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ON THE ASYMPTOTIC EXPANSIONS OF SOLUTIONS OF AN NTH ORDER LINEAR DIFFERENTIAL EQUATION WITH POWER COEFFICIENTS

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Abstract

The asymptotic expansions of solutions of a class of linear ordinary differential equations of arbitrary order n, containing a factor z^m multiplying the lower order derivatives, are investigated for large values of z in the complex plane. Four classes of solutions are considered which exhibit the following behaviour as $|z| \to \infty$ in certain sectors :(i) solutions whose behaviour is either exponentially large or algebraic (involving p ($< n$) algebraic expansions), (ii) solutions which are exponentially small (iii) solutions with a single algebraic expansion and (iv) solutions which are even and odd functions of z whenever $n+m$ is even. The asymptotic expansions of these solutions in a full neigbourhood of the point at infinity are obtained by means of the theory of the solutions in the case $m=0$ developed in a previous paper.

I. INTRODUCTION

This paper is a sequel to the paper (Paris 1980) (hereinafter referred to as I) on the asymptotic expansion of solutions for large $|z|$ of the nth order linear differential equation with integer power coefficients

$$u^{(n)} - z^m \sum_{r=0}^{p} a_r z^r u^{(r)} = 0, \quad z \in \mathbb{C},\ n > p \geq 0,\ m \geq 0. \qquad (1.1)$$

It will be supposed throughout that m, n and p are integers with $n > p \geq 0$, $m \geq 0$ and that $a_0, a_1, \ldots, a_{p-1}$ are arbitrary complex numbers with $a_p = 1$ and a_0 non-zero. Applications of this type of equation arise in a variety of physical problems (eg. boundary layer theory and electromagnetic wave propagation : for references in the case m=0, see I). In spectral theory of linear differential operators, equations of type (1.1) arise in connection with the problem of determining the deficiency indices of symmetric differential operators, both when m=0 (Wood 1971) and, in a more general setting when m is not restricted to a non-negative integer, in (Paris and Wood 1981, 1982, 1984 ; Wood and Paris 1982).

Equation (1.1) represents an extension of the earlier work of Turrittin (1950), Heading (1957) and Braaksma (1971) in the case p=0, and of I in the case m=0. We shall consider four classes of solutions of (1.1) which exhibit the following behaviour as $|z| \to \infty$ in certain sectors: (i) a basic set of solutions which are either exponentially large or algebraic involving p algebraic controlling factors, (ii) solutions which are exponentially small, (iii) solutions which possess an algebraic behaviour with a single controlling factor and (iv) solutions which are either even or odd in z whenever n+m is even. Although the solutions of the first type are generally not of great interest in applications, they are nevertheless of considerable importance as they provide a basis in which to express the other solutions of (1.1) possessing a more useful asymptotic character. Their asymptotic expansion in the case m=0 was obtained in I by means of the asymptotic theory of integral functions of hypergeometric type.

Our aim in this paper is to present a summary of the principal results on the asymptotic expansions of these solutions in a full neighbourhood of the point at infinity. We include, where necessary, only a brief outline of the analysis involved, together with an illustrative example of a third order differential equation. A full discussion of the proof of these results, including the treatment of the particular case n=p and the generalisation to cases with real, non-integer values of m, has been seen for the formally self-adjoint equation in (Paris and Wood 1981) and will be given in detail for the non self-adjoint equation in a forthcoming book (Paris and Wood 1985).

No loss of generality will result in our taking the leading coefficient $a_p=1$, since an equation of type (1.1) with general a_p may always be transformed to the form with $a_p = 1$ by the change of variable $z \to (a_p)^{-1/(n+m)}z$. With the coefficient of $u^{(r)}$ being exactly one power of z higher than the coefficient of $u^{(r-1)}$, the coefficients of the formal power series solutions of (1.1) about z=0 satisfy two-term recurrence relations. The solutions of (1.1) are consequently related to the generalised hypergeometric functions (Scheffé 1942).

To see this, we introduce the polynomial G(s) of degree p associated with (1.1)

$$G(s) = a_0 + \sum_{r=1}^{p} a_r \prod_{k=0}^{r-1} (s-k)$$

$$= \prod_{r=1}^{p} (s+\beta_r), \tag{1.2}$$

where the constants $-\beta_r$ denote the zeros of G(s). We shall suppose throughout that none of the β_r equals a negative integer (we note that, since by hypothesis, a_0 is non-zero there cannot be zero with $\beta_r = 0$), thereby avoiding the complications which arise when discussing linearly independent sets of solutions of (1.1). We let Θ denote the differential operator zd/dz, so that

$$z^r\frac{d^r}{dz^r} = \Theta(\Theta-1)(\Theta-2)\ldots(\Theta-r+1), \quad r=1,2,\ldots .$$

We define N=n+m, P=p+m and multiply (1.1) by z^n to obtain

$$\left\{\vartheta(\vartheta-1)\dots(\vartheta-n+1) - z^N \prod_{r=1}^{p}(\vartheta+\beta_r)\right\}u = 0.$$

With the change of variable

$$\tau = N^{p-n} z^N, \quad \delta \equiv \tau\frac{d}{d\tau} = \frac{1}{N}\vartheta,$$

the above equation becomes the generalised hypergeometric equation

$$\left\{\delta\left(\delta-\frac{1}{N}\right)\cdots\left(\delta-\frac{n-1}{N}\right) - \tau\prod_{r=1}^{p}\left(\delta+\frac{\beta_r}{N}\right)\right\}u = 0. \tag{1.3}$$

Solutions of this equation can be expressed in terms of the ${}_pF_{n-1}(\tau)$ hypergeometric functions (Slater 1966). These solutions, however, are of little interest since, for most values of m,n,p they are exponentially large as $|z| \to \infty$ for all values of argz. In search of solutions which have a more useful asymptotic behaviour at infinity, we turn to the Meijer G-functions, which are themselves linear combinations of the ${}_pF_{n-1}(\tau)$ functions. The advantage of these solutions is that the exponentially large parts of the ${}_pF_{n-1}(\tau)$ functions can be arranged to cancel out in certain sectors of the z-plane, to leave algebraic or exponentially small behaviour.

Of the possible set of Meijer function solutions for (1.3) (Luke 1975) we shall base our solutions on the particular Meijer function

$$G^{n,\ell}_{p,n}\left((-)^{n-p+\ell}\tau \,\middle|\, \begin{matrix} 1-\frac{\beta_1}{N}, \dots, 1-\frac{\beta_p}{N} \\ 0, \frac{1}{N}, \dots, \frac{n-1}{N} \end{matrix}\right), \quad 0 \leqslant \ell \leqslant p. \tag{1.4}$$

The choice of the index ℓ is at our disposal and determines the character of a solution in a certain neighbourhood of infinity. However, the structure of (1.1) is such that it does not prove necessary to apply the detailed theory of these functions.

II. THE SOLUTIONS WHEN m=0

We first consider the case m=0 which was examined in I. As this case is fundamental to the development of the theory of (1.1) in the general case, in this section we give a summary of the principal results. A solution of (1.1) when m=0 is given by the Mellin-Barnes integral

$$U_{n,p}(z)=\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\Gamma(-s)\prod_{j=1}^{p}\Gamma\left(\frac{s}{n}+\frac{\beta_j}{n}\right)\left(-n^{p/n}z\right)^{s}ds,\quad |\arg(-z)|<\tfrac{1}{2}\pi\left(1+\tfrac{p}{n}\right)\tag{2.1}$$

where the path of integration is suitably indented (if necessary) to separate the poles of $\Gamma\left(\frac{s}{n}+\frac{\beta_j}{n}\right)$, situated at $s=-\beta_j-nk$, $k=0,1,2,\ldots(j=1,\ldots,p)$ from those of $\Gamma(-s)$ situated at $s=0,1,2,\ldots$ (provided none of the β_j equals a negative integer). Upon expansion of $\Gamma(-s)$ as a product of n gamma functions by the multiplication formula, the integral (2.1) may be recognised as a constant multiple of the Meijer function (1.4) with index $\ell=p$.

