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DISSIPATIVE SYSTEMS AND BATEMAN'S HAMILTONIAN

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Abstract

It is shown, by using canonical transformations, that one can construct Bateman's Hamiltonian from a Hamiltonian for a conservative system and obtain a clear physical interpretation which explains the ambiguities emerging from its application to describe dissipative systems. (Author)

During the last decades there has been considerable effort in developing a quantum theory for dissipative systems. Also the quantization of classically dissipating systems has received widespread attention. An excellent review article by Dekker[1] covers much of what has been done on the subject to date, and the reader is referred there for a very detailed list of references.

The approach of dissipation in quantum theory seems to have followed two main lines: (a) one starts with the classical equation of motion for a system with dissipation, finds a Lagrangian which yields this equation of motion, and then applies the canonical quantization method based on this Lagrangian; (b) one considers the dissipation as being due to the coupling of two systems, i.e., a dissipative subsystem coupled to another absorptive subsystem, the so-called loss-reservoir. This latter method of introducing dissipation in quantum theory has been treated by various authors in different approaches.[2-5] Notwithstanding, it is about the treatment of the first line that we want to dewell up here.

In the case of linear dissipation, the Lagrangian

$$L(x,\dot{x},t) = e^{\beta t}(m\dot{x}^2/2 - V(x)), \qquad (1)$$

whose associated Hamiltonian is

$$H(x,p,t) = (1/2m)\ e^{-\beta t}p^2 + V(x)e^{\beta t}\ , \qquad (2)$$

where $p = \partial L/\partial\dot{x}$, leads to the equation of motion

$$\ddot{x} + \beta\dot{x} + \frac{1}{m}\ \frac{dV(x)}{dx} = 0\ , \qquad (3)$$

where the positive constants m and β mean respectively the mass of the particle and the friction coefficient. The above Lagrangian was originally found by Bateman[6] and was subsequently rederived by many other authors[7-11]. This Lagrangian has been used by several people[12-15] in order to introduce dissipation in quantum theory through the canonical quantization method. However, various other authors[16-21] have severely criticized this procedure by pointing out some theoretical difficulties which are displayed by this Lagrangian. These difficulties mainly refer to: (a) the violation of the uncertainty principle; (b) the question which concerns the Hamiltonian (2) and the energy it describes.

There is presently some general agreement in that the Hamiltonian (2) (or Lagrangian (1)) is not able to include dissipation in quantum theory, but it remains as a theoretical puzzle the point where it fails. The main purpose of this paper is to present a simple and clear treatment which enables one to get a better insight on this question.

Let us begin by taking a Lagrangian for an one-dimensional conservative system. To be specific we are going to consider the Lagrangian for the simple harmonic oscillator

$$L(x,\dot{x}) = (1/2)(m\dot{x}^2 - kx^2), \qquad (4)$$

where k is the restoring constant. The associated Hamiltonian to (4) is

$$H(x,p,t) = (1/2)(p^2/m + kx^2). \qquad (5)$$

Application of Lagrange's equations to (4) or Hamilton's equation to (5) directly yields

$$\ddot{x} + \omega_0^2 x = 0 \quad , \qquad \omega_0^2 = k/m \ . \tag{6}$$

Now, making a canonical transformation through the generating function

$$F_1(x,p_1) = xp_1 + \frac{m\lambda}{2}x^2 \tag{7}$$

we obtain

$$p = p_1 + m\lambda x \tag{8.a}$$

$$x_1 = x \ , \tag{8.b}$$

where λ is a positive constant. Notice that, according to (8.b), the generating function (7) does not change the variable x and the Hamiltonian (5) transforms into

$$H_1(x_1,p_1) = \frac{p_1^2}{2m} + \frac{m\omega^2}{2}x_1^2 + \lambda x_1 p_1 \ , \tag{9}$$

with

$$\omega^2 = \omega_0^2 + \lambda^2 \ . \tag{10}$$

Next, making a second canonical transformation through the time-dependent generating function

$$F_2(x_1,p_2,t) = x_1 p_2 e^{-\lambda t} \tag{11}$$

we find

$$p_1 = p_2 e^{-\lambda t} \tag{12.a}$$

$$x_2 = x_1 e^{-\lambda t} = x e^{-\lambda t} \ , \tag{12.b}$$

Here, it is of interest to note that, according to (12.b), (11) changes the variable x. The new transformed Hamiltonian becomes

$$H_2(x_2,p_2,t) = (1/2m)e^{-\beta t}p_2^2 + (1/2)m\omega^2x_2^2e^{\beta t}, \quad \beta = 2\lambda, \quad (13)$$

which is identical to Bateman's Hamiltonian (2) when in (2) we replace V(x) by the potential of the harmonic oscillator. (2) is also known in the literature as Kanai's Hamiltonian.

