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MAGNETIC FIELD LINE HAMILTONIAN

By

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MAGNETIC FIELD LINE HAMILTONIAN

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ABSTRACT

The basic properties of the Hamiltonian representation of magnetic fields in canonical form are reviewed. The theory of canonical magnetic perturbation theory is then developed and applied to the time evolution of a magnetic field embedded in a toroidal plasma. Finally, the extension of the energy principle to tearing modes, utilizing the magnetic field line Hamiltonian, is outlined.

I. INTRODUCTION

The Hamiltonian properties of magnetic fields form the basis for much of toroidal plasma physics and have been utilized, although implicitly, in treatments of the subject.[1-5] The relation between Hamiltonian mechanics and magnetic fields has been discussed by several authors,[6-8] but the subject has never been adequately developed to form a basis for theoretical discussions of toroidal plasmas or as a practical computational tool. The basic Hamiltonian theory of magnetic fields is developed in this paper in a form which is particularly useful for the study of toroidal plasmas.

A basic feature of the canonical Hamiltonian treatment of magnetic fields is that all the topological properties of the magnetic field line trajectories are isolated in a single scalar function, the Hamiltonian. The Hamiltonian either has one degree of freedom with dependence on the canonical time or has two degrees of freedom without dependence on the canonical time. A one degree of freedom Hamiltonian will be used in this paper in which the magnetic field is assumed to have a finite toroidal component. The generalization to arbitrary globaly, divergence-free fields utilizes a two degree of freedom Hamiltonian and involves some additional mathematical complexity.[9]

Section II reviews the basic properties of a canonical representation[7] of magnetic fields. A general formulation of canonical perturbation theory of magnetic fields is developed in Sec. III. This formulation of magnetic perturbation theory is used in Sec. IV to obtain equations for the time evolution of magnetic fields embedded in a plasma. The basic results are that the Hamiltonian evolves due to resistivity and would be conserved if the resistivity were zero. The transformation equations between the canonical coordinates of the Hamiltonian and the ordinary spatial coordinates can change on an arbitrarily rapid time scale in order to maintain force balance.

Finally, a generalization of the energy principle[10,11] to tearing modes[12] utilizing the Hamiltonian formulation is outlined in Sec. V. Although not discussed in this paper, the Hamiltonian formulation of the magnetic field is also useful in the evaluation of particle drift orbits.[13]

II. CANONICAL REPRESENTATION

Any globally divergence-free field $\vec{B}(\vec{x})$ can be written in the so-called canonical form[7]

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\theta + \vec{\nabla}\phi \times \vec{\nabla}\chi \tag{1}$$

with ψ and χ single-valued functions of position and θ and ϕ proper angles [a proof is outlined below Eq. (8)]. By a proper angle, we mean that θ and $\theta + 2\pi$ are the same physical position $\vec{x}$. The physical interpretations of ψ,θ,ϕ, and χ are illustrated in Fig. 1. The angles θ and ϕ are poloidal and toroidal angles. The toroidal magnetic flux enclosed by a constant ψ surface is $2\pi\psi$, and the poloidal flux outside a constant χ surface is $2\pi\chi$. However, the constant ψ and constant χ surfaces are generally not identical.

To represent a field using the canonical form it would appear that ψ,θ,ϕ, and χ must be known as functions of position $\vec{x}$ in order to evaluate the gradients. Actually, it is more useful to consider the position as a function of ψ,θ, and ϕ, the so-called canonical coordinates. This can be done as long as the triple product $(\vec{\nabla}\psi \times \vec{\nabla}\theta) \cdot \vec{\nabla}\phi$, which is the toroidal component of the field $\vec{B} \cdot \vec{\nabla}\phi$, is finite. Here, $\vec{B} \cdot \vec{\nabla}\phi$ will be assumed finite, but the general case can be handled at the price of somewhat more difficult mathematics.[9]

To describe a magnetic field $\vec{B}(\vec{x})$ fully using the canonical coordinates, both the poloidal flux function χ and the position $\vec{x}$ must be given as functions of ψ,θ,ϕ. The functions $\vec{x}(\psi,\theta,\phi)$ are called the transformation equations. They can always be made continuous and this continuity will be assumed. If the field is described using ordinary cylindrical coordinates R,ϕ,Z, Fig. 2, then

$$\vec{x}(x,\theta,\phi) = R(\psi,\theta,\phi)\ \hat{R}(\phi) + Z(\psi,\theta,\phi)\hat{z} \tag{2}$$

with $d\hat{R}/d\phi = \hat{\phi}$. In other words, the transformation equations would consist of R and Z as functions of ψ,θ,ϕ.

