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FORCED MAGNETIC RECONNECTION

By

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ABSTRACT

By studying a simple model problem, we examine the time evolution of magnetic field islands which are induced by perturbing the boundary surrounding an incompressible plasma with a resonant surface inside. We find that for sufficiently small boundary perturbations, the reconnection and island formation process occurs on the tearing mode time scale defined by Furth, Killeen, and Rosenbluth. For larger perturbations the time scal^ is that defined by Rutherford. The resulting asymptotic equilibrium is such that surface currents in the resonant region vanish. A detailed analytical picture of this reconnection process is presented.

I. INTRODUCTION

It is well known that a toroidal magnetostatic axisymmetric equilibrium can be specified by giving its boundary, the distribution of mass, and the distribution of rotational transform, assuming the general topology of the magnetic surfaces are nested tori.[1] But in reality, it is probably impossible to set up perfect axisymmetric surfaces. Moreover, when a boundary is perturbed in such a way as to resonate with the closed lines on one of the rational surfaces, the equilibrium with the original topology (nested tori) should have surface currents on this resonant surface.[2] On the other hand, there also exists another class of solutions which have no surface currents in this situation, but the magnetic surfaces do not form nested tori. Instead they have magnetic islands on the resonant surface of the original equilibrium.

To determine which one is the correct solution, it is convenient to do a thought experiment in which the boundary is changed in time from axisymmetry, in a manner as Grad[2] has supposed, and to follow the time evolution of the plasma.

In this paper, we treat a model system suggested by Taylor[3] in which we have a simple slab equilibrium which is changed by perturbing the boundaries a small amount such as to produce island formation at the center of the slab, i.e. the resonant surface (see Fig. 1). The simplicity of this problem is such that we can achieve analytical results.

We first consider a change in the boundary, rapid compared to any resistive time scale, and follow the subsequent evolution of the plasma. There are two natural equilibria consistent with the boundary conditions. The first equilibrium (I) has no other change in topology than a surface current on the resonant surface. The second equilibrium (II) has a different topology possessing islands at the resonant surface, but no surface current. Equilibria intermediate between these two are also possible consistent with the boundary condition.

After the initial change in boundary occurs, the plasma at first moves ideally approaching equilibrium (I) and builds up a concentration of current near the resonant surface. After a while the finite resistivity of the plasma plays a role, reconnection of flux across the resonant surface occurs, and islands start to form. As the reconnection proceeds, the surface current decreases and the plasma tends towards equilibrium (II).

The entire process occurs in four time stages, i.e., phases A, B, C, and D, respectively. We have presented the results of phases A and D in our earlier paper.[4] In this paper, we present all the reconnection phases as well as the nonlinear theory.

Let τ_R and τ_A, defined in Sec. II., denote the resistive diffusion and hydromagnetic time scales. Phase A is the ideal one. It merges in phase B when the first reconnection occurs, but the dynamics of the plasma motion are unaffected by the resistivity. After a time of order $\tau_R^{1/3}\ \tau_A^{2/3}$, the reconnection affects the dynamics and a fully resistive theory must be employed. Finally, after a time long compared to $\tau_R^{1/3}\tau_A^{2/3}$, sufficient reconnection occurs and the simplification of "constant-ψ" can be made, where ψ is the perturbed flux function. This is phase D. During phase D, equilibrium (II) is approached. The evolution of the amount of reconnected flux with time is displayed in Fig. 2.

Most of the complicated dynamics of the plasma occurs near the resonant surface. The equilibrium in the outside region can be parameterized by $\psi(0)$, the amount of reconnected flux. The analytical technique we employ is a boundary layer analysis with asymptotic matching. An important observation we make is the sensitivity of the solution away from the resonant surface to the amount of reconnected flux. Equilibria (I) and (II) are noticeably different. This difference is produced strictly by behavior near the resonant surface.

The model is described in Sec. II. Phases A, B, C, and D are treated in Secs. III, IV, V, and VI, respectively. The treatment of these sections is linear in the displacement of the boundary δ. If $\delta/a > (\tau_A/\tau_R)^{4/5}$, where 2a is the slab thickness, still a quite small δ/a limit, nonlinear theory must be employed. This is carried out in Sec. VII. In Sec. VIII, the case of a slowly changing boundary distortion is considered. Conclusions are drawn in Sec. IX.

II. TAYLOR'S PROBLEM

As shown in Fig. 1, we consider an incompressible plasma which is in equilibrium with a magnetic field of uniform gradient in the x direction and has surrounding perfect conducting walls at $x = \pm a$. The equilibrium magnetic field is given by

$$\vec{B} = B_T \hat{z} + B_y' x \hat{y} , \qquad (1)$$

where $B_y' = B_o/a$, B_T, B_o, and a are constants. After we perturb the boundaries of this equilibrium

$$x = \pm(a - \delta \cos ky) , \qquad (2)$$

the new equilibrium to the first order in δ can be found by taking the curl of the force balance equation $\vec{j} \times \vec{B} = \nabla p$ to get

$$(\vec{B} \cdot \nabla \vec{j})^{(1)} = \vec{B}_0 \cdot \nabla \vec{j}_1 + \vec{B}_1 \cdot \nabla \vec{j}_0 = 0 \quad , \tag{3}$$

where the subscript 0 denotes equilibrium quantities and 1 corresponds to the first order quantities in δ. By symmetry, $\vec{j}_0$, $\vec{j}_1$ are in z-direction and $\vec{B}_1$ is in xy-plane. All quantities are independent of z and the system possesses mirror symmetry about x = 0. Introducing the flux function ψ by

$$\vec{B} = B_T \hat{z} + \hat{z} \times \nabla\psi \ , \tag{4}$$

we have $\psi_0 = B_0/2a \cdot x^2$ and $\psi_1 = \psi_1(x) \cos ky$. Equation (3) can be written as

$$\frac{B_0}{a} \ x \ (\frac{\partial^2}{\partial x^2} \psi_1 - k^2\psi_1) = 0 \ , \tag{5}$$

and the boundary condition [ψ is constant at the perturbed boundary (2)] reduces to

$$\psi_1(a) = B_0 \cdot \delta \ . \tag{6}$$

The general solution of Eq. (5) is

$$\psi_1(x) = A \cosh kx + B \sinh kx \ ; \ x > 0, \tag{7}$$

and $\psi_1(-x) = \psi_1(x)$ by mirror symmetry. From now on, only equations for positive x will be presented. Now there exist two possible solutions with different character. If we require the same topology as the original equilibrium, $\psi(0) = 0$ must be satisfied and we get equilibrium (I).

$$\psi_1(x) = B_o\delta \frac{\sinh kx}{\sinh ka} \tag{8}$$

then

$$B_y = \frac{\partial\psi}{\partial x} = B_o \frac{x}{a} + B_o ka \frac{\cosh kx}{\sinh ka} . \tag{9}$$

B_y has a finite jump corresponding to a surface current.

$$\psi_1(x) = B_y(+o) - B_y(-o) = \frac{2B_o k\delta}{\sinh ka} , \tag{10}$$

at $x = 0$. The second equilibrium solution (II) is

$$\psi_1(x) = B_o\delta \frac{\cosh kx}{\cosh ka} . \tag{11}$$

There is no surface current since $B_{y1} = B_o k\delta \sinh kx/\cosh ka$ is continuous at $x = 0$. However, $\psi(o)$ is finite so the topology has changed and solution (II) has magnetic islands of width $(2a\delta/\cosh ka)^{1/2}$.