On the other hand, introducing the transformation (12.b) into (4) we find

$$L(x_2,\dot{x}_2,t) = e^{\beta t}\left(\frac{m\dot{x}_2^2}{2} - \frac{m\omega^2x_2^2}{2}\right) + \frac{d}{dt}\left(\frac{\lambda m}{2}x_2^2e^{\beta t}\right), \quad (14)$$

which, by neglecting the total time derivative, coincides with Bateman's Lagrangian (1) for V(x) as we have mentioned above. The coordinate transformation (12.b) is a well-known one appearing in mathematical textbooks, used in order to reduce a homogeneous differential equation to its normal form[22]. Here, it can be viewed as part of the canonical transformation (11).

The equation of motion which follows from the Hamiltonian (13) or Lagrangian (14) is easily found to be

$$e^{\beta t}(\ddot{x}_2 + \beta\dot{x}_2 + \omega^2x_2) = 0 \quad (15)$$

or, equivalently,

$$\ddot{x}_2 + \beta\dot{x}_2 + \omega^2x_2 = 0, \quad (16)$$

since the exponencial multiplier in (15) does not introduce any extraneous solutions to the equation of motion described by (16). It is easy to show that (16) can also be obtained by introducing (12.b) into (6).

At this point, it should be emphasized that in the solution from (16) the usual frequency-shift due to presence of the friction coefficient β is cancelled and the "damped" oscillator retains its original frequency, i.e.,

$$\Omega^2 = \omega^2 - (\beta/2)^2 = \omega_0^2 \quad , \tag{17}$$

where (10) has been used.

The foregoing results show that Bateman's Hamiltonian (13) does not describe a dissipative system, since it can be obtained through two canonical transformation applied to a Hamiltonian of a conservative system. Now the question is: how could one account for that (13) gives rise to the equation of motion (16) which, apparently, describes a dissipative system? Some authors[18-20] have attempted to remove this difficulty by interpreting (13) as describing a nondissipative system with variable mass. One of these authors[20] argues that a system with increasing mass classically has an equation of motion identical to that of a system of constant mass subject to a damping force. However, this is not the case in the present work since, according to (4), m is kept constant. So, we have to look for another different interpretation for Bateman's Hamiltonian.

In order to provide a physical interpretation for Bateman's Hamiltonian we are going to return to the transformation (12.b) and to inquire into its physical content. By calculating the acceleration of the particle in the transformed system (12.b) one finds

$$\ddot{x}_2 = \ddot{x} - \{2\ddot{x}e^{-\lambda t/2}\sinh(\lambda t/2) - (2\lambda\dot{x} - \lambda^2 x)e^{-\lambda t}\} , \qquad (18)$$

which shows that $\ddot{x}_2 \neq \ddot{x}$. Hence we can conclude that the transformation (12.b) leads one to a non-inertial frame of reference. The acceleration of the non-inertial frame of reference relative to the inertial one is given by the last term (term in the bracket) in (18).

The previous result allows one to get a clear interpretation of Bateman's Hamiltonian. What this Hamiltonian actually describes is a nondissipative system; the apparent dissipation in (16) is due to the fact that the transformation (12.b) leads one to an accelerated frame of reference and, as consequence, the length scale of an observer located in this non-inertial system is suffering a dilatation as $\ell_2 = \ell\exp(\lambda t)$. This gives an observer, located in this accelerated system, the misleading impression of being observing a damped motion.

Finally, we mention that canonical transformations as these we have employed were used in the past by Kerner[12], for quantizing the damped harmonic oscillator and by Dekker[1], for obtaining Bateman's Hamiltonian (13) from the time-independent Bateman-Morse-Feshbach dual Hamiltonian.

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