The function $\chi(\psi,\theta,\phi)$ has an additional interpretation besides being the poloidal flux function. It is the magnetic field line Hamiltonian. The magnetic field lines, which are also known as the integral curves of the magnetic field, are the solution[5] to the differential equation

$$\frac{d\vec{x}}{d\tau} = \vec{B}(\vec{x}) \tag{3}$$

with τ just a label for the points along a field line trajectory. Consider the field lines in ψ,θ,ϕ coordinates. The change in ψ along a field line is given by $d\psi/d\tau = \vec{\nabla}\psi \cdot d\vec{x}/d\tau$. This means that along a field line

$$\frac{d\psi}{d\phi} = \frac{\vec{B}\cdot\vec{\nabla}\psi}{\vec{B}\cdot\vec{\nabla}\phi} \qquad \frac{d\theta}{d\phi} = \frac{\vec{B}\cdot\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\phi} \quad . \tag{4}$$

The use of the canonical form for $\vec{B}$, Eq. (1), to evaluate $\vec{B}\cdot\vec{\nabla}\psi/\vec{B}\cdot\vec{\nabla}\phi$ and $\vec{B}\cdot\vec{\nabla}\theta/\vec{B}\cdot\vec{\nabla}\phi$ demonstrates that

$$\frac{d\psi}{d\phi} = -\frac{\partial\chi}{\partial\theta} \qquad \frac{d\theta}{d\phi} = \frac{\partial\chi}{\partial\psi} \quad , \tag{5}$$

which are Hamilton's equations. In Hamiltonian language, the poloidal angle θ is the canonical coordinate, the toroidal flux function ψ is the canonical momentum, and the toroidal angle ϕ is the canonical time. Unfortunately, most Hamiltonian mechanics texts identify the canonical time with ordinary clock time. In the magnetic field problem, clock time t is a parameter (see Secs. III and IV) and should not be confused with the canonical time ϕ, which is the toroidal angle.

The importance of the canonical representation of a field $\vec{B}(\vec{x})$ derives from the fact that all topological information on the field line trajectories is contained in the Hamiltonian $\chi(\psi,\theta,\phi)$. That is, questions related to the existence of closed magnetic surfaces, magnetic islands, or stochastic regions can all be answered if the function $\chi(\psi,\theta,\phi)$ is known. This follows from the continuity of the transformation equations and the mathematical result that continuous transformations do not alter topological properties.

The representation of a magnetic field using canonical coordinates is closely related to the well-known magnetic coordinate[2,14,15] and Clebsch representations. The magnetic coordinate representation is identical to the canonical representation except that χ is a function of ψ alone. This representation exists only if all the field line trajectories lie in surfaces. Then, the magnetic coordinates are the action - angle variables of Hamiltonian mechanics. The Clebsch representation,

$$\vec{B} = \vec{\nabla}\psi_o \times \vec{\nabla}\theta_o \quad , \tag{6}$$

can be obtained from the canonical representation by letting $\psi = \psi(\psi_o, \theta_o, \phi)$ and $\theta = \theta(\psi_o, \theta_o, \phi)$ with ψ_o and θ_o the initial values of a field line trajectory. Single-valued transformation equations usually do not exist between the Clebsch coordinates ψ_o, θ_o, ϕ and ordinary space $\vec{x}$. Therefore, the Clebsh representation does not isolate topological information in a single scalar function as does the canonical representation.