Which equilibrium should occur in practice? To decide the answer to this question let us imagine that the distortion δ is set up at a rate slow enough that the situation remains close to equilibrium everywhere except near $x = 0$; that is, δ varies on a time scale long compared to $\tau_A = a/v_A$ where $v_A = B_o/(4\pi\rho)^{1/2}$. We follow the evolution of the equilibrium in time from the original one and determine which equilibrium the plasma approaches. We also assume that δ changes at a rate faster than any resistive time scale $\tau_A{}^s\ \tau_R{}^{1-s}$ where $0 < s < 1$ and $\tau_R = 4\pi a^2/\eta$, η is the plasma resistivity. The resistivity will enable the equilibrium to form islands near $x = 0$.

At all t, the plasma will be in magnetostatic equilibrium everywhere except near $x = 0$ where resistivity and inertia effects are not ignorable. The solution of Eq. (5) with boundary condition Eq. (6) is a combination of (I) and (II) that is determined by matching to the solution near $x = 0$. We express it in terms of $\psi_1(0)$, the amount of flux crossing the boundary $x = 0$ in a quarter period in y;

$$\psi_1(x) = \psi_1(0)\left(\cosh kx - \frac{\sinh kx}{\tanh ka}\right) + B_0\delta\,\frac{\sinh kx}{\sinh ka}\,. \tag{12}$$

We note that equilibrium (I) [Eq. (8)] corresponds to $\psi_1(0)$, and equilibrium (II) [Eq. (11)] corresponds to $\psi_1(0) = B_0\delta/\cosh ka$. Further, the entire equilibrium outside the $x = 0$ region is determined by $\psi_1(0)$.

The same problem has been considered by Hu.[5] However, his initial conditions were different. He started his solution with a surface current already present, namely his initial situation was our equilibrium (I). We show in the appendix that after a short time of order $\tau_A^{2/3}\tau_R^{1/3}$, his surface current opens up and his solution joins our solution in phase C. His specific form of the solution breaks down and must be continued by our solution. Thus, his claim that the system passes to equilibrium (II) in a time of order $\tau_A^{2/3}\tau_R^{1/3}$ is not valid; as shown below the time is of the order of the tearing mode time $\tau_A^{2/5}\tau_R^{3/5}$. To determine the time evolution of $\psi_1(0)$, we must carry out a boundary layer analysis near $x = 0$. The results of this calculation indicate four separate time regions of variation as shown in Fig. 2. Phase A corresponds to the ideal MHD theory, B to MHD plus small resistive corrections, C to tearing mode analysis similar to that given by Furth, Killeen, and Rosenbluth[6] (referred to as FKR) for the nonconstant ψ theory, and D to the constant ψ approximation. We emphasize that the equilibrium which we are considering is stable to the tearing mode. However, a time *dependent* solution of the tearing mode type occurs with controlled amplitude. The bulk of variation happens in phase *D*. *This occurs during* the tearing mode time scale $\tau_T = \tau_A^{2/5}\tau_R^{3/5}$. Note that the reconnection in phase D surprisingly overshoots the island size, growing beyond the final value of (II) and then returning to it.

Equilibrium (I) is never reached, but its form is approached during phase A. These four phases correspond to different asymptotic time regions and it will be shown later that their behavior overlaps smoothly. Before discussing the ideal MHD solution in the next section, we write the resistive MHD equations for future use.

$$\frac{\partial}{\partial t} \vec{B} = \nabla\times(\vec{v} \times \vec{B}) + \frac{\eta}{4\pi} \nabla^2 \vec{B} \; , \tag{13}$$

$$\rho\left(\frac{\partial}{\partial t} \vec{v} + \vec{v} \cdot \nabla \vec{v}\right) = -\nabla p + \vec{j} \times \vec{B} \; . \tag{14}$$

Introducing the stream function ϕ and the vorticity w_z by

$$\vec{v} = \hat{z} \times \nabla\phi \; , \; \omega_z = \hat{z} \cdot \nabla \times \vec{v} = \nabla^2\phi \; ,$$

we may write Eqs. (13) and (14) as

$$\frac{\partial}{\partial t} \psi + \vec{v} \cdot \nabla\psi = \frac{\eta}{4\pi} \nabla^2\psi \; , \tag{15}$$

$$\rho\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right) \omega_z = \vec{B} \cdot \nabla j_z \; . \tag{16}$$

It is easy to see that the y dependence of ϕ and ψ are

$$\phi = \phi(x)\ \sin ky\ ,\ \psi = \psi(x)\ \sin ky, \tag{17}$$

III. IDEAL MHD SOLUTION (A)

We now consider the ideal MHD solution during phase (A) near $x = 0$. Neglecting resistivity, linearizing Eqs. (15) and (16) and assuming $\partial/\partial x \gg k$, we have

$$\frac{\partial}{\partial t}\psi_1(x) - k\ B_o \frac{x}{a}\phi(x) = 0\ , \tag{18}$$

$$\rho \frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}\phi(x) = -\frac{kB_o}{4\pi}\frac{x}{a}\frac{\partial^2}{\partial x^2}\psi_1(x)\ . \tag{19}$$

Eliminating $\phi(x)$ from Eq. (19) by means of Eq. (18) and introducing $\xi = \psi_1/x$, where ξ is proportional to the x component of fluid displacement, we obtain

$$\tau_A^{\ 2}\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial x^2}\xi = -k^2 x \frac{\partial^2}{\partial x^2} x\xi\ . \tag{20}$$

