ABSTRACT

An analysis of the excitation of neutron flux waves in reactor core transients has been performed. A perturbation theory solution has been developed for the time-dependent thermal diffusion equation in which the absorption cross section undergoes a rapid change, as in a PWR rod ejection accident (REA). In this analysis the unperturbed reactor flux states provide the basis for the spatial representation of the flux solution. Using a simplified space-time representation for the cross section change, the temporal integrations have been carried out and analytic expressions for the modal flux amplitudes determined. The first order modal excitation strength is determined by the spatial overlap between the initial and final flux states, and the cross section perturbation. The flux wave amplitudes are found to be largest for rapid transients involving large reactivity perturbations.
EXCITATION OF NEUTRON FLUX WAVES IN REACTOR CORE TRANSIENTS

I. INTRODUCTION

Weinberg and Schweinler have shown that the periodic displacement of a neutron absorber acts as a source of neutron flux waves having the period of the absorber motion. More generally, the spectrum of core harmonics may be excited by a time-dependent change in local nuclear material properties, as illustrated in a recent calculation of the PWR rod ejection accident. In this case the material property change acts as a source of neutron flux waves and the excitation strength of a particular oscillatory flux mode is determined by the space and time dependence of the material property change.

The modes having the largest spatial overlap with the material perturbation (in the integral sense) are most strongly excited. If the material perturbation is slow relative to a particular (non-zero) modal oscillation frequency, the flux contribution of previously emitted waves of this frequency can be considered to be of approximately equal amplitude and random phase, and tend to cancel. In this case the flux is determined by the contribution from the (zero frequency/infinite period) steady-state flux shape and no harmonics or flux waves are excited. If the perturbation is fast the contributions from previously emitted waves are of unequal amplitude and tend not to cancel, and the higher modes are excited.

The flux disturbance propagates away from the source as an outgoing attenuated wave (actually as a wave packet) reflecting off the core boundary and undergoing subsequent rescatterings from the material perturbation at later times. The oscillatory flux modes may be described by a discrete spectrum of frequencies, wave numbers and attenuation lengths. The wave frequencies and attenuation lengths are determined by the local material properties, with the
wave amplitude increasing and the attenuation decreasing with increasing local reactivity.

This local oscillatory flux behavior may have significant effects on the core transient dynamics. In fact, in the rod ejection accident the local flux oscillations at the ejected rod location introduce oscillations into the core reactivity which result in large oscillations in the core power. \(^{(2)}\)

The analysis reported here is intended to further quantify the behavior of these flux waves and is based on the application of time-dependent perturbation methods to the original treatment of flux oscillations by Weinberg and Schweinler. \(^{(1)}\) Use has also been made of the more recent analyses of Perez and Uhrig, \(^{(3)}\) Moore, \(^{(4)}\) and Perez and Booth. \(^{(5)}\) In Section-II the basic flux equation for a thermal group absorption cross section perturbation is presented. In Section-III time-dependent perturbation theory is applied to the basic flux equations and in Section-IV the first order flux solution is determined for a cross section perturbation corresponding to the rod ejection accident (REA). In the Appendix the basic wave solutions are given and the modal frequency and attenuation length are expressed in terms of local material properties.

II. BASIC FLUX EQUATION

The initial core transient is introduced by a rapid change in the core absorption cross section, \(\Sigma_a\), of a uniform just-critical reactor. The system is assumed to undergo an initial core flux transient and then return to a critical state by a readjustment of the core reactivity via feedback mechanisms and/or control rod motion. The perturbation is assumed to have a negligible effect on the core diffusion coefficients, scattering and fission cross sections, and slowing down kernel. In the diffusion approximation the transient thermal neutron flux, \(\phi(x,t)\), is described by the time-dependent equation
there \( \frac{E}{2} \text{and} v \) are thermal neutron diffusion coefficients, absorption and fission cross sections, delayed neutron fraction and velocity at energy \( E_{th} \), respectively. \( \Delta \Sigma_a(t) \) is the change in absorption cross section and \( P \) is the normalised prompt neutron slowing down kernel. In order to simplify the analysis the delayed neutrons have been neglected. This is justified when the transient frequencies, \( \omega \), are large with respect to the reciprocal periods of the delayed neutrons, \( \tau_j \), i.e.

\[
\omega \tau_j >> 1 ,
\]

which will be shown to be the case in the present application.