Given an arbitrary magnetic field $\vec{B}(\vec{x})$, the canonical representation can be established. Let $\vec{x}(\rho, \theta, \phi)$ be any set of smooth transformation equations with θ and ϕ angles. Assume that ρ is zero at the axis of thu θ angle. Then using the canonical form, Eq. (1), one can show[7] that

$$\frac{\partial\psi}{\partial\rho} = \vec{B} \cdot \left(\frac{\partial\vec{x}}{\partial\rho} \times \frac{\partial\vec{x}}{\partial\theta}\right) \qquad \text{and} \qquad \frac{\partial\chi}{\partial\rho} = \vec{B} \cdot \left(\frac{\partial\vec{x}}{\partial\phi} \times \frac{\partial\vec{x}}{\partial\rho}\right) \quad . \tag{7}$$

If these equations are integrated from $\psi = 0$ for a fixed θ and ϕ, the functions $\chi(\psi, \theta, \phi)$ and $\vec{x}(\psi, \theta, \phi)$ can be evaluated. By using a number of θ and ϕ values together with a fast Fourier transform, the functions $\chi(\psi, \theta, \phi)$ and $\vec{x}(\psi, \theta, \phi)$ can be obtained in Fourier series form. A code to carry out such calculations has been written by G. Kuo-Petravic. The proof of Eq. (7) follows from the canonical form for $\vec{B}$, Eq. (1), and the so-called dual relations, Eq. (21).

The vector potential has an important role in the Hamiltonian representation of the magnetic field. The canonical form for the vector potential is

$$\vec{A} = \psi\vec{\nabla}\theta - \chi\vec{\nabla}\phi + \vec{\nabla}G \tag{8}$$

with G the arbitrary gauge function. That is, the curl of $\vec{A}$ gives the canonical form for $\vec{B}$, Eq. (1). The existence of the canonical form for the vector potential follows[16] from Poincaré's theorem that a globally divergence-free field has a single-valued vector potential $\vec{A}(\vec{x})$. Let $\vec{x}(\rho, \theta, \phi)$, as before, be any well-behaved transformation equations with $\rho = 0$ along the axis of the θ angle. Then, any single-valued vector $\vec{A}(\vec{x})$ can be written as

$$\vec{A} = A_\rho \vec{\nabla}\rho + A_\theta \vec{\nabla}\theta + A_\phi \vec{\nabla}\phi \tag{9}$$

with A_ρ, A_θ, and A_ϕ single-valued functions of ρ, θ, and ϕ. Choose $G(\rho, \theta, \phi)$ so that $\partial G/\partial\rho = A_\rho$ and $G = 0$ at $\rho = 0$, then Eq. (9) takes the form of Eq. (8) with ψ and χ being single-valued functions of position.

Canonical transformations maintain the form of Hamilton's Eq. (5) and the canonical form for the vector potential, Eq. (8), and the magnetic field, Eq. (1). The most general canonical transformation, which is known as an extended phase-space transformation, transforms ψ, θ, ϕ, χ into $\bar{\psi}$, $\bar{\theta}$, $\bar{\phi}$, $\bar{\chi}$. Such a transformation can be specified by the generating function $S(\theta, \phi, \bar{\psi}, \bar{\chi})$ with

$$\psi = \frac{\partial S}{\partial\theta} \qquad \bar{\theta} = \frac{\partial S}{\partial\bar{\psi}}$$
$$\chi = -\frac{\partial S}{\partial\phi} \qquad \bar{\phi} = -\frac{\partial S}{\partial\bar{\chi}} \qquad . \tag{10}$$

The representation of the vector potential $\vec{A}(\vec{x})$ using barred coordinates is evaluated by a substitution into Eq. (8),

$$\vec{A} = \bar{\psi}\,\vec{\nabla}\,\bar{\theta} - \bar{\chi}\,\vec{\nabla}\,\bar{\phi} + \vec{\nabla}\,\bar{G}, \tag{11}$$

with the gauge function $\bar{G} = G + S - (\bar{\psi}\bar{\theta} - \bar{\chi}\bar{\phi})$. The generating function is not single valued (S at θ and $\theta + 2\pi$ are not equal), but $\bar{G}$ is. The generating function S naturally depends on four variables while the gauge function $\bar{G}$ has a natural dependence on position, which is only three variables. This is discussed in Sec. III after Eq. (17) and in Ref. 9.