The solution of Eq. (20) must match smoothly to the exterior solution Eq. (8), since $\psi_1(0) = 0$ in ideal MHD theory. Noticing that Eq. (20) is invariant under the transformation

$$x \to \alpha x$$
$$t \to \alpha^{-1} t\ ,$$

we can find a similarity solution $\xi(x, t) = \xi(xt)$ of Eq. (20),

$$\xi = \frac{2}{\pi}\cdot\frac{B_o k\delta}{\sinh ka}\int_o^{\frac{kxt}{\tau_A}} du\,\frac{\sin u}{u}\ ;\ x \ll a\ , \tag{21}$$

where the constant has been adjusted to match Eq. (8). The variations of ξ

given by Eq. (22) is confined to the region $kx \sim \tau_A/t$ which decreases with t. On the other hand, the current

$$4\pi\, j_{z1} = \frac{\partial^2}{\partial x^2}\psi_1 = 2\frac{\partial\xi}{\partial x} + x\frac{\partial^2}{\partial x^2}\xi \tag{22a}$$

has the value

$$4\pi\, j_{z1}(o) = \frac{2}{\pi}\frac{k\delta B_o}{\sinh ka}\cdot\frac{t}{\tau_A} \tag{22b}$$

which increases with t. However, the current is confined to the finite region with width proportional to t^{-1}. Thus, a finite surface current of zero thickness is slowly approached, and thus it takes an infinite time to reach equilibrium (I). On the other hand, if one chooses equilibrium (I) as an initial condition, it is not surprising that some reconnection occurs during an Alfvén time scale as stated by Hu.[5] This is because there is infinite magnetic field gradient which can relax at an explosive rate.

IV. TRANSITION FROM IDEAL MHD TO RESISTIVE EVOLUTION (B)

Ideal MHD analysis in Sec. III shows that the perturbed current gets larger and sharper in time (the height grows as t, and the thickness decreases as t^{-1}). Therefore, no matter how small η is, there will be a time when the $\eta/4\pi\, \partial^2/\partial x^2\, \psi_1(x)$ term in Eq. (15) becomes comparable to other terms. In this section, we investigate the effects of that resistive term on the evolution of $\psi_1(x)$. Linearizing Eq. (15), we have

$$\frac{\partial}{\partial t}\psi_1(x) - \frac{kB_o}{a}\cdot x\phi(x) = \frac{\eta}{4\pi}\frac{\partial^2}{\partial x^2}\psi_1(x)\ , \tag{23}$$

which advances $\psi_1(x)$ forward in time. Imagine that the right-hand side of Eq. (23) becomes non-negligible at some time. Then by substituting the ideal MHD solution Eq. (21) into the right-hand side of Eq. (23), we obtain

$$\frac{\partial}{\partial t}\,\psi_1(o) = \frac{\eta}{4\pi}\,\frac{\partial}{\partial x}\,\xi(o) = \frac{\eta}{\pi^2}\,\frac{B_o\delta k^2}{\sinh\,ka}\;\frac{t}{\tau_A}\;. \tag{24}$$

This can be integrated to give

$$\psi_1(o) = \frac{2}{\pi}\,\frac{B_o\delta k^2 a^2}{\sinh\,ka}\cdot\frac{t^2}{\tau_A\tau_R}\;. \tag{25}$$

Now $\psi_1(o)$ grows algebraically in time due to the resistive effect. This shows that the magnetic field lines begin to reconnect. Although we have neglected the resistive effects on $\phi(x)$ in this section, it will be shown in the next section that Eq. (25) actually describes the small t behavior of $\psi_1(o)$ correctly.

V. RESISTIVE EVOLUTION (NONCONSTANT ψ PHASE) (C)

As time increases, the magnetic field continues to reconnect, the field pattern deviates significantly from the ideal solution, and the velocity pattern is altered. We pass from phase E to phase C in which the full resistive plasma dynamics must be taken into account. After a finite amount of reconnection, it will be possible to simplify this analysis by the assumption of constant ψ in the boundary layer (phase D). In this section, we discuss the intermediate phase C, when the assumption of constant ψ is not valid.

In this intermediate phase, it turns out that the small ka theory of FKR (Appendix D of FKR) is applicable and makes it possible to carry out a smooth transition from phase B to D. First, we introduce the FKR notation to make comparison betwen our theory and theirs easier. Next we introduce Laplace transform Eqs. (15) and (16) in time. Let

$$\tilde{\psi}_1 = \int_0^\infty dt\, e^{-st}\, \psi_1(t)$$

$$\tilde{\phi} = \int_0^\infty dt\, e^{-st}\, \phi(t) \;.$$

Then we have, near $x = 0$ ($\partial/\partial x >> k$),

$$\frac{\partial^2}{\partial\theta^2} \Psi = \varepsilon\Omega\, (4\Psi + \theta U) \tag{26}$$

$$\frac{\partial^2}{\partial\theta^2} U - \frac{1}{4}\, \theta^2 U = \theta\Psi \tag{27}$$

where

$$\theta = \frac{x}{\varepsilon a} \;, \; \Psi = \frac{k}{B_0} \tilde{\psi}_1 \;, \; U = - \frac{\tilde{\phi}}{\nu} \;, \tag{27a}$$

and

$$\varepsilon^4 = \frac{s\tau_A{}^2}{4(ka)^2\tau_R} \;, \; \nu = \frac{sa}{4\varepsilon k^2}, \; \Omega = \frac{\varepsilon\tau_R s}{4} \;. \tag{27b}$$

Equations (26) and (27) can be combined into

$$z''' = \left(\mu + \theta_1{}^2\right) z' + 4\theta_1 z \;, \tag{28}$$

where

$$z \equiv \frac{\partial^2}{\partial\theta^2} \Psi = \varepsilon\Omega(4\Psi + \theta U)$$

$$\theta_1 \equiv \frac{\theta}{(2)^{1/2}}, \ \mu = \beta \ \varepsilon\Omega = \frac{(s\tau_R^{1/3}\tau_A^{2/3})^{3/2}}{ka}, \tag{28a}$$

and prime denotes the derivative with respect to θ_1.

