PPPL-2154 DK-0657-2 PPPL-2154

UC20-G I-18439

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NOVEMBER 1984

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PRINCETON, NEW JERSEY

PREPARED FOR THE U.S. DEPARTMENT OF ENERGY,
UNDER CONTRACT DE-AC02-76-CHO-3073.

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PPPL--2154

DE85 004400

ABSTRACT

Globally divergence-free fields, such as the magnetic field and the vorticity, can be described by a two degree of freedom Hamiltonian. The Hamiltonian function provides a complete topological description of the field lines. The formulation also separates the dissipative and inertial time scale evolution of the magnetic and the vorticity fields.

I. INTRODUCTION

Divergence-free fields, such as the vorticity and the magnetic field, are known to be Hamiltonian systems. However, especially for the vorticity, the Hamiltonian is often considered to have an infinite number of degrees of freedom.[1] In the plasma physics literature, the equivalence of a magnetic field to a two degree of freedom Hamiltonian system has been appreciated,[2,3] but the exact relationship to ordinary canonical theory was not made clear. In this paper, the exact relationship is given between an arbitrary, globally divergence-free field, and canonical Hamiltonian mechanics. This generalizes recent results for magnetic fields with a positive definite toroidal component.[4]

The Hamiltonian representation of a globally divergence-free field consists of two parts, the Hamiltonian H and the ordinary space position $\vec{x}$, both as functions of four canonical coordinates. The Hamiltonian completely describes the topology of the integral curves of the field, usually called field lines. In dissipative fluid mechanics or plasma physics, the Hamiltonian changes only on a slow dissipative time scale. The ordinary space position $\vec{x}$ of a given point in canonical coordinate space evolves on an arbitrarily rapid time scale to maintain force balance. The separation of time scales is one of the attractive features of the canonical Hamiltonian formulation.

To simplify the notation and nomenclature of the paper, the language of magnetic fields will be used. However, the application of the results to other divergence-free fields, in particular the vorticity, requires only a trivial change of notation.

II. CANONICAL REPRESENTATION

The Hamiltonian formulation is based on the proof that any globally divergence-free field can be written in the canonical form[4]

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\theta + \vec{\nabla}\phi \times \vec{\nabla}\chi \quad . \tag{1}$$

The quantities θ and ϕ are proper angles, and ψ and χ are single-valued functions of position.

The canonical representation is distinguished from the well-known Clebsch representation

$$\vec{B} = \vec{\nabla}\psi_o \times \vec{\nabla}\theta_o \tag{2}$$

by the fact that ψ_o will generally be a multivalued function of position and θ_o cannot generally be a proper angle. By a proper angle, we mean the physical position corresponding to θ and $\theta + 2\pi$ are identical.

The simplest existence proof[5] of the canonical representation is based on a theorem, associated with Poincaré, that a globally divergence-free field has a single-valued vector potential $\vec{A}$ with

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad . \tag{3}$$

Consider any set of toroidal coordinates ρ, θ, ϕ which have a finite Jacobian in the spatial region of interest. For definiteness, assume ρ is the radial coordinate, θ is the poloidal angle, and ϕ is the toroidal angle. Then any vector can be written as

$$\vec{A} = A_\rho \vec{\nabla}\rho + A_\theta \vec{\nabla}\theta + A_\phi \vec{\nabla}\phi \quad . \tag{4}$$

This representation can always be rewritten in terms of single-valued functions G, ψ, and χ of ρ, θ, and ϕ as

$$A = \psi\vec{\nabla}\theta - \chi\vec{\nabla}\phi + \vec{\nabla}G \quad , \tag{5}$$

which has the canonical form for $\vec{B}$ as its curl.

The functions ψ and χ have a simple physical interpretation. The toroidal magnetic flux inside a constant ψ surface is $2\pi\psi$. The poloidal magnetic flux outside a constant χ surface is $2\pi\chi$. Of course, the constant ψ and constant χ surfaces need not have the same shape.

The quantities ψ, θ, ϕ, χ will be the canonical coordinates of the Hamiltonian. The canonical transformations of classical mechanics are the transformations which preserve the canonical form for $\vec{B}$.

