution $T_{t+\Delta t}$ computed at the end of an increment Δt. This condition makes algorithm (62) preferable when compared to the method (60).

Using once more the linearity of f (equation (70)) we deduce from (67) that

$$\frac{\Delta t}{a} f + f_T \cdot \Delta T \leq 0 \quad \forall \quad x \in B \tag{72}$$

Hence, it suffices to assume that $a \geq \Delta t$ to assure plastic admissibility of the final stresses. In fact, for these values of a, it follows from equation (72) that

$$f(T_{t+\Delta t}) = f + f_T \cdot \Delta T \leq 0 \quad \forall \quad x \in B \tag{73}$$

For a linear yield function equation (68) can be written as

$$\left(\frac{\Delta t}{a} f + f_T \cdot \Delta T\right) \Delta\lambda = 0 \quad \forall \quad x \in B \tag{74}$$

We examine first the algorithm obtained for the particular value $a = \Delta t$. In this case (74) reads

$$\Delta\lambda \, f(T_{t+\Delta t}) = 0 \quad \forall \quad x \in B \tag{75}$$

This means that the true constitutive relation in rates (15), that enforces $\dot{\lambda} = 0$ if $f < 0$, is substituted in the approximate method by a relation in increments that imposes null plastic deformation ($\Delta\lambda = 0$) whenever the final stresses are elastic ($f(T_{t+\Delta t}) < 0$). Just as points with elastic stress in the beginning of the step may undergo plastic deformation ($\Delta\lambda > 0$) if they reach a stress on the plastic limit at the end of the increment ($f(T_{t+\Delta t}) = 0$).

Hence, if the actual process for stress and strains during the load increment ΔL consists of plastic deformation followed by elastic local unloading, setting final stress with negative value of f, we conclude from previous remarks that this step cannot be correctly reproduced in the finite increments approximation (62). Obviously, this difficulty is not present if we assume that the actual local process during the load step is either elastic, elastic unloading, elastic followed by progressive yielding or progressive yielding (if the stress in a point reaches the limit surface $f(T) = 0$ it remains in this condition until the end of the step). The approximate solution is able to represent elastic local unloading when no plastification has occurred in the step, that is to say when elastic local unloading starts from the initial state of stress which must belong to the plastic limit.

The previous discussion suggests that the exact incremental solution of an elastoplastic problem with linear yield function, and in the case that nonprogressive plastification is ruled out in the step, is found by solving (62) with $a = \Delta t$, that is

$$\min_{\substack{\Delta u \in V \\ \Delta\lambda \in \Lambda}} \bar{\pi}(\Delta u, \Delta\lambda) \tag{76}$$

where

$$\bar{\pi}(\Delta u,\Delta\lambda) = \int_B [\frac{1}{2}\, \mathbb{D}\, \mathcal{D}(\Delta u)\cdot\mathcal{D}(\Delta u) + \frac{1}{2}\, \mathbb{D}\, f_T\cdot f_T(\Delta\lambda)^2 -$$

$$- \mathbb{D}\, \mathcal{D}(\Delta u)\cdot f_T\, \Delta\lambda - f\, \Delta\lambda]\, dB - \Delta L(\Delta u) \tag{77}$$

This is the principle proposed by Maier [4,1] in the framework of holonomic incremental elastiplasticity with piecewise linear constitutive relations. In fact, under the assumptions of linearity of f and progressive yielding, the constitutive equations (14) and (15) are also valid for increments, and can be recognized as optimality conditions for (77) proving that this formulation gives indeed the exact incremental solution.

We continue the discussion of algorithm (62) considering now the case $a > \Delta t$. For a linear yield function the final state of stress fulfills

$$f(T_{t+\Delta t}) < \frac{\Delta t}{a} f + f_T \Delta T \quad \forall \quad x \in B \tag{78}$$

It was allready mentioned that for $a > \Delta t$ final stresses are plastically admissibles, but equations (78) and (74) imply that a plastic deformation ($\Delta\lambda > 0$) can occur even for points where initial and final stresses remain inside the elastic domain.

Until now we considered the algorithms obtained by performing one minimization of $\bar{\pi}_a$, with a fixed value $a \geq \Delta t$, in every step of the loading. We pointed out that this method for $a = \Delta t$ becomes appropriate to proceed in finite increments because it preserves equilibrium and plastic admissibility.

A similar analysis on formulation (60) shows that equilibrium is assumed for the final state although plastic admissibility can be violated if the initial state is elastic. This conclusion stands for each plastic mode in the case of piecewise linear yield limit.

Let us consider next the iterative application of the minimization of $\bar{\pi}_a$ for fixed Δt and $a \to 0$. This is the standard procedure corresponding to the inclusion of a penalty term in the objective function. This algorithm gives increments that lead to inadmissible final stresses; however, it reproduces in the limit the true constitutive relations.

Finally, let us make some remarks on the possibility of applying algorithm (62) for a nonlinear yield function. Obviously, there is no way to assure plastic admissibility at the end of the step using the initial values of f and its gradient f_T. For $a \geq \Delta t$ final plastic admissibility is expected in the foresight of a linear approximation of f. It may be advantageous to take parameter a greater than Δt in some amount, with the aim of increasing the chances to have admissible final stresses, even though this can lead to admit plastic strain increment for a process having elastic initial and final stresses. Besides, equilibrium equation (64) is only imposed for a linear approximation of the increments in stresses, hence the nonlinearity of a material characteristic, namely the yield function, affects the equilibrium condition of the approximate solution.

6. CONCLUSIONS

Previous section contains a discussion of numerical methods in elastoplastic analysis. It has been enphatized the distinctions between nondifferentiable optimization approach (Principle I) and differentiable optimization formulations (Principle II and III).

The methods stated in (60) and (62), involving two fields, formulate quadratic optimization problems. Under the assumptions of linearity of yield function and progressive yielding both methods guarantee equilibrium of stresses reached at the end of a load step, and also determine automatically if local elastic unloading occurs. The latter condition distinguish these methods from those that enforce arbitrary tangent relation between stress and strain. In addition, only the algorithm (62) can be adapted to preserve plastic admissibility of the stresses in a finite step.

If a discretization of the continuum body is required, the aforementioned conditions must be restated taking into account the characteristics of this additional approximation.

It was pointed out in the present paper that the formulation (77), which has been successfully used in incremental elastoplastic analysis [1], can be directly derived from a principle in rates obtained by means of general procedures of optimization theory. Besides, when the numerical method (77) is used, it may be misleading to invoque Principle II because it is related with a discrete version of the type (60).

The behavior of different formulations with respect to equilibrium and plastic admissibility can also be discussed in a purely statical description of the elastoplastic analysis. The corresponding principles for rates and increments of stresses present similar situation than that encountered so far in kinematical formulations. The Prager principle [8] is a proper characterization of stress rates whose discrete version, obtained by Euler approximation of time derivatives, constitutes the dual problem of (60). Hence, it can handle local elastic unloading and equilibrium but plastic admissibility may be violated.

There is a statical discrete formulation dual of (77), first proposed by Maier [1]. This formulation can also be derived by direct time discretization of an optimality condition for rates, which is in this case the evolutionary variational inequation

$$\int_B \mathbb{D}^{-1}\dot{T}\cdot(T^* - T)\,dB \geq 0 \qquad \forall \quad T^* \in K \tag{79}$$

This approach, described for example by Johnson [10], is in some sense the dual statical procedure for the derivation of problem (62) from Principle III.

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