We further normalize Z by $Z = \mu/2\ \Psi(o) \cdot \bar{Z}$, then $\bar{Z}$ is subject to the boundary conditions

$$\bar{Z}(o) = 1, \ \bar{Z}'(o) = 0, \ \bar{Z}(\infty) = 0. \tag{28b}$$

We must match the inside solution for $d\Psi/dx$ as $x \to \infty$,

$$\lim_{x \to \infty} \frac{d\Psi}{dx} = \int_o^\infty dx \frac{d^2}{dx^2} \Psi = \frac{\mu}{2^{1/2}\varepsilon a} \Psi(o) \int_o^\infty d\theta_1 \cdot \bar{Z}$$

to the exterior solution as $x \to 0$.

$$\frac{d}{dx} \Psi(o) = \frac{k^2\delta}{s \cdot \sinh(ka)} - \frac{k\ \Psi(o)}{\tanh(ka)},$$

which is the Laplace transform of Eq. (12) at x = 0. We thus obtain

$$\Psi(o) = \frac{k^2\delta}{\sinh(ka)} \frac{1}{\{[k/\tanh(kd)] + [\mu/(2^{1/2}\varepsilon a)I(\mu)]\}s}, \tag{29}$$

where

$$I(\mu) = \int_o^\infty d\theta_1 \cdot \bar{Z} \tag{29a}$$

Equation (29) gives the Laplace transform of the amount of reconnected flux $\Psi(o)$. [To evaluate it we must solve Eq. (28) with boundary conditions

Eq. (28b) and then evaluate $I(\mu)$ of Eq. (29a).] First, let $t \to 0$, $\mu \to \infty$. For $\mu >> 1$, Eq. (28) has the asymptotic solution

$$\bar{Z} = \frac{1}{(1 + \theta_1{}^2/\mu)^2} \tag{30}$$

which satisfies conditions (28b). (Z''' is clearly negligible.) Evaluation of $I(\mu)$ yields

$$I(\mu) = \frac{\pi}{2}\,\mu^{1/2} \quad . \tag{30a}$$

Substitution of this in Eq. (29) yields

$$\Psi(o) = \frac{\pi}{4}\,\frac{k^3a^2\delta}{\sinh\,(ka)} \cdot \frac{1}{\tau_R\tau_A s^3} \; . \tag{31}$$

Inverse Laplace transforming Eq. (31), we get

$$\Psi_1(o) \approx \frac{2}{\pi}\,\frac{B_o\delta k^2a^2}{\sinh\,ka}\,\frac{t^2}{\tau_A\tau_R} \; , \tag{32}$$

which is exactly the same as Eq. (25) in phase B. In fact, phase B is actually a subset of phase C for small t. (This asymptotic limit applies when $\mu >> 1$ or from Eq. (28a) $t << \tau_R^{1/3}\tau_A^{2/3}$).

On the other hand, when $t \to \infty$, $\mu \to 0$, Eq. (28) reduces to

$$Z''' = \theta_1{}^2\,Z' + 4\theta_1 Z \; , \tag{33}$$

which is valid when $t >> \tau_R{}^{1/3}\tau_A{}^{2/3}$. Equation (33) is actually equivalent to the constant ψ approximation of phase D treated in the next section. This is clear from Eqs. (26) and (27). If one takes Ψ as a constant in Eq. (27), expresses U in terms of Z by $U = Z/\varepsilon\Omega - 4\Psi$ where Ψ is again a constant, substitutes the result in Eq. (27), and differentiates once, one gets Eq. (33). Thus phase D is also a subset of phase C, but the solution of Eq. (33) is more difficult and we consider its solution in the next section.

It is interesting to note that phase A is also actually a subset of phase C. If one Laplace transforms $d^2\psi/dx^2$ making use of Eqs. (21) and (22), we find it agrees with Eq. (30) if the scale factor $\Psi(0)$ is replaced with Eq. (31). Thus, in point of fact the ideal solution for ξ and $\partial^2/\partial x^2\ \psi$ persists even during the reconnection phase B. Therefore, one could consider these phases to overlap. There is no true ideal phase as reconnection commences immediately according to Eq. (32), although at first it is so slow that it does not change the ideal dynamics.

VI. CONSTANT Ψ APPROXIMATION (D)

After reconnection has been developed sufficiently so that $\varepsilon\,\Delta' \ll 1$, we can apply the constant ψ approximation. As indicated in the last section, this approximation is valid if $t \gg \tau_A{}^{2/3}\tau_R{}^{1/3}$. Making use of the Laplace transform of Eq. (12), we have

$$\tilde{\Psi}_1(o) = \frac{B_o\delta/s}{\cosh(ka)+(\Delta'/2k)\sinh(ka)}\ . \tag{34}$$

(We have Laplace transformed the amplitude of distortion of the boundary δ so that $\tilde{\delta} = \delta/s$ where δ denotes its final value.) Equation (34) provides the exterior region information needed to complete the boundary layer analysis. Following FKR, we obtain

$$\Delta' = \frac{1}{\varepsilon a} \int_{-\infty}^{\infty} d\theta \, \frac{\partial^2/\partial\theta^2 \, \Psi}{\Psi(o)} = \frac{12\Omega}{a} = \frac{3}{(2ka)^{1/2} \, a} \left(s\tau_R^{3/5}\tau_A^{2/5}\right)^{5/4} . \tag{35}$$

Combining Eqs. (34) and (35) we can determine $\tilde{\psi}_1(o)$.

$$\tilde{\psi}_1(o) = \frac{B_o\delta/s}{\cosh(ka) + \left[3\sinh(ka)/(2ka)^{3/2}\right] \cdot p^{5/4}} , \tag{36}$$

where $p = s \, \tau_R^{3/5}\tau_A^{2/5}$.

Inverting the Laplace transform we find

$$\psi_1(o, t) = \frac{1}{2\pi i} \int_C ds \, \tilde{\psi}_1(o) \, e^{st}$$

$$= \frac{B_o\delta}{\cosh ka} \cdot \frac{1}{2\pi i} \int_C dp \, \frac{e^{p\tau}}{p\left(1 + \lambda \cdot p^{5/4}\right)} , \tag{37}$$

where

$$\tau = t/\tau_R^{3/5}\tau_A^{2/5} , \tag{37a}$$

$$\lambda = \frac{3}{2^{3/2}} \frac{\tanh(ka)}{(ka)^{3/2}} . \tag{37b}$$

and the contour C in Eq. (37) is the Bromwich contour for inversion of the Laplace transform. C can be di torted as in Fig. 3. Because of the fractional exponent 5/4 appearing in Eq. (37), there is a branch cut along the negative real axis and the integrand has poles at $p = 0$ and $p = \lambda^{-4/5} \exp(\pm 4\pi i/5)$ (A and B in Fig. 3). Thus, $\psi_1(o, t)$ can be reduced to a real form

$$\psi_1(o,\ t) = \frac{B_o \cdot \delta}{\cosh\ ka}\left[1 - 4/5\left(e^{P_A\tau} + e^{P_B\tau}\right)\right.$$