The integral (2.1) may be evaluated by swinging round the path of integration into the infinite semi-circle in the right-half plane, when $|\arg(-z)|<\frac{1}{2}\pi\left(1+\frac{p}{n}\right)$, and application of the residue theorem to find the series expansion †

$$U_{n,p}(z)=\sum_{k=0}^{\infty}\frac{(n^{p/n}z)^k}{k!}\prod_{j=1}^{p}\Gamma\left(\frac{k}{n}+\frac{\beta_j}{n}\right),\tag{2.2}$$

which holds for all arg z by analytic continuation. The solution $U_{n,p}(z)$ is an integral function of z and is uniformly and absolutely convergent throughout the finite z-plane. Since the differential equation (1.1) is unchanged when m=0 if z replaced by ωz, where ω denotes an nth root of unity, we see that, with the above restrictions on β_r, a fundamental system of solutions is given by

$$U_{n,p}(\omega_j z),\quad \omega_j=\exp(2\pi ij/n),\quad j=0,1,\ldots,n-1.\tag{2.3}$$

We now state the first asymptotic expansion theorem for $U_{n,p}(z)$. We define the parameters

$$\kappa=1-\frac{p}{n},\quad \vartheta=\frac{1}{n}\sum_{r=1}^{p}\beta_r-\frac{1}{2}p,\tag{2.4}$$

† We employ a different normalisation of the solutions when m=0 to that given in I.

where, from (1.2),

$$\sum_{r=1}^{p} \beta_r = a_{p-1} - \tfrac{1}{2}p(p-1),$$

and introduce the formal asymptotic sums

$$E(z) = (2\pi)^{\frac{1}{2}p} K^{-\frac{1}{2}} n^{-\vartheta} z^{n\vartheta/(n-p)} \exp\left[\frac{(n-p)}{n} z^{n/(n-p)}\right] \sum_{k=0}^{\infty} c_k \left[\frac{(n-p)}{n} z^{n/(n-p)}\right]^{-k}, \quad (2.5)$$

$$H(z) = n \sum_{s=1}^{p} (n^{p/n} z)^{-\beta_s} \; S_{n,p}(\beta_s; z), \quad (2.6)$$

where

$$S_{n,p}(\beta_s; z) = \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \Gamma(nk+\beta_s) \prod_{j=1}^{p}{}' \Gamma\left(\frac{\beta_j-\beta_s}{n} - k\right) (n^{p/n} z)^{-nk} \quad (2.7)$$

with the prime denoting the omission of the term corresponding to j=s in the product. The coefficients c_k are independent of z with $c_0=1$ and in (2.6) it has been assumed that no two of the β_j either coincide or differ by an integer multiple of n. Then we have the following

Theorem 1 . For $|z| \to \infty$, the solution $U_{n,p}(z)$ of (1.1), in the case m=0 and n > p, has the asymptotic expansion given by

$$\begin{aligned} U_{n,p}(z) &\sim E(z) + H(z e^{\mp\pi i}) \quad &\text{in} \quad |\arg z| \le \tfrac{1}{2}\pi(1-\tfrac{p}{n}) \\ U_{n,p}(z) &\sim H(z e^{\mp\pi i}) \quad &\text{in} \quad |\arg(-z)| < \tfrac{1}{2}\pi(1+\tfrac{p}{n}) \end{aligned} \quad (2.8)$$

where the upper or lower sign is taken according as arg z is > 0 or < 0 respectively. The expansion of the fundamental system, $U_{n,p}(\omega_j z)$, j=0,1, ...,n-1 in (2.3) follows immediately by simple rotation of the argument by $2\pi j/n$.

This result was established in I by applying the powerful asymptotic theory of integral functions of hypergeometric type, developed principally by Ford (1936), Newsom (1938), Wright (1940) and Braaksma (1964), to the series expansion of $U_{n,p}(z)$ in (2.2).

For $k \geq 2$, the coefficients c_k in the exponential expansion $E(z)$ are defined by an n-term recurrence relation given in I. The complexity of the terms does not readily permit the explicit formulation of any but the coefficient c_1, given by

$$n\kappa c_1 = Q(n) - Q(p) - \kappa(p-1)\left[\vartheta + \tfrac{1}{2}\tfrac{p}{n}(p-2)\right]a_{p-1} - \kappa^2 a_{p-2} \tag{2.9}$$

where

$$Q(k) = \tfrac{1}{8}k(k-1)\left[\left\{2\vartheta - 1 + \tfrac{p}{n}(k-1)\right\}^2 + \tfrac{1}{3}(k+1)(1-\kappa^2) - 1\right].$$

The algebraic expansion $H(z)$, when some of the β_j coincide or differ by integer multiples of n, is obtained from

$$H(z) = \frac{1}{2\pi i}\sum_k \int^{(s_k+)} \Gamma(-s)\prod_{j=1}^{p}\Gamma\left(\frac{s}{n}+\frac{\beta_j}{n}\right)\left(n^{p/n}z\right)^s ds. \tag{2.10}$$

The integral is along a contour encircling the pole s_k in the positive sense and the summation is to be carried out over all the poles s_k resulting from $\Gamma\left(\frac{s}{n}+\frac{\beta_j}{n}\right)$, $(j=1,\ldots,p)$ but <u>excluding</u> those of $\Gamma(-s)$. In this case, certain poles in (2.10) will be of higher order, and their residues must be evaluated accordingly and will involve terms in $\log z$.

The solution $U_{n,p}(z)$ is thus seen to be exponentially large as $|z| \to \infty$ in the sector $|\arg z| < \frac{1}{2}\pi(1-p/n)$, while in the sector $|\arg(-z)| < \frac{1}{2}\pi(1+p/n)$ its asymptotic expansion comprises p algebraic expansions, each with the controlling factor $z^{-\beta_j}$, $j=1, \ldots, p$. On the rays $\arg z = \pm\frac{1}{2}\pi(1-p/n)$, the exponential term in (2.5) is oscillatory and the expansion of $U_{n,p}(z)$ is therefore of the mixed type, where both the exponential and algebraic expansions must be retained. On crossing $\arg z=0$, where the algebraic expansions (2.6) are of maximum degree of subdominancy, the coefficients of $z^{-\beta_s}$ multiplying the algebraic expansions change discontinuously by the factor $\exp(2\pi i\beta_s)$. This is the Stokes phenomenon, where the coefficients of the subdominant algebraic terms change discontinuously on $\arg z=0$ in order to preserve the single valuedness of $U_{n,p}(z)$ as z describes a circuit about the origin.

Solutions of (1.1) when m=0 which possess an exponentially small or algebraic expansion (with the single controlling factor $z^{-\beta_j}$) in certain sectors of the z-plane at infinity will be denoted by $V_{n,p}(z)$ and $W_{n,p}(\beta_j ; z)$ respectively. To within constant multiples, these solutions correspond to the Meijer function (1.4) with index ℓ equal to 0 and 1 respectively, and are defined by the integrals

$$V_{n,p}(z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\pi^{p}\,\Gamma(-s)}{\prod_{r=1}^{p}\Gamma(1-\frac{\beta_r}{n}-\frac{s}{n})}\left(e^{-\pi i(n-p)/n}\, n^{p/n} z\right)^{s} ds \tag{2.11}$$

$$W_{n,p}(\beta_j ; z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\pi^{p-1}\,\Gamma(-s)\Gamma(\frac{s}{n}+\frac{\beta_j}{n})}{\prod_{r=1}^{p}{}'\,\Gamma(1-\frac{\beta_r}{n}-\frac{s}{n})}\left(e^{-\pi i(n-p+1)/n}\, n^{p/n} z\right)^{s} ds, \quad (p\geq 1) \quad j=1,2,\dots,p \tag{2.12}$$

where the paths of integration are indented at s=0 to lie to the left of all the poles of $\Gamma(-s)$ but, in the second integral, to the right of the poles of $\Gamma(\frac{s}{n}+\frac{\beta_j}{n})$ (β_j not equal to a negative integer). As usual the prime denotes the omission of the term corresponding to r=j. These integral representations are valid in the sectors

$$\frac{1}{2}\frac{\pi}{n}(n-p) < \arg z < \frac{3}{2}\frac{\pi}{n}(n-p) + \frac{2\pi\ell}{n}, \quad (\ell=0,1) \tag{2.13}$$

which we shall refer to as the principal sectors of $V_{n,p}(z)$, ($\ell=0$) and $W_{n,p}(\beta_j;z)$ ($\ell=1$). We observe that the solution $W_{n,p}(\beta_j ; z)$ only represents a new solution of (1.1) for p > 2, since, when p=1, $W_{n,1}(\beta_1 ; z) = U_{n,1}(z)$.