III. TIME DEPENDENT PERTURBATION THEORY

If we define the temporal displacement operator, \( H \), by writing equation (1)

\[
(H - \frac{\partial}{\partial t}) \phi = 0 ,
\]

we can define the unperturbed operator, \( H_o \), by the equation

\[
H = H_o - v \Delta \Sigma_a .
\]

An integral equation for the flux solution, \( \phi \), may be obtained by introducing the unperturbed Green's function defined by the equation

\[
(H_o - \frac{\partial}{\partial t})G_o(t-t') = \delta(t-t')\Sigma ,
\]
where $I$ is the spatial identity operator with matrix elements, $\delta(x-x')$. $G_o^{(+)}$ and $G_o^{(-)}$ are the retarded and advanced Green's functions, respectively, and satisfy the initial conditions

$$G_o^{(\pm)}(t-t') = 0 \quad t-t' < 0 ,$$

and

$$G_o^{(-)}(t-t') = 0 \quad t-t' > 0 .$$

The flux solution may then be determined by the integral equation

$$\phi(x,t) = \phi_o^{(\pm)}(x,t) + \nu \int G_o^{(\pm)}(x,x';t-t') \Delta \Sigma_g (x',t') \phi(x',t') d^3 x' dt' ,$$

where $\phi_o^{(\pm)}(t)$ is the solution to the homogeneous form of Equation (4) satisfying the appropriate temporal boundary condition.

A spatial representation of $\phi(t)$ may be obtained by introducing the eigenfunctions, $\psi_m$, of the operator $H_o$,

$$H_o \psi_m = -i \omega_m \psi_m .$$

These functions are assumed to be complete and satisfy the orthogonality condition

$$\langle \psi_k | \psi_k \rangle = \int \psi_k^*(x) \psi_k(x) dx = \delta_{kk} .$$

*In the present analysis the core transient is introduced by a change in the absorption cross section, $\Sigma_o$, and the flux is expanded in powers of $\Delta \Sigma_g$. If the perturbation enters through a different group constant, a similar expansion may be developed in the change in that constant.
$G_{o}^{(\pm)}(t)$ may then be expanded

$$G_{o}^{(\pm)}(x, x', t-t') = \sum_{m} \psi_{m}(x) G_{m0}^{(\pm)}(t-t') \psi_{m}(x') ,$$  \hspace{1cm} (10)

or introducing the Fourier transform of $G_{m0}^{(\pm)}(t-t')$,

$$G_{o}^{(\pm)}(x, x', t-t') = \frac{1}{\sqrt{2\pi}} \sum_{m} \int_{-\infty}^{+\infty} \psi_{m}(x) \psi_{m}(x') e^{-i\omega(t-t')} G_{m0}^{(\pm)}(\omega) d\omega .$$  \hspace{1cm} (11)

Inserting this expansion for $G_{o}^{(\pm)}$ in Equation (5) together with the resolvent expansion for the spatial identity operator,

$$I = \sum_{m} |\psi_{m} \rangle \langle \psi_{m}| ,$$  \hspace{1cm} (12)

and making use of the orthogonality of the $\psi_{m}$ we find

$$G_{m0}^{(\pm)}(\omega) = \frac{i}{\nu \sqrt{2\pi}} \frac{1}{\omega - \omega_{m}} .$$  \hspace{1cm} (13)

The time dependent Green's function may then be determined by inverting Equation (13),

$$G_{m0}^{(\pm)}(t-t') = \frac{i}{2\nu} \int_{-\infty}^{+\infty} e^{i\omega(t-t')} \frac{1}{\omega - \omega_{m}} \int_{-\infty}^{+\infty} d\omega .$$  \hspace{1cm} (14)