$$\left. + \frac{\lambda}{(2)^{1/2}\pi}\int_o^\infty du\, \frac{u^{1/4}\ e^{-\tau u}}{1 - (2)^{1/2}\lambda u^{5/4} + \lambda^2 u^{5/2}}\right] , \qquad (37c)$$

and evaluated numerically. This gives the curve of Fig. 2. The asymptotic results for $\psi_1(o)$ are

$$\psi_1(o,\ t) \sim \frac{B_o\delta}{\cosh\ ka}\ \frac{8(2)^{1/2}}{5\pi}\ \Gamma\left(\frac{3}{4}\right)\frac{\tau^{5/4}}{\lambda} , \qquad \tau << 1$$

$$\psi_1(o,\ t) \sim \frac{B_o\delta}{\cosh\ ka}\left[1 + \frac{\lambda}{2^{1/2}\pi}\ \Gamma\left(\frac{5}{4}\right)\ \tau^{-5/4}\right] , \qquad \tau >> 1 . \qquad (38)$$

Note from Fig. 2 that the amount of reconnected flux $\psi_1(o,\ t)$ overshoots that of solution (II) and then approaches it from above. This is also evident from Eq. (38). The origin of this overshooting is in the amount of motion stirred up by our initial conditions. Phase D concludes the evolution of the reconnected flux.

To summarize, if $t << \tau_A^{2/3}\tau_R^{1/3}$, flux reconnects at the rate of Eq. (25) or Eq. (32). This is phase B or early phase C. During this time the fluid velocities are given by the ideal theory of phase A. For $t \sim \tau_A^{2/3}\tau_R^{1/3}$, the full theory of phase C must be employed. We are not able to give an explicit solution for $\psi_1(o,\ t)$ in this phase. For $t >> \tau_A^{2/3}\tau_R^{1/3}$, the constant ψ approximation of phase D, or late phase C, applies and $\psi_1(o,t)$ given by Eq. (37) or Eq. (37c) applies. This result simplifies to Eq. (38a) when $t << \tau_A^{2/5}\tau_R^{3/5}$ and to Eq. (38b) when $t >> \tau_A^{2/5}\tau_R^{3/5}$. For $t \sim \tau_A^{2/5}\tau_R^{3/5}$ the more complicated formula Eq. (37) must be used and the results are given in Fig. 2.

An important point is that a time long compared to the tearing mode time scale $\tau_A^{2/5}\tau_R^{3/5}$ is required to reach equilibrium (II).

VII. NONLINEAR THEORY

When the magnetic island grows to a size exceeding the resistive layer width and sizable nonlinear eddy currents arise which produce third order $\vec{j} \times \vec{B}$ forces opposing the fluid flow pattern, the linear tearing mode analysis breaks down. This effect should be considered when

$$\frac{\delta}{a} \gtrsim \left(\frac{\tau_A}{\tau_R}\right)^{4/5} , \tag{38}$$

which is equivalent to the island width being greater than resistive layer width. Our previous ideal MHD solutions (A) are still valid even when Eq. (38) is satisfied as long as $\delta/a \ll 1$, but a nonlinear treatment is necessary for the resistive evolution.

Following Rutherford's approach,[7] we can derive the nonlinear time evolution equation for $\psi_1(o)$. Including the nonlinear eddy current we have

$$\frac{\partial}{\partial t} \psi_1 - \left(\frac{\partial \phi}{\partial y}\right)_\psi B_y' x = \eta j_z - \eta j_{zo} , \tag{39}$$

where ϕ is the stream function.

Other nonlinear terms do not give secular effects on ψ_1 due to periodicity in y. Neglecting inertia in Eq. (16), we have

$$j_z = j_z(\psi) . \tag{40}$$

We can annihilate the information about ϕ in Eq. (39) by averaging over y along constant ψ_o surfaces. By doing this, we have

$$j_z(\psi) = j_{zo} + \eta^{-1} \frac{\langle \partial/\partial t \psi_1 / (\psi - \psi_1)^{1/2} \rangle_y}{\langle (\psi - \psi\ )^{-1/2} \rangle_y} , \tag{41}$$

where $\langle\ldots\rangle_y$ means the average over one period in y and $\psi = \psi_o(x) + \psi_1(t)\cos ky$. The constant ψ approximation is used here. It can be shown that the lowest harmonic solution in k_y dominates (the one corresponding to our periodic perturbation at the boundary). Performing the standard matching procedure of Ref. 7, we have

$$\Delta'\psi_1^{1/2} = \frac{16\pi A}{\eta(2B_y')^{1/2}}\frac{\partial}{\partial t}\psi_1 , \tag{42}$$

$$A \equiv \int_{-1}^{\infty} dw \frac{\langle \cos ky/(w-\cos ky)^{1/2}\rangle_y^2}{\langle (w-\cos ky)^{-1/2}\rangle_y} = 0.7 .$$

Solving Δ' from Eq. (34), and restoring dimensions to the variables, we have

$$\frac{\partial z}{\partial T} = \frac{1}{z^2} - 1 , \tag{43}$$

where

$$T = t/\tau_{NL},\ \tau_{NL} = \frac{2^{3/2}A}{ka}\frac{\sinh ka}{\cosh^{3/2}ka}\left(\frac{\delta}{a}\right)^{1/2}\tau_R ,$$

$$z = \left[\frac{\psi_1(o)}{B_o\delta/\cosh ka}\right]^{1/2} ,$$

so z is proportional to the island width. Equation (43) shows that z increases in time and reaches 1, which corresponds to equilibrium (II), on a time scale $\tau_{NL} \sim (\delta/a)^{/12}\ \tau_R$. Therefore the overshooting of the island width disappears if $(\delta/a) > (\tau_A/\tau_R)^{4/5}$ or $\tau_{NL} > \tau_A^{2/5}\tau_R^{3/5}$. Integrating Eq. (43),

$$T = -z + \tanh^{-1} z . \tag{44}$$

Asymptotic limits are

$$\psi_1(o,\ t) \sim \frac{B_o\delta}{\cosh(ka)}\ \left(\frac{3t}{\tau_{NL}}\right)^{2/3}; \qquad t \ll \tau_{NL},$$

$$\psi_1(o,\ t) \sim \frac{B_o\delta}{\cosh(ka)}\ \tanh^2\left(\frac{t}{\tau_{NL}}\right); \qquad t \gg \tau_{NL}. \tag{45}$$

VIII. TIME DEPENDENCE OF δ

Throughout this paper we have taken δ, the distortion of the boundary, to vary rapidly in time compared to any resistive time scale. The distortion leads to an amount of reconnected flux $\psi(o) = \psi_{II}(o) \equiv B_o\delta/\cosh(ka)$, the reconnected flux of equilibrium (II).