Application of the residue theorem and appeal to analytic continuation shows that

$$V_{n,p}(z) = \pi^{p}\sum_{k=0}^{\infty}\frac{(-)^{k}}{k!}\frac{\left(e^{-\pi i(n-p)/n}\, n^{p/n} z\right)^{k}}{\prod_{r=1}^{p}\Gamma(1-\frac{\beta_r}{n}-\frac{k}{n})} \tag{2.14}$$

$$W_{n,p}(\beta_j ; z) = \pi^{p-1}\sum_{k=0}^{\infty}\frac{(-)^{k}\,\Gamma(\frac{k}{n}+\frac{\beta_j}{n})}{k!\,\prod_{r=1}^{p}{}'\,\Gamma(1-\frac{\beta_r}{n}-\frac{k}{n})}\left(e^{-\pi i(n-p+1)/n}\, n^{p/n} z\right)^{k} \tag{2.15}$$

for all arg z. These solutions are both integral functions of z, uniformly and absolutely convergent throughout the finite z-plane. Note that for integer values of β_r, there is no contribution to the sums (2.14) and (2.15) whenever $k+\beta_r$ is a positive integer multiple of n.

For $|z| \to \infty$, $V_{n,p}(z)$ possesses the expansion

$$V_{n,p}(z) \sim \left(\tfrac{1}{2}\pi\right)^{\frac{1}{2}p} \kappa^{\cdots-\vartheta} \left\{e^{-\pi i} z^{n/(n-p)}\right\}^{\vartheta} \exp\left[\frac{(n-p)}{n} z^{n/(n-p)}\right] \sum_{k=0}^{\infty} c_k \left[\frac{(n-p)}{n} z^{n/(n-p)}\right]^{-k} \tag{2.16}$$

valid in the extended sector

$$-\frac{\pi\epsilon}{n} < \arg z < \frac{2\pi}{n}(n-p) + \frac{\pi\epsilon}{n}, \quad \epsilon = \begin{cases} \frac{1}{2} & \text{if } n-p=1 \\ 1 & \text{if } n-p \geq 2 \end{cases} \tag{2.17}$$

where the parameters, κ, ϑ are defined in (2.4) and the coefficients c_k are those appearing in E(z). Thus $V_{n,p}(z)$ is exponentially small as $|z| \to \infty$ in the principal sector (2.13) with $\ell = 0$. The behaviour in the rest of the sector (2.17) is exponentially large, while on the boundaries, arg z = $\frac{1}{2}\pi\kappa$, $\frac{3}{2}\pi\kappa$ of the exponentially small sector, $V_{n,p}(z)$ is exponentially oscillatory. We remark that for cases with $n \gg p$, the exponentially small sector approaches the half-plane $\mathrm{Re}(z) > 0$.

The expansion of $W_{n,p}(\beta_j ; z)$ in its principal sector may be obtained by the usual method, described in (Slater 1966), of estimating Mellin-Barnes integrals by displacing the path of integration in (2.12) to the left over the poles of $\Gamma\left(\frac{s}{n} + \frac{\beta_j}{n}\right)$. For $|z| \to \infty$ in the principal sector (2.13) with $\ell = 1$,

$$W_{n,p}(\beta_j;z) \sim \pi^{p-1} n \left[n^{p/n} z e^{-\pi i(n-p+1)/n}\right]^{-\beta_j} \sum_{k=0}^{\infty} \frac{(-)^{k(n-p)}\,\Gamma(nk+\beta_j)}{k! \prod_{r=1}^{p}{}' \Gamma\left(k+1+\frac{\beta_j-\beta_r}{n}\right)} \left(n^{p/n} z\right)^{-nk}, \tag{2.18}$$

$$j = 1, 2, \ldots, p.$$

This expansion is seen to consist of a single algebraic expansion with the controlling factor $z^{-\beta_j}$.

The expansion of $V_{n,p}(z)$ and $W_{n,p}(\beta_j ; z)$, outside the sectors (2.17) and (2.13) respectively, can be obtained by employing the following expansion theorems [I].

Theorem 2 . *When $m=0$, the solutions $V_{n,p}(z)$ and $W_{n,p}(\beta_j\,;\,z)$ of (1.1) may be expressed as linear combinations of solutions of the fundamental system $U_{n,p}(\omega_j z)$ in (2.3) in the form*

$$V_{n,p}(z) = \sum_{r=0}^{p} B_{r,0}\, U_{n,p}\left(z\, e^{2\pi i(p-r)/n}\right)$$
$$W_{n,p}(\beta_j; z) = \sum_{r=0}^{p-1} B_{r,1}\, U_{n,p}\left(z\, e^{2\pi i(p-r-1)/n}\right), \quad j=1,2,\dots,p. \tag{2.19}$$

The coefficients $\beta_{r,\ell}$ are defined by

$$\prod_{r=\ell+1}^{p} \sin\frac{\pi}{n}(s+\beta_r) = \sum_{r=0}^{p-\ell} B_{r,\ell}\, \exp[(p-\ell-2r)\pi i s/n], \quad \ell=0,1 \tag{2.20}$$

where it is to be understood that when $\ell=1$, the factor omitted in the product of series is that corresponding to $r=j$. In particular we have

$$B_{0,\ell} = (2i)^{\ell-p} \exp\Big[\frac{\pi i}{n}\sum_{r=\ell+1}^{p}\beta_r\Big], \quad B_{p-\ell,\ell} = (-2i)^{\ell-p}\exp\Big[-\frac{\pi i}{n}\sum_{r=\ell+1}^{p}\beta_r\Big].$$

The expansions in (2.19) may be obtained by application of the reflection formula for the gamma function in (2.11) and (2.12) and use of (2.20), together with analytic continuation to all arg z. As the expansion of $U_{n,p}(\omega_j z)$ is known for all arg z from Theorem 1, it follows that the complete asymptotic expansion of $V_{n,p}(z)$ and $W_{n,p}(\beta_j\,;\,z)$ beyond the sectors (2.17) and (2.13) may be constructed from (2.19).

A discussion of the three different situations which can arise according as to whether the central angle $\pi\kappa$ of the sector in which $U_{n,p}(z)$ is exponentially large [cf. (2.8)] is less than, equal to or greater than $2\pi/n$ (that is, according as n-p is less than, equal to or greater than 2) is discussed in I. We simply remark here that, from (2.19), the asymptotic behaviour of $V_{n,p}(z)$ and $W_{n,p}(\beta_j\,;\,z)$ $(j = 1,2,\dots,p)$ when $n-p=1$ is characterised by alternate exponential and algebraic sectors, each of central angle π/n. When $n-p=2$, the exponential sectors are adjacent, while for $n-p \geq 3$ the exponential sectors overlap.

A second convenient set of n fundamental solutions of (1.1), when $m=0$ and $n > p$, is given by taking the p algebraic-type solutions, $W_{n,p}(\beta_j;z)$

together with n-p exponentially small solutions, $V_{n,p}(\omega_j z)$. These functions provide the only fundamental set of solutions whose members possess distinct algebraic and exponential behaviour for large $|z|$. We may write this fundamental set as

$$\begin{aligned} &W_{n,p}(\beta_j; z) \quad ; \; j=1,2,\dots,p \\ &V_{n,p}(\omega_j z)\;,\; \omega_j = \exp(2\pi i j/n)\,;\; \text{any } n\text{-}p \text{ distinct values of } j \in \{0,1,\dots,n-1\}. \end{aligned} \tag{2.21}$$

Finally, when m=0 and the order n of the differential equation is even, we define even and odd solutions by $E_{n,p}(z)$ and $O_{n,p}(z)$. Such solutions are of considerable interest in physical applications, where problems are often encountered involving symmetry or anti-symmetry under the transformation $z \to -z$. These solutions are defined by the integrals[I]

$$\begin{aligned} \mathbb{E}_{n,p}(z) &= \frac{\pi^{\frac{1}{2}}}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\Gamma(-s)}{\Gamma(\frac{1}{2}+s)} \prod_{j=1}^{p} \Gamma\left(\frac{2s}{n}+\frac{\beta_j}{n}\right)\left(-\tfrac{1}{4}n^{2p/n}z^2\right)^s ds \\ \mathcal{O}_{n,p}(z) &= \frac{\pi^{\frac{1}{2}}}{2\pi}\int_{-\infty i}^{\infty i} \frac{\Gamma(\frac{1}{2}-s)}{\Gamma(1+s)} \prod_{j=1}^{p} \Gamma\left(\frac{2s}{n}+\frac{\beta_j}{n}\right)\left(-\tfrac{1}{4}n^{2p/n}z^2\right)^s ds \end{aligned} \tag{2.22}$$

valid in the sector $|\arg(-z^2)| < \pi p/n$. The paths of integration are suitably indented to lie to the right of all the poles at $s = -\frac{1}{2}\beta_j - \frac{1}{2}nk$, $k=0,1,2,\dots$ $(j=1,2,\dots,p)$, but to the left of those of $\Gamma(-s)$ and $\Gamma(\frac{1}{2}-s)$. By expanding the ratio of gamma functions as in (2.1), these solutions may be shown to be constant multiples of the Meijer function (1.4) with $\ell = p$, but with the first superscript index n replaced by $\frac{1}{2}$ n and the parameters r/n, $r=0,1,2,\dots,n-1$ reordered into groups corresponding to even and odd r.