The poles on the real axis at $\omega = \pm \omega_{m}$ have been displaced into the lower (upper) $\omega$-half plane in order to satisfy the retarded (advanced) Green's function initial conditions.† Evaluating this integral by closing the contour in the upper (lower) half plane for the advanced (retarded) Green's function and substituting the result in Equation (10) yields
where $S(x)$ is the step-function

$$S(x) = \begin{cases} 
1 & x > 0 \\
0 & x < 0 
\end{cases}$$

Inserting this expression for $G^{(\pm)}_0$ into Equation (7), the integral equation for $\phi$ may be written

$$\phi(x, t) = \phi_0(x, t) + \sum_{\omega} \psi_m(x) e^{-i\omega x}$$

$$\times \int_{-\infty}^{\infty} \langle \psi_m | \Delta E_\omega(x') | \phi(t') \rangle e^{i\omega x} S(x(t-t')) dt' .$$

For convenience the perturbation, $\Delta E_\omega$, is taken to be zero for $t < 0$. Making use of the definition of $S(x)$ in Equation (17), it follows that $\phi^{(+)}(\phi^{(-)})$ is determined by the initial (final) state,

$$\phi_0^{(+)}(x, 0) = \phi_0^{(-)}(x, 0) = \phi_{\text{initial}}(x) ,$$

and

$$\lim_{t \to \pm\infty} \phi_0^{(-)}(x, t) = \lim_{t \to \pm\infty} \phi_0^{(+)}(x, t) = \phi_{\text{final}}(x) .$$

$^\dagger$In order to satisfy the (Equation (6)) temporal boundary conditions when states in the bottom $\omega$-half plane are included, the contour of integration for $G^{(+)}$ [$G^{(-)}$] is taken above (below) the corresponding poles at $\omega = \omega_m$.

$^*$Here and in the following the flux is understood to be the real part of $\phi$. 

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Since in the present case the temporal boundary condition is specified in terms of the initial critical state

\[ \phi_{\text{initial}}(x) = \psi_0(x), \quad (19) \]

Equation (17) will be solved using the retarded Green's function, \( G(t') \). Expanding the unperturbed flux, \( \phi_o^{(\ast)} \), in terms of the solutions to the unperturbed form of Equation (3),

\[ \phi_o^{(\ast)}(x,t) = \sum m a_m \psi_m(x) e^{-i\omega_m t}, \quad (20) \]

and combining Equations (18)-(20) we find

\[ \phi^{(\ast)}(x,0) = \sum m a_m \psi_m(x) = \psi_0(x), \quad (21) \]

Using the orthogonality property of the \( \psi_m \) and recalling that \( \omega_0 = 0 \), it follows that

\[ \phi^{(\ast)}(x,t) = \psi_0(x), \quad (22) \]

The appropriate form of Equation (17) is now

\[ \phi(x,t) = \psi_0(x) + \sum m \psi_m(x)e^{-i\omega_m t} \]

\[ \times \int_0^t \langle \psi_m | \Delta F(t') | \phi(t') \rangle e^{+i\omega_m t'} dt'. \quad (23) \]
A Neumann perturbation expansion for $\phi$ may be obtained by iterating Equation (23). The first order solution is given by

$$\phi(x,t) = \psi_0(x) + \sum_m \psi_m(x)e^{-i\omega_m t} \int_0^t \Delta \Sigma_{a,m0} (t') dt' ,$$

and the $n$-th order term is

$$\phi^{(n)}(x,t) = \sum_m \psi_m(x)e^{-i\omega_m t} \int_0^{t_n} \int_0^{t_{n-1}} \ldots \int_0^{t_1} \Delta \Sigma_{a,m} (t_1) \Delta \Sigma_{a,m} (t_{n-1}) \ldots \Delta \Sigma_{a,m} (t_1)_{m0} ,$$

where

$$\Delta \Sigma_{a,mn} (t) = \langle \psi_m | \Delta \Sigma_{a} (t) | \psi_n \rangle e^{i\omega_{mn} t} ,$$

and

$$\omega_{mn} = \omega_m - \omega_n .$$

This expression is the $n$-th order term in a multiple scattering expansion in which the cross section perturbation acts as a source of neutron flux waves. The flux waves produced at $(x,t)$ propagate unperturbed (via $G_0$) between subsequent scatterings off the perturbation, $\Delta \Sigma_{a}$.