It is plausible that if $\delta(t)$ varies on a time scale long compared to $\tau_A^{2/5}\tau_R^{3/5}$, the tearing mode time scale, $\psi(o,\ t)$ would always be approximately equal to $\psi_{II}(o,\ t)$ corresponding to the local $\delta(t)$. It is interesting to inquire to what extent $\psi(o,\ t)$ can track $\Psi_{II}(o,\ t)$ as $\delta(t)$ varies at various rates.

To examine this question, we take

$$\delta(t) = e^{s_o t}, \tag{46}$$

from $t = -\infty$ and assume all linearized quantities are proportional to $\exp(s_o t)$. The solutions are obtained formally by taking the Laplace transforms evaluated at $s = s_o$ to be the constant coefficients multiplying $\exp(s_o t)$. From Eq. (29) we have

$$\psi(o,\ t) = \frac{\psi_{II}(o,\ t)}{1 + \left[\tanh(ka)\mu_o/2^{1/2}ka\varepsilon_o\right]\cdot I(\mu_o)}, \tag{47}$$

where μ_o and ε_o denote μ and ε with s replaced by s_o. For $s_o >> (\tau_A^{2/3}\tau_R^{1/3})^{-1}$, we obtain

$$\psi(o, t) = \frac{\psi_{II}(o, t)}{\pi \tanh(ka)/2(ka)^2 \cdot s_o^2\tau_R\tau_A} . \tag{48}$$

Note that this may be written as

$$\Psi(o, t) = \Psi_{II}(o, t - \tau_d) , \tag{49}$$

where τ_d is the delay in reaching equilibrium (II), and

$$\tau_d = s_o^{-1} \cdot [\log(s_o^2\tau_R\tau_A) + \log(\frac{\pi}{2} \frac{\tanh(ka)}{(ka)^2})] . \tag{50}$$

On the other hand, if $s_o << \tau_A^{2/3}\tau_R^{1/3}$, we obtain

$$\psi(o, t) = \frac{\psi_{II}(o, t)}{1 + [\tanh(ka)\mu_o/2^{1/2}ka\ \varepsilon_o]\cdot I(\infty)}$$

$$= \frac{\psi_{II}(o, t)}{1 + [3 \tanh(ka)/2^{3/2}(ka)^{3/2}](s_o\tau_R^{3/5}\tau_A^{2/5})^{5/4}} .$$

If we write Eq. (51) in the form Eq. (49), the time delay in this case will be

$$\tau_d = s_o^{-1} \log[1 + \frac{3}{2^{3/2}} \frac{\tanh(ka)}{(ka)^{3/2}} (s_o\tau_R^{3/5}\tau_A^{2/5})^{5/4}]. \tag{52}$$

From Eq. (51) we can see that if $s_o << (\tau_R^{3/5}\tau_A^{2/5})^{-1}$, $\psi(o, t) \approx \psi_{II}(o, t)$ as suggested above.

For the nonlinear problem, it is also possible to include a time dependent δ. Equation (43) is replaced by

$$\frac{dz}{d\Gamma} = \frac{f(t)}{z^2} - 1 , \qquad (53)$$

where $\delta(t) = \delta \cdot f(t)$ and δ denotes the final value of distortion such that $f(\infty) = 1$, $f(o) = 0$. Again one finds that $\psi(o, t) \approx \psi_{II}(o, t)$ if the time scale of evolution of δ is larger than $(\delta/a)^{1/2}\tau_R$. The larger the perturbation the more slowly must δ vary to satisfy $\psi(o, t) \approx \psi_{II}(o, t)$.

IX. CONCLUSIONS

It is now generally appreciated that in geometries that are not perfectly symmetric, the corresponding magnetostatic equilibria will not have smooth magnetic surfaces, but will be subject to break up into islands. The size of these islands will depend on the amplitude of perturbations in boundary conditions resonant with the rotational transform on the surface as well as the global equilibria. As the boundary conditions and the global equilibrium change, the size of the islands will change and affect the global equilibrium itself. The change in island size results from magnetic reconnection and the question arises, will reconnection occur as fast as required by the changing boundary conditions and equilibrium?

In order to bring out the physics of this problem, we have considered a particularly simple model problem which is tractable analytically. We have found that reconnection and changing island size occur on the tearing mode time scale for small island size and on the nonlinear time scale, $\tau_{NL} \sim (\delta/a)^{1/2}\tau_R$ for larger island size. The equilibrium always tries to form islands that remove any surface currents on the resonant surface, and this leads to important corrections to the global equilibrium.

Hu[5] had considered the same model problem but concluded that reconnection proceeds somewhat faster because he did not carry his solution to the saturated island state but only a short distance towards it. We were able to continue his solution to the saturated island state and again found reconnection on the tearing mode time scale.

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REFERENCES

1 M.D. Kruskal and R.M. Kulsrud, Phys. Fluids 1, 265 (1958).

2 H. Grad, P.N. Hu, and D.C. Stevens, Proc. Nat. Acad. Sci. USA 72, 3789 (1975).

3 J.B. Taylor, Private Communication.

4 R.M. Kulsrud and T.S. Hahm, Phys. Scripta T2;2, 525 (1982).

5 P.N. Hu, Phys. Fluids 26, 2234 (1983).

6 H. P. Furth, J. Killeen, and M.N. Rosenbluth, Phys. Fluids 6, 459 (1963).

7 P.H. Rutherford, Phys. Fluids 16, 1903 (1973).

APPENDIX: RESISTIVE EVOLUTION OF SURFACE CURRENT

In this appendix, we consider the case when equilibrium (I) is given as an initial state. Although this assumption is rather unphysical, it will elucidate the relation between our results and those of Hu.[5] Since the initial equilibrium has a surface current, the magnetic diffusion term ($\eta/4\pi$ $\nabla^2\psi$) in Eq. (15) is large from the starting moment. We expect intuitively that at first, the surface current will diffuse outward without inducing any considerable plasma motion. It takes an Alfvén time to diffuse through a resistive skin depth ($\sim \eta^{1/2}$). This can obviously be seen from dimensional arguments. This essentially summarizes the fast time scale results of Hu. The amount of reconnected flux in this time scale is $\sim \eta^{1/2}\delta$, which is negligible.