Application of the residue theorem yields the series expansions

$$\begin{aligned} \mathbb{E}_{n,p}(z) &= \pi^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(\frac{1}{2}n^{p/n}z)^{2k}}{k!\,\Gamma(k+\frac{1}{2})} \prod_{j=1}^{p} \Gamma\left(\frac{2k+\beta_j}{n}\right) \\ \mathcal{O}_{n,p}(z) &= \pi^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(\frac{1}{2}n^{p/n}z)^{2k+1}}{k!\,\Gamma(k+\frac{3}{2})} \prod_{j=1}^{p} \Gamma\left(\frac{2k+1+\beta_j}{n}\right). \end{aligned} \tag{2.23}$$

These solutions are integral functions of z, uniformly and absolutely convergent throughout the finite z-plane and define $E_{n,p}(z)$ and $O_{n,p}(z)$ for all arg z by analytic continuation. We remark that

$$U_{n,p}(z) = E_{n,p}(z) + O_{n,p}(z) \quad (n \text{ even}).$$

From the asymptotic theory of integral functions of hypergeometric type, the asymptotic expansion of $E_{n,p}(z)$ and $O_{n,p}(z)$ is given in I by the following

Theorem 3. For $|z| \to \infty$, the asymptotic expansions of the even and odd solutions, $E_{n,p}(z)$, $O_{n,p}(z)$ of (1.1) when m=0 and n is even, are given by

$$\begin{aligned} E_{n,p}(z),\ O_{n,p}(z) &\sim E_{e,o}(z) + H_{e,o}(ze^{\mp\pi i}) \quad \text{in } |\arg z| \leq \tfrac{1}{2}\pi(1-\tfrac{p}{n}) \\ E_{n,p}(z),\ O_{n,p}(z) &\sim H_{e,o}(ze^{\mp\pi i}) \quad \text{in } \tfrac{1}{2}\pi(1-\tfrac{p}{n}) < |\arg z| \leq \tfrac{1}{2}\pi \end{aligned} \tag{2.24}$$

where the upper or lower signs are chosen according as arg z > 0 or < 0 respectively and the subscripts e,o refer to the even and odd solutions. The exponential expansions, $E_{e,o}(z)$ are defined by

$$E_e(z) = E_o(z) = \tfrac{1}{2}E(z), \tag{2.25}$$

where E(z) is given in (2.5), and, in the case that the β_r are all distinct and do not differ by integer multiples of n, the algebraic expansions, $H_{e,o}(z)$ are given by

$$\begin{aligned} H_e(ze^{\mp\pi i}) &= n\sum_{s=1}^{p}\left(n^{p/n} z e^{\mp\frac{1}{2}\pi i}\right)^{-\beta_s}\cos\tfrac{1}{2}\pi\beta_s\ S_{n,p}(\beta_s;z) \\ H_o(ze^{\mp\pi i}) &= \pm ni\sum_{s=1}^{p}\left(n^{p/n} z e^{\mp\frac{1}{2}\pi i}\right)^{-\beta_s}\sin\tfrac{1}{2}\pi\beta_s\ S_{n,p}(\beta_s;z) \end{aligned} \tag{2.26}$$

where $S_{n,p}(\beta_s;z)$ is defined in (2.7). The expansion of the fundamental system

$$E_{n,p}(\omega_j z),\ O_{n,p}(\omega_j z),\quad \omega_j = \exp(2\pi i j/n)\ ;\ j = 0,1,\ldots,\tfrac{1}{2}n-1 \tag{2.27}$$

follows immediately by simple rotation of the argument by $2\pi j/n$.

The solutions $E_{n,p}(z)$ and $O_{n,p}(z)$ are consequently exponentially large as $|z| \to \infty$ in the sectors $|\arg(\pm z)| < \frac{1}{2}\pi(1-p/n)$ and possess an algebraic behaviour in the sectors $|\arg(\pm iz)| < \frac{1}{2}\pi p/n$. On the dividing rays, $\arg z = \pm\frac{1}{2}\pi(1-p/n)$ and $\arg z = \pm\frac{1}{2}\pi(1+p/n)$, the expansions are of the mixed type where both the exponential and algebraic expansions must be retained. By suitable choice of j in (2.27), it is possible to orientate the sectors in which the behaviour of these solutions is algebraic. Of particular interest in physical problems are even and odd solutions which are algebraic as $z \to \pm\infty$. When $\frac{1}{2}n$ is even, it is always possible to choose $j = \frac{1}{4}n$, for example, so that the solutions $E_{n,p}(iz)$ and $O_{n,p}(iz)$ are algebraic as $|z| \to \infty$ in the sectors $|\arg(\pm z)| < \frac{1}{2}\pi p/n$. When $\frac{1}{2}n$ is odd, it is only possible to choose j in (2.27) such that the algebraic sectors, $|\arg(\pm i\omega_j z)| < \frac{1}{2}\pi p/n$ enclose the real axis provided $p \geq 3$.

III. THE SOLUTIONS FOR m A POSITIVE INTEGER

We extend the definition of the four classes of solutions of (1.1) when $m=0$, $U_{n,p}(z)$, $V_{n,p}(z)$, $W_{n,p}(\beta_j\,;z)$ and $E_{n,p}(z)$, $O_{n,p}(z)$, discussed in the preceding section, to include the general case of positive integer values of m. We shall denote these solutions by the addition of a superscript index m, which is omitted in the case $m=0$.

Recalling that $N=n+m$, $P=p+m$, we define the solution of (1.1), which reduces to (2.1) when $m=0$, by

$$U^m_{n,p}(z) = \frac{(-\pi)^m}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(-s)\,\frac{\prod_{j=1}^{p}\Gamma\left(\frac{s+\beta_j}{N}\right)}{\prod_{j=1}^{m}\Gamma\left(1-\frac{j}{N}-\frac{s}{N}\right)}\left(N^{P/N} z\, e^{-n\pi i/N}\right)^s ds \tag{3.1}$$

valid in the sector

$$|\arg(z e^{-n\pi i/N})| < \tfrac{1}{2}\pi(n+p)/N. \tag{3.2}$$

The path of integration is as in (2.1) and the factor $(-\pi)^m$ has been introduced for convenience. By expanding $\Gamma(-s)$ as a product of N gamma functions, as at (2.1), and noting that the last m of these functions cancel with the product in the denominator of (3.1), we see that this solution is a constant multiple of the Meijer function (1.4) with index $\ell=p$.

The solution $U^m_{n,p}(z)$ is an integral function of z, except for those values of the β_j for which the poles of the gamma functions cannot be separated. We see this as in Section 2 by swinging round the path of integration into an infinite semi-circle in the right-half plane and applying the residue theorem to find

$$U^m_{n,p}(z) = (-\pi)^m \sum_{k=0}^{\infty} \frac{\left(N^{P/N} z\, e^{m\pi i/N}\right)^k}{k!}\,\frac{\prod_{j=1}^{p}\Gamma\left(\frac{k+\beta_j}{N}\right)}{\prod_{j=1}^{m}\Gamma\left(1-\frac{j}{N}-\frac{k}{N}\right)} \tag{3.3}$$

for all arg z by analytic continuation. We observe that there is no contribution to the above sum whenever k+j is a positive integer multiple of N. Since (1.1) is unaltered if we replace z by ωz, where ω denotes an Nth root of unity, it follows that $U^m_{n,p}(\omega z)$ is also a solution. The theory of linear differential equations tells us that only n of the N solutions $U^m_{n,p}(\omega_j z)$, where $\omega_j=\exp(2\pi ij/N)$, $j=0,1,\ldots,N-1$, can be linearly independent. We therefore choose n suitable values of j to form a fundamental set.