III. TRANSFORMATION EQUATIONS

The mapping between the canonical coordinates ψ, θ, ϕ, χ and the ordinary position vector $\vec{x}$ plays a central role in the theory. Actually the position vector should be considered to have four components

$$\vec{X} = \vec{x} + H\hat{h} \quad . \tag{6}$$

The fourth component is the Hamiltonian H. The unit vector $\hat{h}$ is orthogonal to the three basis vectors of ordinary space and physical space corresponds to H = 0. The Jacobian of the mapping can be chosen to be unity

$$\frac{\partial \vec{x}, H}{\partial \psi, \theta, \phi, \chi} = 1 \quad . \tag{7}$$

If ψ and θ are any two distinct canonical coordinates, then the theory of partial differentiation implies

$$\frac{\partial \psi}{\partial \vec{x}} \cdot \frac{\partial \vec{x}}{\partial \psi} = 1 \text{ and } \frac{\partial \psi}{\partial \vec{x}} \cdot \frac{\partial \vec{x}}{\partial \theta} = 0 \quad . \tag{8}$$

These relations and the unit Jacobian imply the useful, so-called dual, relations

$$\frac{\partial \vec{x}}{\partial \psi} = (\vec{\nabla}\theta \times \vec{\nabla}\phi)\left(\frac{\partial \chi}{\partial H}\right)_{\vec{x}} + (\vec{\nabla}\chi \times \vec{\nabla}\theta)\left(\frac{\partial \phi}{\partial H}\right)_{\vec{x}} + (\vec{\nabla}\phi \times \vec{\nabla}\chi)\left(\frac{\partial \theta}{\partial H}\right)_{\vec{x}}$$

$$- \hat{h}\, \vec{\nabla}\theta \cdot (\vec{\nabla}\phi \times \vec{\nabla}\chi) \tag{9}$$

and

$$\frac{\partial \psi}{\partial \vec{x}} = \left(\frac{\partial \vec{x}}{\partial \theta} \times \frac{\partial \vec{x}}{\partial \phi}\right)\frac{\partial H}{\partial \chi} + \left(\frac{\partial \vec{x}}{\partial \chi} \times \frac{\partial \vec{x}}{\partial \theta}\right)\frac{\partial H}{\partial \phi} + \left(\frac{\partial \vec{x}}{\partial \phi} \times \frac{\partial \vec{x}}{\partial \chi}\right)\frac{\partial H}{\partial \theta} - \hat{h}\,\frac{\partial \vec{x}}{\partial \theta} \cdot \left(\frac{\partial \vec{x}}{\partial \phi} \times \frac{\partial \vec{x}}{\partial \chi}\right) \quad . \tag{10}$$

The gradients, like $\vec{\nabla}\psi$, are ordinary three-dimensional gradients. Even permutations of the ψ, θ, ϕ, χ labels also give valid relations.

IV. FIELD LINE EQUATIONS

The field lines, which are the integral curves of the field, are defined by

$$\frac{d\vec{x}}{d\tau} \equiv \vec{B} \tag{11}$$

with the $\hat{h}$ component of $\vec{B}$ zero. Using

$$\frac{d\psi}{d\tau} = \frac{\partial\psi}{\partial\vec{x}} \cdot \frac{d\vec{x}}{d\tau} \quad , \tag{12}$$

the canonical form for $\vec{B}$, Eq. (1), and the $\hat{h}$ component of $\partial\vec{x}/\partial\theta$ [see Eq. (9)], one finds

$$\frac{d\psi}{d\tau} = \frac{\partial H}{\partial\theta} \quad . \tag{13}$$

Similar algebric manipulations give the remainder of Hamilton's equations

$$\frac{d\theta}{d\tau} = -\frac{\partial H}{\partial\psi} \quad , \quad \frac{d\phi}{d\tau} = \frac{\partial H}{\partial\chi} \quad , \quad \text{and } \frac{d\chi}{d\tau} = -\frac{\partial H}{\partial\psi} \tag{14}$$

proving that H is indeed the field line Hamiltonian.

V. TIME DEPENDENCE

In addition to the canonical time τ, which is related to the distance along the field lines, there is ordinary time t, which will be called just time. Time is a parameter in the Hamiltonian theory for there is no quantity canonically conjugate to it.

Any divergence-free field obeys a Faraday's law equation

$$\frac{\partial\vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} \quad \text{or} \quad \vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\Phi \tag{15}$$

with $\vec{\nabla}\Phi$ the single-valued electric potential. For either the magnetic field or the vorticity there is a relation, Ohm's law, which is usually taken to have the form

$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{\nabla} \times \vec{B} \tag{16}$$

with $\vec{u}$ the fluid velocity and η the resistivity or viscosity. The parallel component to $\vec{B}$ of Ohm's law determines the topological evolution of the field. The perpendicular components are related only to the velocity difference between the fluid and the canonical coordinates.