The canonical transformations, which will be of primary interest, are the infinitesimal canonical transformations. They are defined by

$$S = \bar{\Phi}\theta - \bar{\chi}\phi + \varepsilon\, s(\bar{\Phi},\ \theta,\ \phi,\ \bar{\chi}) \tag{12}$$

with ε an infinitesimal quantity [see Eq. (23)]. Other important transformations include

$$S = \bar{\chi}\theta - \bar{\Phi}\phi\ , \tag{13}$$

which reverses the roles of the poloidal and the toroidal vari- les, and

$$S = \bar{\Phi}\theta - (\bar{\chi} + N\bar{\Phi})\,\frac{\phi}{M}\ , \tag{14}$$

with M and N integers, which establishes a helical coordinate system with $M\theta - N\phi = M\bar{\theta}$. In classical mechanics, it is customary to consider only ordinary canonical transformations, which do not alter the canonical time. These transformations are of the form

$$S = S_o\,(\bar{\Phi},\ \theta, \phi) - \bar{\chi}\phi\ . \tag{15}$$

III. CANONICAL PERTURBATION THEORY

The specification of field line topology by a single scalar function χ makes the canonical formulation particularly useful for the study of evolving magnetic fields. In this section, the mathematical properties of the canonical formulation are developed for a magnetic field which depends not only on position $\vec{x}$ but also on an arbitrary parameter t, which will be called time. In Secs. IV and V, these mathematical properties, which are summarized by Eq. (29), are applied to plasma physics problems.

There are three basic ways to describe a magnetic field which depends on a parameter t. First, the vector potential can be specified as a function of position and time, $\vec{A}(\vec{x},t)$. Second, the quantities ψ, θ, ϕ, and χ can be given as functions of $\vec{x}$ and t. Third, the Hamiltonian and the transformation equations can be given as functions of the canonical coordinates and time, $\chi(\psi, \theta, \phi, t)$ and $\vec{x}(\psi, \theta, \phi, t)$. Since a magnetic field is uniquely defined by any one of the three, mathematical relations exist which determine the other two descriptions of the field if any one is given. The main subject of this section is the derivation of these relations. In addition to their other applications, these relations define magnetic perturbation theory in canonical form.

The three magnetic field descriptions are related by their partial derivatives with respect to time. Unfortunately, the notation for partial differentiation is either cumbersome or incomplete. For example, $\partial\chi/\partial t$ can be interpreted as either $\partial\chi(\vec{x},t)/\partial t$ or $\partial\chi(\psi, \theta, \phi, t)/\partial t$. Since both interpretations occur for χ and for the gauge function G, a subscript $\vec{x}$ or c for canonical will be used. For example, $(\partial\chi/\partial t)_c$ means $\partial\chi(\psi, \theta, \phi, t)/\partial t$. However, to simplify the notation, $\partial\vec{A}/\partial t$ means $\partial\vec{A}(\vec{x},t)/\partial t$. Derivatives of the canonical coordinates ψ, θ, ϕ with respect to time should be interpreted so that $\partial\psi/\partial t = \partial\psi(\vec{x}, t)/\partial t$. The time derivative of the transformation equations is $\partial\vec{x}(\psi, \theta, \phi, t)/\partial t$.