IV. ROD EJECTION ACCIDENT

The rod ejection accident is initiated by a reduction in the control rod neutron absorption cross section. The initial increase in core power introduces fuel and moderator temperature feedback which tends to offset the initial decrease in absorption cross section. These cross section changes are ultimately balanced and the system returns to a steady-state. In the following the cross
section space-time dependence will be taken to be separable and, in order to simplify the resulting expressions, the control rod and feedback cross sections will be assumed to have the same time dependence. The change in absorption cross section resulting from a rod ejection at $x_0$ at time $t = 0$ may then be represented ($\lambda$ and $\mu$ are positive constants)

$$\Delta \Sigma_a(t) = -\Delta \Sigma_T \delta(x-x_0) - \alpha_F(x) s(t) e^{-\lambda t}(1-e^{-\mu t})(1+\gamma e^{i\omega t}) ,$$

(28)

where $\Delta \Sigma_T \delta(x-x_0)$ is the decrease in control rod absorption cross section and $\Delta \Sigma_F(x)$ is the resulting increase in feedback cross section. The harmonic time dependence of frequency $\omega$ and strength $\gamma$ is included in $\Delta \Sigma_a(t)$ to account for the effects of oscillations in the fuel temperature. These may result from either flux oscillations or from power/fuel-temperature feedback oscillations.

In the case of feedback oscillations, $\omega$ is determined by the fuel heat transfer model, the Doppler reactivity coefficients and weighting and the neutron kinetics parameters.

Inserting this expression for $\Delta \Sigma_a(t)$ in the first order solution (Equation (24)) and integrating over $\ell$ we find:

$$\phi(x,t) = \psi_o(x) - i\Delta \Sigma_T \sum_{m} \psi_m(x) e^{-i\omega_m t} \times \left[ (\psi_m | \delta(x-x_0) | \psi_o ) - (\psi_m | \alpha_F(x) | \psi_o ) \right]$$

$$\times \left[ (e^{i\omega_m t - \lambda t} - 1) - (e^{i\omega_m t - (\lambda+\mu) t} - 1) \right]$$

$$\times \gamma \left[ (e^{i(\omega_m - \omega) t - \lambda t} - 1) - (e^{i(\omega_m - \omega) t - (\lambda+\mu) t} - 1) \right] .$$

Perturbation theory is applied here to describe the small flux oscillations about the asymptotic flux shape, which take place at large values of $t$ when $\Delta \Sigma_a(t)$ is small.
Neglecting decaying terms, the amplitude of the m-th mode flux wave, $A_m$, is given by

$$A_m(\omega_m; \lambda, \mu) = -i\Delta \langle \psi_m | \delta(x-x_o) | \psi_o \rangle - \langle \psi_m | \alpha(x) | \psi_o \rangle$$

$$\times \left[ \frac{1}{\omega_m+i(\lambda+\mu)} - \frac{1}{\omega_m+i\lambda} \right] + \gamma \left( \frac{1}{\omega_m-\omega+i(\lambda+\mu)} - \frac{1}{\omega_m-\omega+i\lambda} \right].$$

As a result of the assumption of separability the flux response is reduced to a product of spatial and temporal factors. The spatial factor is the overlap integral between the initial state, $\psi_o$, the final state, $\psi_m$, and the spatial distribution of the cross section perturbation.

Examination of the temporal factor indicates that the higher modes are not excited as strongly as the lower modes (i.e., $A_m \rightarrow 0$). For very slow perturbations and in the case of strong feedback ($\mu \ll \lambda$) the amplitude becomes vanishingly small and no waves are excited (i.e., $A_m \rightarrow 0$). If the perturbation is fast ($\mu \gg \omega, \omega_m, \lambda$) flux waves are readily excited with the lower modes being more strongly excited than the higher modes, especially in the case of weak feedback. It is also noteworthy that the terms resulting from the fuel temperature oscillation exhibit resonance behavior near $\omega \sim \omega_m$.