The expansion of $U^m_{n,p}(z)$ as $|z|\to\infty$ in the principal sector (3.2) then follows from (3.1) by displacing the path of integration to the left to find

$$U^m_{n,p}(z) \sim (-\pi)^m N \sum_{s=1}^{p} \left(N^{p/N} z e^{-n\pi i/N}\right)^{-\beta_s} S^m_{n,p}(\beta_s; z e^{-\pi i}) \tag{3.4}$$

where $S^m_{n,p}(\beta_s;z)$, in analogy with (2.7), denotes the formal asymptotic sum

$$S^m_{n,p}(\beta_s;z) = \sum_{k=0}^{\infty} \frac{(-)^k \Gamma(Nk+\beta_s)}{k! \prod_{j=1}^{m} \Gamma\left(k+1+\frac{\beta_s-j}{N}\right)} \prod_{j=1}^{p}{}' \Gamma\left(\frac{\beta_j-\beta_s}{N}-k\right) \left(N^{p/N} z e^{m\pi i/N}\right)^{-Nk}. \tag{3.5}$$

We see that this expansion consists of p expansions each with controlling factor $z^{-\beta_r}$, $r=1,2,\ldots,p$. When m=0 this is the algebraic expansion $H(ze^{-\pi i})$ given in (2.8). The $e^{\pi i}$ term in (2.8) does not arise here because arg $z>0$ in the region of validity of (3.1).

To obtain the asymptotic expansion of $U^m_{n,p}(z)$ valid in the rest of the z-plane, it is convenient to express $U^m_{n,p}(z)$ as a sum of m+1 solutions of an equation of type (1.1) of higher order N=n+m but with the absence of the term z^m multiplying the lower derivatives. To do this, we replace the product of gamma functions in the denominator of (3.1) by a product of sines in the numerator using the reflection formula, to give

$$U^m_{n,p}(z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(-s) \prod_{j=1}^{p} \Gamma\left(\frac{s+\beta_j}{N}\right) \prod_{j=1}^{m} \left\{\Gamma\left(\frac{j+s}{N}\right) \sin\frac{\pi}{N}(s-n+1-j)\right\} \left(N^{p/N} z e^{-n\pi i/N}\right)^s ds$$

$$= \sum_{r=0}^{m} \frac{B_r}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(-s) \prod_{j=1}^{p} \Gamma\left(\frac{s+\beta_j}{N}\right) \left(-N^{t/N} z e^{2\pi i(m-r)/N}\right)^s ds, \tag{3.6}$$

where we have combined the two products of gamma functions by defining

$$\beta_{p+j} = j \quad , \quad j = 1,2,.., m \tag{3.7}$$

and where the product of sines has been expanded as

$$\prod_{j=1}^{m} \sin \frac{\pi}{N}(s-n+1-j) = \sum_{r=0}^{m} B_r \, e^{(m-2r)\pi i s/N} .$$

The coefficient B_r (the Stokes multipliers) are defined by

$$B_r = (-2)^{-m} \exp\left[\frac{\pi i}{2N}(n-1)(2r-m)\right] \prod_{j=1}^{r} \frac{\sin \frac{\pi}{N}(m-j+1)}{\sin \frac{\pi j}{N}} , \tag{3.8}$$

with the usual convention that empty products are taken to be unity.

But we know from (2.2) that, for $N > P \geq 0$,

$$U_{N,P}(z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(-s) \prod_{j=1}^{P} \Gamma\left(\frac{s+\beta_j}{N}\right)\left(-N^{P/N} z\right)^{s} ds , \quad |\arg(-z)| < \tfrac{1}{2}\pi\left(1+\tfrac{P}{N}\right).$$

Thus (3.6) enables us to derive the expansion theorem

$$U^{m}_{n,p}(z) = \sum_{r=0}^{m} B_r \, U_{n+m,\,p+m}\left(z e^{2\pi i(m-r)/N}\right) \tag{3.9}$$

valid for all arg z. The asymptotic expansion of the solution $U^{m}_{n,p}(z)$ beyond the principal sector (3.2), where its expansion is given by (3.4) , may now be constructed from that of the m+1 $U_{N,P}(\omega_j z)$ functions in (3.9) and application of Theorem 1, with n and p both incremented by m.

We remark that we have expressed the solution $U^{m}_{n,p}(z)$ of the nth order equation (1.1) in terms of the solutions $U_{N,P}$ of an equation of the form (1.1) of higher order N, involving P lower derivatives but with effectively m=0. This connection may be exhibited by differentiation of (1.1) m times and use of the additional parameters β_{p+j} defined in (3.7). Adopting the form (1.2), we may write (1.1) as

$$u^{(n)} - z^m \prod_{r=1}^{p} (\Theta + \beta_r) u = 0.$$

Successive differentiation with respect to z yields

$$u^{(n+1)} - z^{m-1}(\Theta+m)\prod_{r=1}^{p}(\Theta+\beta_r)u = 0$$

and, after the mth differentiation,

$$u^{(n+m)} - (\Theta+1)(\Theta+2)\ldots(\Theta+m)\prod_{r=1}^{p}(\Theta+\beta_r)u = 0.$$

This may be rearranged as

$$u^{(N)} - \prod_{r=1}^{P}(\Theta+\beta_r)u = 0\,,\ \beta_{p+r} = r\quad (r=1,2,\ldots,m) \tag{3.10}$$

which, as we have seen in Section 2, possesses the solutions $U_{N,P}(\omega_j z)$, $\omega_j = \exp(2\pi ij/N)$. We shall call (3.10) the <u>related</u> <u>differential</u> <u>equation</u> associated with (1.1).

The solution $U^{m}_{n,p}(z)$ considered so far is capable of exhibiting only algebraic or exponentially large behaviour as $|z| \to \infty$. The extension of the exponentially small and algebraic-type solutions in Section 2, for positive integer m, are given by constant multiples of (1.4) with the indices $\ell = 0$ and 1 respectively. After suitable manipulation of the gamma functions, as in (3.1), and introduction of the additional parameters β_{p+j} in (3.7), we have the integral representations

$$V^{m}_{n,p}(z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\pi^{P}\,\Gamma(-s)\,\left(N^{P/N} z e^{-\pi i(n-p)/N}\right)^{s}}{\prod_{r=1}^{P}\Gamma(1-\frac{\beta_r}{N}-\frac{s}{N})}\,ds \tag{3.11}$$

$$W^{m}_{n,p}(\beta_j\,;z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\pi^{P-1}\,\Gamma(-s)\,\Gamma(\frac{s}{N}+\frac{\beta_j}{N})}{\prod_{r=1}^{P}{}'\,\Gamma(1-\frac{\beta_r}{N}-\frac{s}{N})}\left(N^{P/N} z e^{-\pi i(n-p+1)/N}\right)^{s} ds \quad (p\geq 1) \tag{3.12}$$

$$j=1,2,\ldots,p$$

valid in the principal sectors

$$\tfrac{1}{2}\tfrac{\pi}{N}(n-p) < \arg z < \tfrac{3}{2}\tfrac{\pi}{N}(n-p) + \frac{2\pi\ell}{N}\qquad (\ell=0,1). \tag{3.13}$$