To begin the derivation of the relations, assume $\vec{A}(\vec{x}, t)$ is given and that we wish to find ψ, θ, ϕ, and χ as functions of $\vec{x}$ and t. This can be accomplished by time differentiating the canonical form for $\vec{A}$, Eq. (5), to obtain

$$\frac{\partial \vec{A}}{\partial t} = - \frac{\partial \theta}{\partial t} \vec{\nabla}\phi + \frac{\partial \phi}{\partial t} \vec{\nabla}\theta - \left(\frac{\partial \chi}{\partial t}\right)_{\vec{x}} \vec{\nabla}\phi + \frac{\partial \phi}{\partial t} \vec{\nabla}\chi + \vec{\nabla}s \tag{16}$$

with the function s defined by

$$s \equiv \left(\frac{\partial G}{\partial t}\right)_{\vec{x}} + \phi \frac{\partial \theta}{\partial t} - \chi \frac{\partial \phi}{\partial t} . \tag{17}$$

The function s is the generating function for infinitesimal canonical transformations [see the discussion of Eq. (23) below] and can be specified arbitrarily due to the freedom of the gauge function, G. Actually, s is not just an arbitrary function of ϕ, θ, ϕ, and t as one would suppose, but rather an arbitrary function of five variables, $s(\phi, \theta, \phi, \chi, t)$. To interpret the χ dependence of s mathematically, a distinction must be made between the independent variable χ and the field line Hamiltonian $\chi_H(\phi, \theta, \phi, t)$. Except when calculating derivatives of s, the variable χ always has the value χ_H. Derivatives with respect to χ are to be interpreted as $\partial s/\partial\chi = \partial s(\phi, \theta, \phi, \chi, t)/\partial\chi$ evaluated at $\chi = \chi_H$. The condition

$$\frac{\partial \phi}{\partial t} = - \frac{\partial s}{\partial \chi} \tag{18}$$

on the χ dependence is required to make s an infinitesimal generating function. The freedom of an extra condition on s arises from the use of four quantities ϕ, θ, ϕ, and χ to represent a vector potential which, even in a specified gauge, has only three independent components. That is, even with $\phi(\vec{x}, t)$ a specified function, an arbitrary vector potential can be written in the canonical form, Eq. (8). Viewing s as a function of ϕ, θ, ϕ, χ, and t, with χ evaluated at $\chi = \chi_H$, one has

$$\frac{\partial\vec{A}}{\partial t} = \left(\frac{\partial s}{\partial\psi} - \frac{\partial\theta}{\partial t}\right)\vec{\nabla}\psi + \left(\frac{\partial s}{\partial\theta} + \frac{\partial\psi}{\partial t}\right)\vec{\nabla}\theta + \left[\frac{\partial s}{\partial\phi} - \left(\frac{\partial\chi}{\partial t}\right)_{\vec{x}}\right]\vec{\nabla}\phi \ . \tag{19}$$

To obtain $\partial\psi/\partial t$, $\partial\theta/\partial t$, and $(\partial\chi/\partial t)_{\vec{x}}$ in terms of $\partial\vec{A}/\partial t$, some theorems of partial differentiation theory are required. There are simple relations between derivatives of the canonical coordinates with respect to the position $\vec{x}$, like $\vec{\nabla}\psi(\vec{x}, t)$, and the derivatives of the position $\vec{x}$ with respect to the canonical coordinates. Letting ψ and θ be any two canonical coordinates, partial differential theory implies the orthogonality relations

$$\frac{\partial\vec{x}}{\partial\psi}\cdot\vec{\nabla}\psi = 1 \text{ and } \frac{\partial\vec{x}}{\partial\phi}\cdot\vec{\nabla}\theta = 0 \ . \tag{20}$$

The orthogonality relations can be used to obtain the so-called dual relations

$$\frac{\partial\vec{x}}{\partial\psi} = J\,\vec{\nabla}\theta \times \vec{\nabla}\phi \quad \text{and} \quad \vec{\nabla}\psi = \frac{1}{J}\frac{\partial\vec{x}}{\partial\theta}\times\frac{\partial\vec{x}}{\partial\phi} \tag{21}$$

with the Jacobian J satisfying

$$J = \frac{\partial\vec{x}}{\partial\psi}\cdot\left(\frac{\partial\vec{x}}{\partial\theta}\times\frac{\partial\vec{x}}{\partial\phi}\right) = \frac{1}{\nabla\psi\cdot(\vec{\nabla}\theta\times\vec{\nabla}\phi)} \ . \tag{22}$$

Even permutations of the ψ, θ, ϕ coordinate labels also give valid equations.