There are three mathematically equivalent ways of describing the time evolution of the field. The vector potential can be specified as a function of position and time, $\vec{A}(\vec{X},t)$. The position can be given as a function of the canonical coordinates and time, $\vec{X}(\psi,\theta,\phi,\chi,t)$. The canonical coordinates may be known as functions of position and time. The relations among these three forms have considerable physical importance. These three relations are

$$\left(\frac{\partial\vec{A}}{\partial t}\right)_{\vec{X}} = \left(\frac{\partial H}{\partial t}\right)_c \vec{\beta} + \vec{v} \times \vec{B} + \vec{\nabla}s \tag{17}$$

$$\left(\frac{\partial\vec{X}}{\partial t}\right)_c = -\left[\frac{\partial\vec{X}}{\partial\psi}\left(\frac{\partial\psi}{\partial t}\right)_{\vec{X}} + \frac{\partial\vec{X}}{\partial\theta}\left(\frac{\partial\theta}{\partial t}\right)_{\vec{X}} + \frac{\partial\vec{X}}{\partial\phi}\left(\frac{\partial\phi}{\partial t}\right)_{\vec{X}} + \frac{\partial\vec{X}}{\partial\chi}\left(\frac{\partial\chi}{\partial t}\right)_{\vec{X}}\right] \tag{18}$$

$$\left(\frac{\partial\vec{A}}{\partial t}\right)_{\vec{X}} = \left(\frac{\partial\psi}{\partial t}\right)_{\vec{X}} \frac{\partial\theta}{\partial\vec{X}} - \left(\frac{\partial\theta}{\partial t}\right)_{\vec{X}} \frac{\partial\psi}{\partial\vec{X}} + \left(\frac{\partial\phi}{\partial t}\right)_{\vec{X}} \frac{\partial\chi}{\partial\vec{X}} - \left(\frac{\partial\chi}{\partial t}\right)_{\vec{X}} \frac{\partial\phi}{\partial\vec{X}} + \frac{\partial s}{\partial\vec{X}} \quad . \tag{19}$$

The vector $\vec{\beta}$ is

$$\vec{\beta} = \left(\frac{\partial\theta}{\partial H}\right)_{\vec{x}} \vec{\nabla}\psi - \left(\frac{\partial\psi}{\partial H}\right)_{\vec{x}} \vec{\nabla}\theta + \left(\frac{\partial\chi}{\partial H}\right)_{\vec{x}} \vec{\nabla}\phi - \left(\frac{\partial\phi}{\partial H}\right)_{\vec{x}} \vec{\nabla}\chi \tag{20}$$

and has the important property of $\vec{\beta} \cdot \vec{B} = 1$. The scalar s, which is a canonical generating function, is

$$s = \left(\frac{\partial G}{\partial t}\right)_{\vec{c}} - \vec{v} \cdot \vec{A} - \left(\frac{\partial H}{\partial t}\right)_c \left[\psi\left(\frac{\partial \theta}{\partial H}\right)_{\vec{x}} - \chi\left(\frac{\partial \phi}{\partial H}\right)_{\vec{x}} - \left(\frac{\partial G}{\partial H}\right)_{\vec{x}}\right] \quad . \tag{21}$$

The velocity $\vec{v}$ of the canonical coordinates is defined by

$$\left(\frac{\partial \vec{X}}{\partial t}\right)_c = \vec{v} + \hat{h}\left(\frac{\partial H}{\partial t}\right)_c \quad . \tag{22}$$

The subscript "c" means the canonical coordinates are held constant. The derivation of these relations requires the use of Eq. (5) for $\vec{A}$, the dual relations, Eqs. (9) and (10), and the identity

$$\left(\frac{\partial f}{\partial t}\right)_c = \left(\frac{\partial f}{\partial t}\right)_{\vec{x}} + \left(\frac{\partial \vec{X}}{\partial t}\right)_c \cdot \frac{\partial f}{\partial \vec{X}} \quad . \tag{23}$$