It will be shown that as t reaches ${\tau_R}^{1/3}{\tau_A}^{2/3}$, plasma motion builds up, the magnetic convection term becomes non-negligible, and the results of Hu cannot be applied any further. We note that up to this time scale, the outside ideal MHD solution is still given by equilibrium (I). Then, when ${\tau_R}^{1/3}{\tau_A}^{2/3} \lesssim t \lesssim {\tau_R}^{3/5}{\tau_A}^{2/5}$, the reconnection process goes through phase (C) in our main text and approaches phase (D) as $t \to \tau_R^{3/5}\tau_A^{2/5}$.

We now show explicitly that this is actually the case. When we Laplace transformed the resistive MHD equations in Sec. V, ψ_1 at t = 0 was set to zero corresponding to our initial condition (i.e., unperturbed equilibrium). Now, we suppose the surface current is already present at the singular surface [i.e., initial state is given by equilibrium (I)]. Then the Laplace transformed equations are slightly modified due to the finite value of $\psi_1(t = 0)$. Equation (28) should thus be replaced by

$$Z''' = \left(\mu + {\theta_1}^2\right) Z' - \mu \frac{\partial g}{\partial \theta} (t = 0) + 4\theta_1 \cdot Z \,, \tag{A.1}$$

where

$$Z \equiv \frac{\partial^2}{\partial\theta^2}\Psi + g(t = 0) ,$$

$$g(t = 0) = \frac{\varepsilon\Omega}{s}\frac{k}{B_o}\Psi_1(t = 0) ,$$

and other variables are defined in the main text. Using equilibrium (I), Eq. (8),

$$\frac{\partial}{\partial\theta} g(t = 0) = \mu^{1/2}\,\bar{\delta}\cdot \mathrm{sgn}(x) , \tag{A.2}$$

where $\bar{\delta}$ is the magnitude of perturbation at the wall appropriately normalized to be independent of s and η. Then, Eq. (4.1) becomes

$$Z''' = (\mu + \theta_1^{\,2})Z' - \mu^{3/2}\cdot\bar{\delta}\cdot \mathrm{sgn}(x) + 4\theta_1\cdot Z . \tag{A.3}$$

When $\mu >> 1$, (i.e., $t << \tau_R^{\,1/3}\tau_A^{\,2/3}$), Eq. (A.3) can be approximated by

$$Z''' = \mu Z' - \mu^{3/2}\cdot\bar{\delta}\cdot \mathrm{sgn}(x) , \tag{A.4}$$

which is equivalent to a magnetic diffusion equation with the initial surface current. Therefore, we can see that Hu's results are actually valid as long as $t << \tau_A^{\,2/3}\tau_R^{\,1/3}$. When $\mu \sim 1$, we cannot explicitly solve Eq. (A.3) due to its complexity. However, if $\mu << 1$ (i.e., $t >> \tau_A^{\,2/3}\tau_R^{\,1/3}$), Eq. (A.3) can be approximated by

$$Z''' = \theta^2_{\;1}\cdot Z^1 + 4\theta_1\cdot Z , \tag{A.5}$$

and as shown in the main text, we have the constant ψ behavior [phase (D)] *given in Sec. VI. We also conclude from* Eq. (A.3) that as $t >> \tau_R^{1/3}\tau_A^{2/3}$, the initial conditions are completely forgotten and the later part of phase (C) and the phase (D) describe the reconnection process for Hu's problem.

In summary , if we start with the surface current [equilibrium (I)], our phases (A) and (B) should be replaced by magnetic diffusion (without convection). However, as $t \gtrsim \tau_R^{1/3}\tau_A^{2/3}$, the plasma motion cannot be ignored and the reconnection process is correctly described by our results for phases (C) and (D).

FIGURE CAPTIONS

FIG. 1. The geometry of the problem.

FIG. 2. The evolution of $\psi(o, t)$ and the four time phases A, B, C and D.

FIG. 3. The Bromwich Contour for the inversion of the Laplace transform of Eq. (37).

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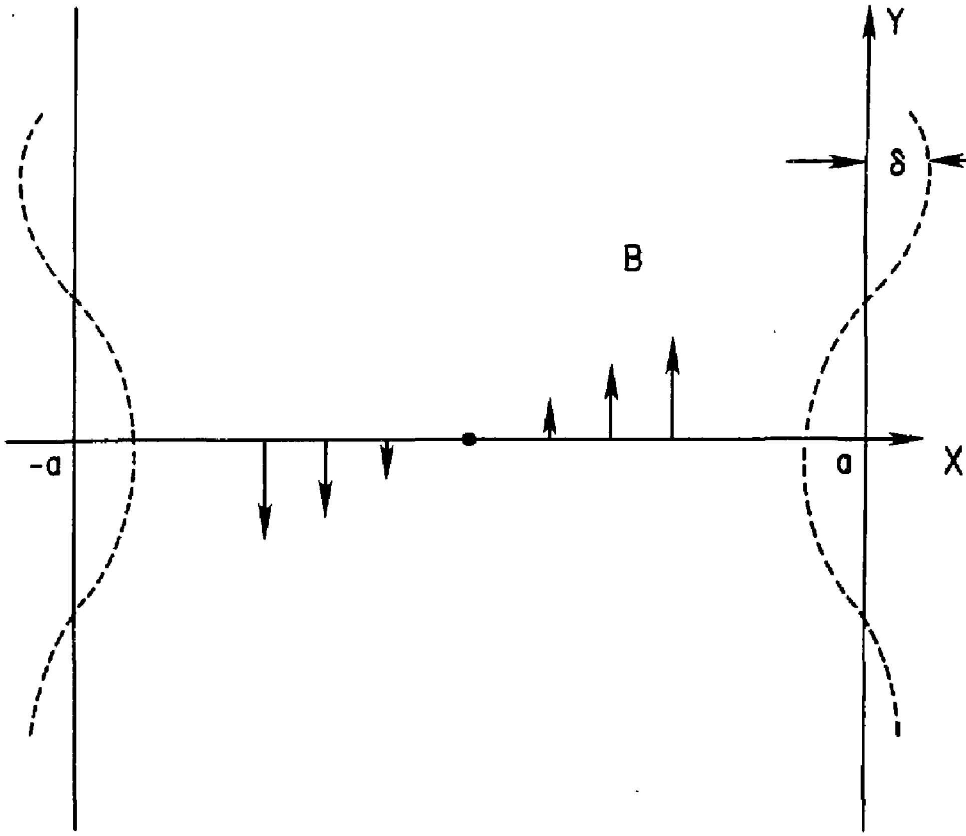