On comparing with (2.11) and (2.12), we see at once that

$$\begin{aligned} V^{m}_{n,p}(z) &= V_{n+m,\,p+m}(z) \\ W^{m}_{n,p}(\beta_j;z) &= W_{n+m,\,p+m}(\beta_j;z) \end{aligned} \qquad (3.14)$$

in their respective principal sectors, and hence for all arg z by analytic continuation. Thus we have succeeded in expressing the solutions $V^{m}_{n,p}(z)$ and $W^{m}_{n,p}(\beta_j;z)$ in terms of the single corresponding solution of the related Nth order equation (3.10) with m effectively equal to zero. It then follows that the asymptotic expansions of a fundamental system of these solutions, given by the p functions $W^{m}_{n,p}(\beta_j;z)$, $j=1,2,\ldots,p$ together with any n-p functions $V^{m}_{n,p}(\omega_j z)$, $\omega_j = \exp(2\pi i j/N)$, $j=0,1,\ldots,N-1$, can be obtained immediately from (2.16), (2.18) and Theorems 1 and 2, with n and p both incremented by m and

$$\vartheta = -\tfrac{1}{2}P + \frac{1}{N}\sum_{r=1}^{P}\beta_r = -\tfrac{1}{2}p + \frac{1}{N}\sum_{r=1}^{p}\beta_r - \frac{1}{2}\frac{m}{N}(n-1).$$

We note that when p=0 or 1

$$V^{m}_{n,0}(z) = (-\pi)^{-m}\,U^{m}_{n,0}(z)\,,\quad W^{m}_{n,1}(\beta_1;z) = (-)^{m}\,U^{m}_{n,1}(z). \qquad (3.15)$$

Finally, even and odd solutions of (1.1) can be found whenever N=n+m is even. We can achieve this in two different ways which we consider separately below : either when n,m are both even or when both are odd. In the case when n,m are both even we shall write m=2m'. The even and odd solutions in this case which reduce to (2.22) when m=0 may be shown to be

$$E^{m}_{n,p}(z) = \frac{\pi^{\frac{1}{2}}}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(-s)}{\Gamma(\frac{1}{2}+s)}\prod_{j=1}^{m'}\frac{\Gamma\left(\frac{2s+2j-1}{N}\right)}{\Gamma\left(1-\frac{2j}{N}-\frac{2s}{N}\right)}\prod_{j=1}^{p}\Gamma\left(\frac{2s+\beta_j}{N}\right)\left(\tfrac{1}{4}N^{2P/N}z^2e^{-n\pi i/N}\right)^s ds \qquad (3.16)$$

$$= \pi^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(\frac{2k+\beta_j}{N}\right)}{k!\;\Gamma(k+\frac{1}{2})}\prod_{j=1}^{m'}\frac{\Gamma\left(\frac{2k+2j-1}{N}\right)}{\Gamma\left(1-\frac{2k}{N}-\frac{2j}{N}\right)}\left(\tfrac{1}{2}N^{P/N}z\,e^{m\pi i/2N}\right)^{2k} \qquad (3.17)$$

$$\mathcal{O}^m_{n,p}(z) = \frac{\pi^{\frac{1}{2}}}{2\pi}\int_{-\infty i}^{\infty i}\frac{\Gamma(\frac{1}{2}-s)}{\Gamma(1+s)}\prod_{j=1}^{m'}\frac{\Gamma(\frac{2s+2j}{N})}{\Gamma(1-\frac{2s}{N}-\frac{2j-1}{N})}\prod_{j=1}^{p}\Gamma\left(\frac{2s+\beta_j}{N}\right)\left(\tfrac{1}{4}N^{2P/N}z^2e^{-n\pi i/N}\right)^s ds \tag{3.18}$$

$$= \pi^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\frac{2k+1+\beta_j}{N})}{k!\,\Gamma(k+\frac{3}{2})}\prod_{j=1}^{m'}\frac{\Gamma(\frac{2k+2j+1}{N})}{\Gamma(1-\frac{2k}{N}-\frac{2j}{N})}\left(\tfrac{1}{2}N^{P/N}z\,e^{m\pi i/2N}\right)^{2k+1}, \tag{3.19}$$

where the paths of integration are as in (2.22) and the integrals are both valid in the sector $|\arg(e^{-n\pi i/N}z^2)| < \pi p/N$. The series expansions define the solutions for all arg z by analytic continuation.

Omitting the details, which consist of employing the additional β_{p+j} in (3.7) and application of the reflection formula for the gamma function, we may express $E^m_{n,p}(z)$ and $O^m_{n,p}(z)$ as sums of the corresponding even and odd solutions of the related higher order equation (3.10) with effectively m=0. We find the expansion theorems in the case n,m both even, valid for all arg z,

$$\begin{aligned} E^m_{n,p}(z) &= \sum_{r=0}^{m'} B^e_r\, E_{n+m,p+m}\left(z e^{\pi i(m-2r)/N}\right) \\ \mathcal{O}^m_{n,p}(z) &= \sum_{r=0}^{m'} B^o_r\, \mathcal{O}_{n+m,p+m}\left(z e^{\pi i(m-2r)/N}\right), \end{aligned} \tag{3.20}$$

where the coefficients $B^{e,o}_r$ are given by

$$\begin{aligned} B^e_r &= (2\pi)^{-\frac{1}{2}m}\exp\left[\frac{\pi i}{4N}(n-2)(4r-m)\right]\prod_{j=1}^{r}\frac{\sin\frac{\pi}{N}(m-2j+2)}{\sin\frac{2\pi j}{N}} \\ B^o_r &= \exp\left[\frac{\pi i}{2N}(4r-m)\right]B^e_r. \end{aligned} \tag{3.21}$$

In the case n,m both odd, we set m=2m'-1. A similar procedure shows that the even and odd solutions in this case may be defined by

$$E^m_{n,p}(z) = \frac{\pi^{\frac{1}{2}}}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(-s)}{\Gamma(\frac{1}{2}+s)}\frac{\prod_{j=1}^{m'}\Gamma(\frac{2s+2j-1}{N})}{\prod_{j=1}^{m'-1}\Gamma(1-\frac{2j}{N}-\frac{2s}{N})}\prod_{j=1}^{p}\Gamma\left(\frac{2s+\beta_j}{N}\right)\left(\tfrac{1}{4}N^{2P/N}z^2e^{-(n+1)\pi i/N}\right)^s ds \tag{3.22}$$

$$= \pi^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\frac{2k+\beta_j}{N})}{k!\,\Gamma(k+\frac{1}{2})}\frac{\prod_{j=1}^{m'}\Gamma(\frac{2k+2j-1}{N})}{\prod_{j=1}^{m'-1}\Gamma(1-\frac{2j}{N}-\frac{2k}{N})}\left(\tfrac{1}{2}N^{P/N}z\,e^{\pi i(m-1)/2N}\right)^{2k} \tag{3.23}$$

$$\mathcal{O}^m_{n,p}(z) = \frac{\pi^{\frac12}}{2\pi}\int_{-\infty i}^{\infty i} \frac{\Gamma(\frac12 - s)}{\Gamma(1+s)}\, \frac{\prod_{j=1}^{m'-1}\Gamma\left(\frac{2s+2j}{N}\right)}{\prod_{j=1}^{m'}\Gamma\left(1-\frac{2s}{N}-\frac{2j-1}{N}\right)} \prod_{j=1}^{p} \Gamma\left(\frac{2s+\beta_j}{N}\right)\left(\tfrac14 N^{2P/N} z^2 e^{-\pi i(n-1)/N}\right)^s ds \tag{3.24}$$

$$= \pi^{\frac12}\sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma\left(\frac{2k+1+\beta_j}{N}\right)}{k!\,\Gamma(k+\frac32)}\, \frac{\prod_{j=1}^{m'-1}\Gamma\left(\frac{2k+1+2j}{N}\right)}{\prod_{j=1}^{m'}\Gamma\left(1-\frac{2j}{N}-\frac{2k}{N}\right)} \left(\tfrac12 N^{P/N} z\, e^{\pi i(m+1)/2N}\right)^{2k+1}, \tag{3.25}$$

where the integrals (3.22) and (3.24) are defined in the sectors $|\arg\,(e^{-\pi i(n-1)/N}z^2)| < \pi\,(p \pm 1)/N$ respectively. The corresponding expansion theorems when n,m are both odd, valid for all arg z, are then

$$\begin{aligned} \mathbb{E}^m_{n,p}(z) &= \sum_{r=0}^{m'-1} B^e_r\, \mathbb{E}_{n+m,p+m}\left(z e^{\pi i(m-2r-1)/N}\right) \\ \mathcal{O}^m_{n,p}(z) &= \sum_{r=0}^{m'} B^o_r\, \mathcal{O}_{n+m,p+m}\left(z\, e^{\pi i(m-2r+1)/N}\right) \end{aligned} \tag{3.26}$$

where the coefficients $B^{e,o}_r$ are now defined by

$$\begin{aligned} B^e_r &= (2\pi)^{\frac12-\frac12 m} \exp\left[\frac{\pi i}{4N}(n-1)(4r-m+1)\right] \prod_{j=1}^{r} \frac{\sin\frac{\pi}{N}(m-2j+1)}{\sin\frac{2\pi j}{N}} \\ B^o_r &= (2\pi)^{\frac12-\frac12 m} \exp\left[\frac{\pi i}{4N}(n-1)(4r-m-1)\right] \prod_{j=1}^{r} \frac{\sin\frac{\pi}{N}(m-2j+3)}{\sin\frac{2\pi j}{N}}\ . \end{aligned} \tag{3.27}$$

The above expansions in (3.20) and (3.26) may be used to construct the asymptotic expansions of $E^m_{n,p}(\omega_j z)$ and $0^m_{n,p}(\omega_j z)$, $\omega_j = \exp\,(2\pi i j/N)$, as $|z| \to \infty$ for all arg z from the known expansions of $E_{n,p}(z)$ and $\theta_{n,p}(z)$ given in Theorem 3, with n and p both incremented by m.