In addition, the behavior of s off the physical plane H = 0 must be specified by

$$\vec{B} \cdot \vec{v} = (\partial s/\partial H)_{\vec{x}} \quad . \tag{24}$$

Let us consider the application of these relations to the time development of the field or equivalently to canonical perturbation theory. A noncanonical treatment of magnetic perturbation theory has been given by Cary and Littlejohn.[6] Assume $\vec{A}(\vec{x},t)$ is given as well as the initial position $\vec{X}(\psi,\theta,\phi,\chi)$. The problem is to find $(\partial \vec{X}/\partial t)_c$. The freedom of gauge G or generating function $s(\psi,\theta,\phi,\chi)$ implies that the answer is not unique, but all possible answers are canonically equivalent. The only equation relevant to the topological evolution is the component of Eq. (17) parallel to $\vec{B}$,

$$\vec{B} \cdot \left(\frac{\partial \vec{A}}{\partial t}\right)_{\vec{x}} = \left(\frac{\partial H}{\partial t}\right)_c + \vec{B} \cdot \vec{\nabla} s \quad . \tag{25}$$

If a function s exists such that this equation has a solution with $(\partial H/\partial t)_c$ zero, then the topology of the field lines clearly does not change. In general, this is not possible since the field lines may change topology. One can then choose s so that $(\partial H/\partial t)_c$ is minimized consistent with s and $(\partial H/\partial t)_c$ being conveniently smooth functions of ψ, θ, ϕ, χ.

Once s is chosen, the time dependence of the canonical coordinates is given by Eq. (19). Using the orthogonality relations, Eq. (8), one finds for example

$$\left(\frac{\partial\psi}{\partial t}\right)_{\vec{x}} = \frac{\partial\vec{x}}{\partial\theta} \cdot \left(\frac{\partial\vec{A}}{\partial t}\right)_{\vec{x}} - \frac{\partial s}{\partial\theta} \quad . \tag{26}$$

This equation and the analogous ones for the other canonical coordinates are the well-known infinitesimal canonical transformation equations when the field is not changing, $(\partial\vec{A}/\partial t)_{\vec{x}} = 0$. Finally, Eq. (18) can be used to obtain $(\partial\vec{x}/\partial t)_c$, which was the desired result.

Actually, Faraday's law and Ohm's law determine only the Hamiltonian $H(\psi,\theta,\phi,\chi,t)$, not the three-dimensional position vector $\vec{x}(\psi,\theta,\phi,\chi,t)$. Using Faraday's law, Eq. (15), Ohm's law, Eq. (16), and Eq. (17), one finds

$$\left(\frac{\partial H}{\partial t}\right)_c \vec{\beta} + (\vec{v} - \vec{u}) \times \vec{B} + \vec{\nabla}(s + \Phi) = -\eta\vec{\nabla} \times \vec{B} \quad . \tag{27}$$

The components of this equation perpendicular to $\vec{B}$ serve only to set $\vec{v} - \vec{u}$, the difference between the canonical coordinate and the fluid velocity. To determine $\vec{v}$ or $\vec{u}$ an additional equation is required, which is generally force balance. The three position vector $\vec{x}(\psi,\theta,\phi,\chi,t)$ evolves arbitrarily rapidly to maintain force balance, including inertial forces. The Hamiltonian and the topology, on the other hand, evolve at a rate determined by the component of

Ohm's law parallel to the field. The separation of the two time scales, one dissipative and the other inertial is an important feature of the method.

The equations, which have been derived, have physical meaning only in the $H = 0$ plane. They can be considerably simplified if a canonical coordinate, say ϕ, satisfies $|\vec{B}\cdot\vec{\nabla}\phi| > 0$. Then, ψ,θ,ϕ,H can be used as the independent variables. The function $\chi(\psi,\theta,\phi,H)$, with $H = 0$, becomes a one degree of freedom, time dependent, Hamiltonian with ϕ the canonical time.[4]

A field $\vec{E}$, which is not divergence-free, can be fitted into the Hamiltonian formulation of this paper if and only if a finite solution λ exists to the equation

$$\vec{E} \cdot \vec{\nabla}\lambda = \vec{\nabla} \cdot \vec{E} \quad . \tag{28}$$

If λ exists, $\vec{E}\,\exp(-\lambda)$ is a divergence-free vector and a trivial modification of the Jacobian, Eq. (7), will absorb the $\exp(-\lambda)$ factor.

ACKNOWLEDGMENTS

This work was supported by the U.S. DoE Contract No. DE-AC02-76-CHO-3073.

Part of this work was carried out while the author was visiting the Culham Laboratory. The author wishes to acknowledge both the hospitality and the stimulating discussions which occurred during that visit.

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