Fig. 1

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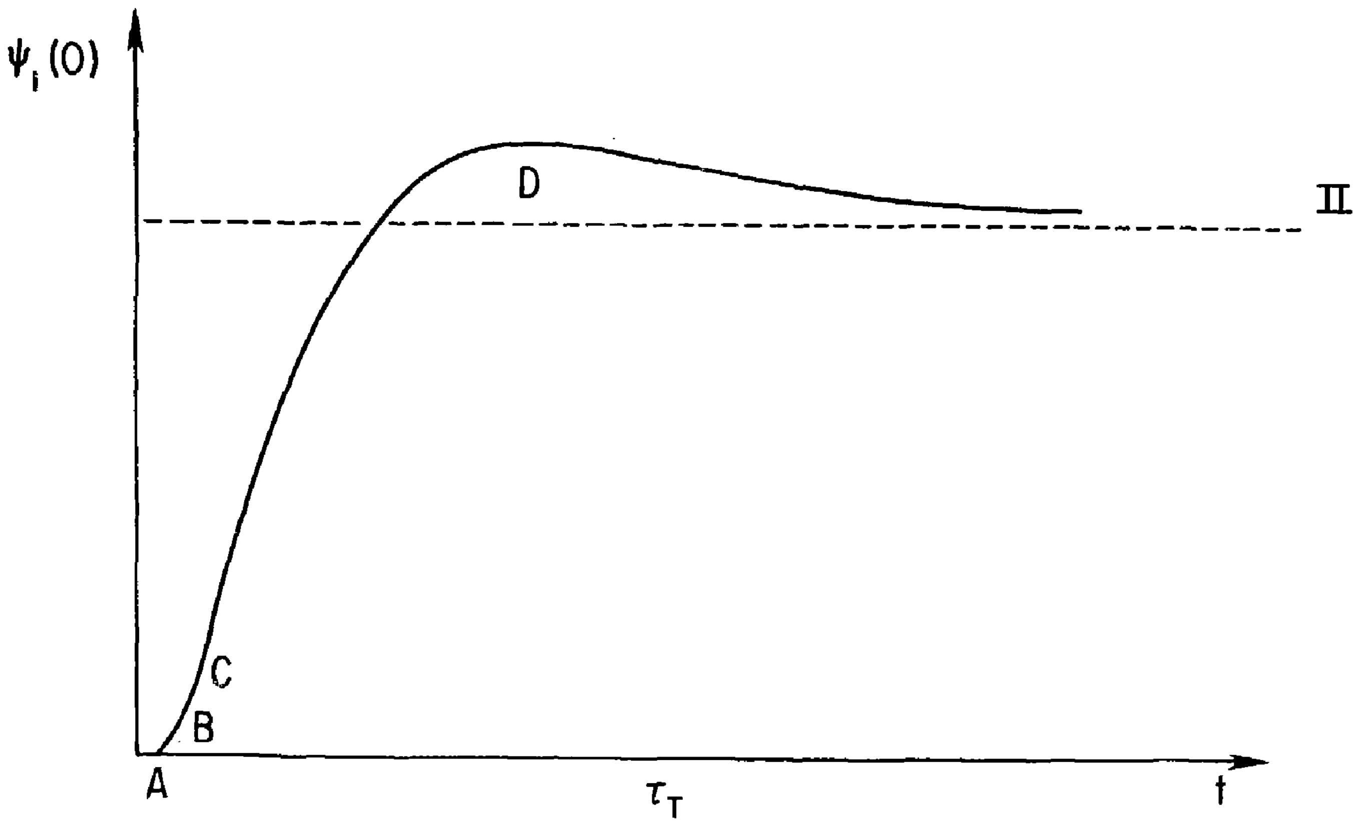

Fig. 2

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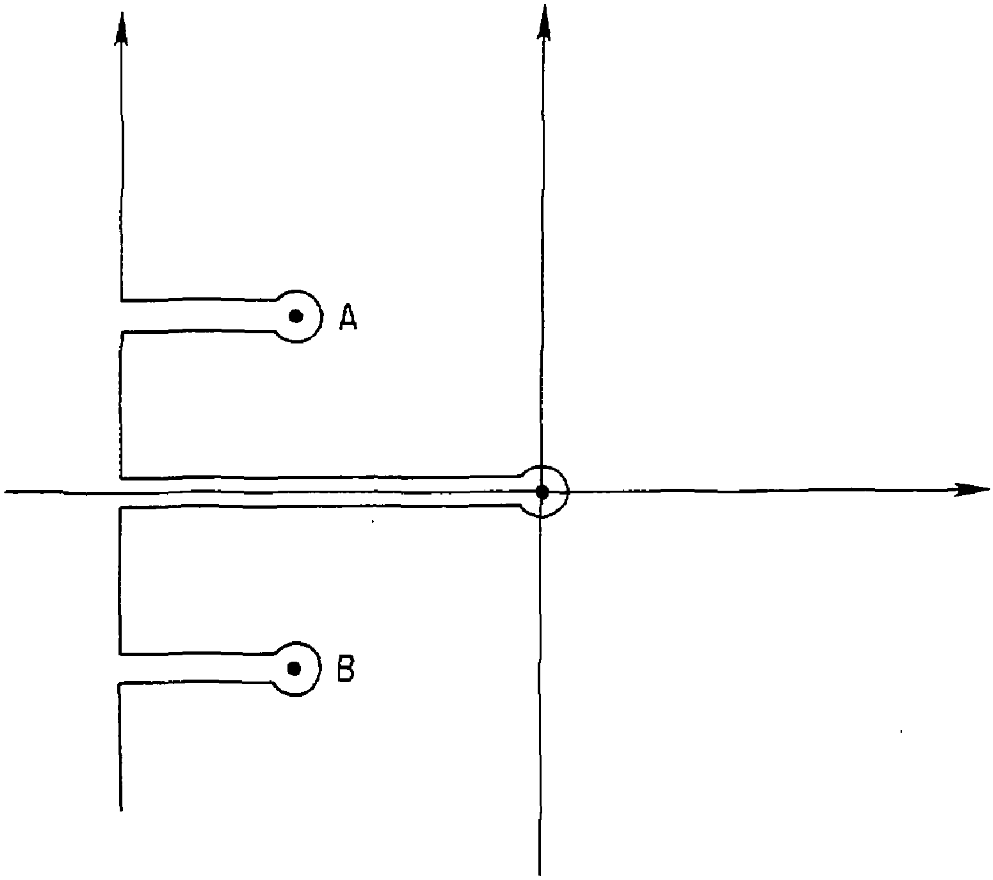

Fig. 3

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