The dependence of the excitation amplitude on the local core reactivity, $\rho$, may be determined by inserting $\omega_m(\rho)$ (Equation (A-10)) in Equation (30) with the result, in the case of fast transients ($\mu \gg \omega, \omega_m, \lambda$),

$$A_m \propto \left( \frac{2M}{E} \right)^{\frac{2}{3}} \left( \left( \frac{M}{m} \right)^2 - \rho \right)^{\frac{1}{2}} - 1 + \gamma \left( \frac{2M}{E} \right)^{\frac{2}{3}} \left( \left( \frac{M}{m} \right)^2 - \rho \right)^{\frac{1}{2}} - \omega i \lambda - 1.$$

As the local core reactivity increases the modal amplitudes increase and the flux waves are more readily excited.
The frequencies of the oscillation modes having periods close to the rod ejection time are typically \( \omega \approx 10^{-1} \text{ to } 100 \text{ sec}^{-1} \), well above the delayed neutron frequencies, \( \omega_d \approx 1 \times 10^{-1} \text{ sec}^{-1} \), justifying the neglect of the delayed neutrons and well below the reactor frequency, \( \omega_r = 2\pi \nu \Sigma_e \approx 2.6 \times 10^5 \text{ sec}^{-1} \), satisfying Equation (A-9) and justifying the use of expressions (A-5) and (A-10).

The excitation amplitude, \( A_{\text{mo}} \), decreases with increasing \( m \) (see Equation (30)) and the oscillatory flux is dominated by the low order harmonic response. The spatial attenuation of the flux oscillations with increasing \( r \) is given in cylindrical geometry by \( \omega e^{-r/a} \sqrt{r} \). While moderator feedback will be negligible during the initial power excursion, with time it can be expected to become increasingly more negative and the local reactivity, \( \rho \), will tend to zero. This will result in a decrease in the wave amplitude (see Equation (31)), and in the modal attenuation length, \( a_m \) (Equation (A-11)), and an attenuation of the oscillation amplitude with time.
APPENDIX

NEUTRON FLUX BASIS STATES

The asymptotic form of the oscillatory solutions to Equation (8) may be written (in the indicated geometry)

\[ \psi_m(x) = e^{\pm iB_m r} = e^{\pm \left( \frac{B_m}{a_m^m} \right) r}, \quad \text{(spherical) (A-1)} \]

\[ \psi_m(x) = e^{\pm iB_m r} = e^{\pm \left( \frac{B_m}{a_m^m} \right) r}, \quad \text{(cylindrical) (A-2)} \]

and

\[ \psi_m(x) = e^{\pm iB_m x} = e^{\pm \left( \frac{B_m}{a_m^m} \right) x}, \quad \text{(planar) (A-3)} \]

Inserting these expressions in Equation (8) and using the diffusion slowing down kernel satisfying the relation

\[ \int P(x, x'; \nu) \psi_m(x') dx' = (1 + \tau B_m^2)^{-1} \psi_m(x) \]

\[ = (1 - \tau B_m^2) \psi_m(x)^+, \quad \text{(A-4)} \]

where \( \tau \) is the neutron age, the following expressions for the attenuation length, \( a_m \), and wave number, \( k_m \), result (1)

\[ a_m = \frac{\sqrt{2M}}{\left( \sqrt{\sigma^2 + \omega^2} - \rho \right)^{1/2}}, \quad \text{(A-5)} \]

+The truncation of this series is an excellent approximation in the present application.
and

\[ \eta_m = \left( \frac{1}{2} \right)^{-1} \left( \frac{1}{\sqrt{\rho^2 + \omega_m^2}} \right)^{1/2} . \]  \hspace{1cm} (A-6)

\( M^2 \) is the migration area and the reactor period, \( L_o \), and local prompt core reactivity, \( \rho \), are given by

\[ L_o = (\nu \Sigma_a)^{-1} , \]  \hspace{1cm} (A-7)

and

\[ \rho = \nu(1-\beta) \Sigma_f/\Sigma_a - 1 . \]  \hspace{1cm} (A-8)