We note that because of the disposition of the gamma functions in the integrands and the choice of the path of integration, the integral representations (3.16) and (3.18) are valid only for $p > 0$, and (3.24) for $p > 1$. In the cases not covered, we must select the alternative path of integration $\infty - ic$, $\infty + ic$ $(c > 0)$ for the Meijer function described in (Luke 1975). The resulting series and asymptotic expansions remain unaltered.

IV. AN EXAMPLE

To illustrate the asymptotic theory of (1.1) given in Section 3, we consider the specific example, with n=3, m=p=1,

$$u^{(3)} - z(zu' + au) = 0, \qquad a \neq 0, -1, -2, \ldots . \tag{4.1}$$

This third order differential equation arises in magnetohydrodynamic boundary layer theory (Glasser et al. 1975) and in the stability of the stellar wind (Kahn 1981). The case m=0 has been considered previously by Langer (1955) and in [I] .

We find from (1.2) and (2.4) that β_1=a and $\vartheta = \frac{1}{4}a - \frac{3}{4}$. The basic solution is then, from (3.1) and (3.3),

$$U^1_{3,1}(z) = \frac{-1}{2i}\int_{-\infty i}^{\infty i} \frac{\Gamma(-s)\Gamma(\frac{1}{4}s+\frac{1}{4}a)}{\Gamma(\frac{3}{4}-\frac{1}{4}s)} \left(2ze^{-\frac{3}{4}\pi i}\right)^s ds, \quad \tfrac{1}{4}\pi < \arg z < \tfrac{5}{4}\pi$$

$$= -\pi \sum_{k=0}^{\infty} \frac{(2ze^{\frac{1}{4}\pi i})^k}{k!} \frac{\Gamma(\frac{1}{4}k+\frac{1}{4}a)}{\Gamma(\frac{3}{4}-\frac{1}{4}k)} .$$

In (Paris and Wood 1985), it is shown that $U^1_{3,1}$ (z) can also be represented in terms of the Bessel function by the Laplace integral

$$U^1_{3,1}(z) = -2^{\frac{3}{2}-\frac{1}{2}a}\pi \int_0^{\infty} \exp(zte^{\frac{1}{4}\pi i})\, t^{\frac{1}{2}a-\frac{1}{2}} J_{\frac{1}{4}a-\frac{1}{4}}(\tfrac{1}{2}t^2)\, dt, \quad Re(a)>0,\ Re(ze^{\frac{1}{4}\pi i})<0.$$

The asymptotic expansion of $U^1_{3,1}$ (z) in the principal sector $(\frac{1}{4}\pi, \frac{5}{4}\pi)$ is found from (3.4) and (3.5) to be

$$-4\pi\left(2ze^{-\frac{3}{4}\pi i}\right)^{-a} \sum_{k=0}^{\infty} \frac{\Gamma(4k+a)}{k!\,\Gamma(\frac{1}{4}a+\frac{3}{4}+k)} (2z)^{-4k}.$$

Outside the principal sector, we must use the expansion theorem (3.9)

$$U^1_{3,1}(z) = B_0\, U_{4,2}(ze^{\frac{1}{2}\pi i}) + B_1\, U_{4,2}(z)$$

with $B_o = -\frac{1}{2} e^{-\pi i/4}$, $B_1 = -\frac{1}{2} e^{\pi i/4}$. We know from Theorem 1 that

$$U_{4,2}(ze^{\frac{1}{2}\pi i}) \sim E(ze^{\frac{1}{2}\pi i}) + H(ze^{\frac{1}{2}\pi i} \cdot e^{\mp\pi i}) \quad \text{in} \quad (-\tfrac{3}{4}\pi, -\tfrac{1}{4}\pi)$$

and

$$U_{4,2}(z) \sim E(z) + H(ze^{\mp\pi i}) \qquad \text{in} \quad (-\tfrac{1}{4}\pi, \tfrac{1}{4}\pi),$$

where E and H are the exponential and algebraic type of asymptotic sums defined in (2.5), (2.6) and (2.7). In particular, on setting n=4, p=2, $\vartheta = \frac{1}{4} a - \frac{3}{4}$, $\beta_1 = a$ and, from (3.7), $\beta_2 = 1$, we have

$$E(z) = 2^{3-\frac{1}{2}a} \pi \, z^{\frac{1}{2}a-\frac{3}{2}} \, e^{\frac{1}{2}z^2} \sum_{k=0}^{\infty} c_k \left(\tfrac{1}{2}z^2\right)^{-k},$$

where c_o=1, and

$$H(z) = 4(2z)^{-a} \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \Gamma(4k+a) \Gamma(\tfrac{1}{4}-\tfrac{1}{4}a-k)(2z)^{-4k}$$

$$+ 4(2z)^{-1} \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \Gamma(4k+1) \Gamma(\tfrac{1}{4}a-\tfrac{1}{4}-k)(2z)^{-4k}.$$

Note that in $H(ze^{\frac{1}{2}\pi i} \cdot e^{\mp\pi i})$ the upper or lower sign is chosen according as arg $(ze^{\frac{1}{2}\pi i})$ is > 0 or < 0 respectively. We see from (2.9) that $c_1 = \frac{3}{32}(a-3)^2$; in this particular example, a closed form representation for the coefficients c_k in terms of the Gauss hypergeometric function can be given as [1]

$$c_k = \frac{1}{k!} \left(\tfrac{3}{4}-\tfrac{1}{4}a\right)_k \left(\tfrac{5}{4}-\tfrac{1}{4}a\right)_k \, {}_2F_1\left(-k, \tfrac{1}{4}a+\tfrac{1}{4}; \tfrac{5}{4}-\tfrac{1}{4}a; \tfrac{1}{2}\right). \tag{4.2}$$

The leading behaviour of $U^1_{3,1}(z)$ as $|z| \to \infty$ in the exponential sectors $(-\frac{1}{4}\pi, \frac{1}{4}\pi)$ and $(-\frac{3}{4}\pi, -\frac{1}{4}\pi)$ is therefore given by

$$U^1_{3,1}(z) \sim -2^{2-\frac{1}{2}a} \pi e^{\frac{1}{4}\pi i} z^{\frac{1}{2}a-\frac{3}{2}} e^{\frac{1}{2}z^2} \quad \text{in} \quad (-\tfrac{1}{4}\pi, \tfrac{1}{4}\pi)$$

$$U^1_{3,1}(z) \sim -2^{2-\frac{1}{2}a} \pi e^{-\frac{1}{4}\pi i} (iz)^{\frac{1}{2}a-\frac{3}{2}} e^{-\frac{1}{2}z^2} \quad \text{in} \quad (-\tfrac{3}{4}\pi, -\tfrac{1}{4}\pi).$$

On the rays arg $z = -\frac{3}{4}\pi$, $\pm\frac{\pi}{4}$, which devide the adjacent sectors, the asymptotics are a combination of algebraic and exponential oscillatory expansions.