It has been assumed in the derivation of these expressions that the modal frequency, \( \omega_m \), is small relative to the reactor frequency, \( \omega_r \),

\[ \omega_m << \omega_r = 2\pi L_o^{-1} = 2\pi \nu \Sigma_a , \]  \hspace{1cm} (A-9)

and \( \rho \gg 0 \). These relations will be shown to be satisfied in the present analysis in Section-V. Inverting Equation (A-6), \( \omega_m \) may be expressed in terms of \( \eta_m \)

\[ \omega_m = 2\pi \nu \Sigma_a^{-1} (\Sigma_a \eta_m^2 - \rho)^{1/2} , \]  \hspace{1cm} (A-10)

and the attenuation length may be written

\[ a_m = M[(\Sigma_a \eta_m^2 - \rho)]^{-1/2} . \]  \hspace{1cm} (A-11)

\textit{\footnote{This dependence might be used to infer core nuclear material properties from measurements of wave frequency and attenuation length.}}
The phase velocity, \( v_m = \omega_m / k_m \), is mode dependent and the waves are in general dispersive. The wave packet group velocity, \( v^g = d\omega / dk \), may be determined from Equation (A-10).

The flux contribution from the \( m \)-th mode oscillatory solution may be expressed as the sum of an outgoing and reflected wave:

\[
\psi_m = e^{a_m \Delta m} e^{- \frac{2i\delta}{m} - ik_m r} - e^{- \frac{2i\delta}{m} + ik_m r},
\]

where the amplitudes of the outgoing and reflected waves are taken to be equal at the extrapolated core radius, \( R \). The outgoing wave is assumed to undergo a phase shift of \( 2\delta \) upon reflection at the core boundary. Demanding that the flux vanish at the core radius, \( R \), requires that

\[
\delta_m = k_m R,
\]

and \( \psi_m \) may then be written

\[
\psi_m = e^{a_m \Delta m} e^{- \frac{(r-R)}{m} + ik_m (r-R)} - e^{- \frac{(R-r)}{m} - ik_m (r-R)}.
\]

The \( \psi_m \) form a complete set of basis functions over the required interval for the wave numbers

\[
k_m = (m+1) \frac{R}{R} \quad m = 0, 1, 2, \ldots.
\]

\( ^{\text{Only the spherical results are given in the following. The cylindrical results may be obtained by replacing the } r^{-1} \text{ attenuation with } r^{-1/2}. \)
When the attenuation length is large \((a_m >> R)\) \(\psi_m\) may be approximated

\[
\psi_m \approx \frac{\sin k (r-R)}{r}.
\]

It follows that for large \(m\), \(a_m \to 0\), \(\omega_m \approx k_m^2\) and the higher modes are more strongly spatially attenuated and have higher phase and group velocities.

It is also important to note the behavior of \(a_m\), \(\omega_m\) and \(v_m^p\) with increasing local reactivity, \(\rho\). As \(\rho\) increases (1) \(a_m\) increases and the waves are less strongly spatially attenuated and (2) \(\omega_m\) and \(v_m^p\) decrease and the waves are more readily excited (see Equation (31)). For values of \(\rho > (Mk_m)^2\), the \(m\)-th mode flux may become undamped in space and time. The reactivity \(\rho\) for which the \(j\)-th and \(m\)-th modes are resonant may be determined by demanding \(\omega_j = n\omega_m\) \((n = 1, 2, 3, \ldots)\) and is given by

\[
\rho_{jm} = \left(\frac{M\omega_m}{R}\right)^2 \frac{(j+1)^2}{(m+1)^2} \frac{n^2}{(m+1)^2} - 1.
\]

The non-oscillatory temporally damped solutions of the form \(e^{-\omega_m t}\) are not attenuated spatially \((a_m \to \infty)\), and the modal period is given by

\[
\tau = 2\pi \omega_m ((Mk_m)^2 - \rho)^{-1}.
\]
REFERENCES