The solution which is exponentially small at infinity is given by (3.11) to be

$$V_{3,1}^{1}(z) = \frac{\pi}{2i}\int_{-\infty i}^{\infty i}\frac{\Gamma(-s)\,(2ze^{-\frac{1}{2}\pi i})^{s}}{\Gamma(1-\frac{1}{4}a-\frac{1}{4}s)\,\Gamma(\frac{3}{4}-\frac{1}{4}s)}\,ds$$

valid in its principal sector $(\frac{1}{4}\pi, \frac{3}{4}\pi)$. The relationship (3.14) gives

$$V_{3,1}^{1}(z) = V_{4,2}(z)$$

with asymptotic expansion as $|z|\to\infty$ in the extended sector $(-\frac{1}{4}\pi, \frac{5}{4}\pi)$ given by (2.16)

$$2^{1-\frac{1}{2}a}\pi\,(e^{-\frac{1}{2}\pi i}z)^{\frac{1}{2}a-\frac{3}{2}}\,e^{\frac{1}{2}z^{2}}\sum_{k=0}^{\infty}c_k\,(\tfrac{1}{2}z^{2})^{-k}, \tag{4.3}$$

where the c_k are as defined in (4.2). Thus $V_{3,1}^{1}(z)$ is exponentially small in the principal sector $(\frac{1}{4}\pi, \frac{3}{4}\pi)$. We may obtain the asymptotic expansion of $V_{3,1}^{1}(z)$ outside its extended sector $(-\frac{1}{4}\pi, \frac{5}{4}\pi)$ by use of Theorem 2, which in this case is

$$V_{3,1}^{1}(z) = \sum_{r=0}^{2} B_{r,0}\,U_{4,2}(ze^{(2-r)\pi i/2}),$$

where the coefficients $B_{r,0}$ are given by (2.20) with $p = 2$, namely

$$B_{0,0} = -\frac{1}{4}e^{\frac{1}{4}\pi i(a+1)}, \quad B_{1,0} = \frac{1}{2}\cos\frac{\pi}{4}(a-1), \quad B_{2,0} = -\frac{1}{4}e^{-\frac{1}{4}\pi i(a+1)}.$$

This will be a combination of algebraic, $H(z)$ and exponential, $E(z)$ expansions as for $U_{3,1}^{1}(z)$. In the sectors $(\frac{3}{4}\pi, \frac{5}{4}\pi)$ and $(-\frac{1}{4}\pi, \frac{1}{4}\pi)$ it follows that $V_{3,1}^{1}(z)$ is exponentially large at infinity with expansion given by (4.3). In the remaining sector $(-\frac{3}{4}\pi, -\frac{1}{4}\pi)$ the expansion is algebraic in character.

As in the case m=0, the algebraic type solution $W^{1}_{3,1}(\beta_1\,;z)$ requires no discussion, since from (3.15) it is identically equal to the basic solution $-U^{1}_{3,1}(z)$. But, unlike the case m=0, it is possible, when m=1, to construct even and odd solutions, even though the differential equation (4.1) is of odd order. Since n=3 and m=1, we are in the case n,m both odd of the preceding section. With m'=1, the even solution is, from (3.22) and (3.23), given by

$$E^{1}_{3,1}(z) = \frac{\pi^{\frac12}}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\Gamma(-s)\,\Gamma(\frac12 s+\frac14 a)\,\Gamma(\frac12 s+\frac14)}{\Gamma(\frac12+s)}\,(-z^2)^s\,ds\ , \quad 0<\arg z<\tfrac12\pi$$

$$= \pi^{\frac12}\sum_{k=0}^{\infty}\frac{\Gamma(\frac12 k+\frac14 a)\,\Gamma(\frac12 k+\frac14)}{k!\,\Gamma(k+\frac12)}\,z^{2k}.$$

According to the expansion theorem (3.26), this is equal to the even solution $E_{4,2}(z)$ of the related fourth order equation (3.10) with m=0, as may be verified directly by inspection of (2.22).

The odd solution is defined by (3.25) and (3.26) to be, for all z,

$$\mathcal{O}^{1}_{3,1}(z) = \pi^{\frac12}\sum_{k=0}^{\infty}{}' \frac{\Gamma(\frac12 k+\frac14 a+\frac14)}{k!\,\Gamma(k+\frac32)\,\Gamma(\frac12-\frac12 k)}\,(ze^{\frac12\pi i})^{2k+1}$$

$$= \frac{1}{2\pi}\left\{e^{-\frac14\pi i}\,\mathcal{O}_{4,2}(ze^{\frac12\pi i}) + e^{\frac14\pi i}\,\mathcal{O}_{4,2}(z)\right\},$$

where $O_{4,2}(z)$ is the odd solution of the related fourth order equation (3.10) with m=0 and the prime indicates summation over even values of k. Note that, because p=1 here, the integral representation (3.24) for $O^{1}_{3,1}(z)$ has null range of validity and may not be employed (see the remark at the end of Section 3).

The asymptotic expansions of $E_{4,2}(z)$ and $O_{4,2}(z)$ are known from Theorem 3. From (2.24) and (2.26), we obtain

$$E'_{3,1}(z) = E_{4,2}(z) \sim E_e(z) + H_e(ze^{\mp\pi i}) \quad \text{in } [-\tfrac{1}{4}\pi, \tfrac{1}{4}\pi]$$

$$\sim H_e(ze^{\mp\pi i}) \qquad \text{in } (\tfrac{1}{4}\pi, \tfrac{1}{2}\pi) \text{ and } (-\tfrac{1}{2}\pi, -\tfrac{1}{4}\pi),$$

where $E_e(z) = \frac{1}{2}E(z)$, as defined in (2.5) , and

$$H_e(ze^{\mp\pi i}) = 4(2ze^{\mp\frac{1}{2}\pi i})^{-a}\cos\tfrac{1}{2}\pi a \sum_{k=0}^{\infty} \frac{(-)^k}{k!}\Gamma(4k+a)\Gamma(\tfrac{1}{4}-\tfrac{1}{4}a-k)(2z)^{-4k}.$$

Similarly, we have

$$O_{4,2}(z) \sim E_o(z) + H_o(ze^{\mp\pi i}) \quad \text{in } [-\tfrac{1}{4}\pi, \tfrac{1}{4}\pi]$$

$$\sim H_o(ze^{\mp\pi i}) \qquad \text{in } (\tfrac{1}{4}\pi, \tfrac{1}{2}\pi) \text{ and } (-\tfrac{1}{2}\pi, -\tfrac{1}{4}\pi),$$

where $E_o(z) = E_e(z)$, and

$$H_o(ze^{\mp\pi i}) = \pm 4i(2ze^{\mp\frac{1}{2}\pi i})^{-a}\sin\tfrac{1}{2}\pi a \sum_{k=0}^{\infty} \frac{(-)^k}{k!}\Gamma(4k+a)\Gamma(\tfrac{1}{4}-\tfrac{1}{4}a-k)(2z)^{-4k}$$

$$\pm 4i(2ze^{\mp\frac{1}{2}\pi i})^{-1} \sum_{k=0}^{\infty} \frac{(-)^k}{k!}\Gamma(4k+1)\Gamma(\tfrac{1}{4}a-\tfrac{1}{4}-k)(2z)^{-4k}.$$

We now see that both even and odd solutions are exponentially large in the sectors $(-\pi/4, \pi/4)$ and $(3\pi/4, 5\pi/4)$ and of algebraic type at infinity in the sectors $(-3\pi/4, -\pi/4)$ and $(\pi/4, 3\pi/4)$. On the dividing rays $\arg z = \pm 3\pi/4$, $\pm\pi/4$ the expansions are of mixed type, where both algebraic and exponential oscillatory terms must be considered.

It is interesting to contrast these expansions with the corresponding results for the equation n=3, p=1, m=0 discussed in I. There we had exponentially small, algebraic and exponentially large sectors each of angular width $2\pi/3$. The effect of introducing the multiplier z, acting on the lower order derivatives in the differential equation, is to reduce the angle of the basic sector to $\pi/2$. The sectors remain touching, but now with two adjacent algebraic sectors and one each of exponentially large and small behaviour.

We remark finally that solutions of (4.1) which are real on the real z-axis (a real) can be found from the above either by suitable rotation of the argument z by multiples of $\pi/2$ or by forming suitable linear combinations. The equation (4.1) with $a_2=-1$, instead of $a_2=1$, can be similarly dealt with by rotation of z by $\pi/4$.

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