Using the orthogonality relations and Eq. (19) for $\partial\vec{A}/\partial t$, one finds that

$$\frac{\partial\psi}{\partial t} = -\frac{\partial s}{\partial\theta} + \frac{\partial\vec{A}}{\partial t}\cdot\frac{\partial\vec{x}}{\partial\theta} \qquad \frac{\partial\theta}{\partial t} = \frac{\partial s}{\partial\psi} - \frac{\partial\vec{A}}{\partial t}\cdot\frac{\partial\vec{x}}{\partial\psi}$$

$$\left(\frac{\partial\chi}{\partial t}\right)_{\vec{x}} = \frac{\partial s}{\partial\phi} - \frac{\partial\vec{A}}{\partial t}\cdot\frac{\partial\vec{x}}{\partial\phi} \qquad \frac{\partial\phi}{\partial t} = -\frac{\partial s}{\partial\chi} \ . \tag{23}$$

If $\partial\vec{A}/\partial t = 0$, these are just the equations for infinitesimal canonical transformations in the extended phase space ψ, θ, ϕ, χ, which validates the identification of s as an infinitesimal generating function [see Eq. (12)].

The equations relating $\partial\psi/\partial t$, $\partial\theta/\partial t$, and $\partial\phi/\partial t$ with $\partial\vec{A}/\partial t$ can be used to determine a more important relation, the relation between $\partial\vec{x}/\partial t$ and $\partial\vec{A}/\partial t$. If the trivial relation $(\partial\vec{x}/\partial t)_{\vec{x}} = 0$ is expressed in ψ, θ, ϕ coordinates, one obtains the result that

$$\frac{\partial\vec{x}}{\partial t} = - \frac{\partial\vec{x}}{\partial\psi}\frac{\partial\psi}{\partial t} - \frac{\partial\vec{x}}{\partial\theta}\frac{\partial\theta}{\partial t} - \frac{\partial\vec{x}}{\partial\phi}\frac{\partial\phi}{\partial t} . \tag{24}$$

Substituting $\partial\psi/\partial t$, $\partial\theta/\partial t$, and $\partial\phi/dt$ from Eq. (23), one obtains the desired relation

$$\frac{\partial\vec{x}}{\partial t} = \left(\frac{\partial s}{\partial\theta} - \frac{\partial\vec{A}}{\partial t}\cdot\frac{\partial\vec{x}}{\partial\theta}\right)\frac{\partial\vec{x}}{\partial\psi} - \left(\frac{\partial s}{\partial\psi} - \frac{\partial\vec{A}}{\partial t}\cdot\frac{\partial\vec{x}}{\partial\psi}\right)\frac{\partial\vec{x}}{\partial\theta} + \frac{\partial s}{\partial\chi}\frac{\partial\vec{x}}{\partial\phi} . \tag{25}$$

This is a very important equation. If initial transformation equations, $\vec{x}(\psi, \theta, \phi, 0)$ are known, this equation determines the transformation equations for all future time, in any field $\vec{A}(\vec{x}, t)$, with an arbitrary choice of canonical coordinates. The freedom of canonical coordinates is the freedom to choose the infinitesimal generating function s. The function $\partial\vec{x}/\partial t$ can be physically interpreted as the velocity of a (ψ, θ, ϕ) point through ordinary space $\vec{x}$.

An even more important equation than the relation between $\partial\vec{x}/\partial t$ and $\partial\vec{A}/\partial t$ is the equation in $(\partial\chi/\partial t\ )_c$, which determines the evolution of the Hamiltonian in canonical coordinates. Clearly $\chi(\psi, \theta, \phi, t)$ is the fundamental function of the theory, since it determines the field line topology at each point in time. To obtain the expression for $(\partial\chi/\partial t)_c$, consider the cross product $(\partial\vec{x}/\partial t) \times \vec{B}$. Using Eq. (24) for $\partial\vec{x}/\partial t$ and the dual relations, Eq. (21), one obtains

$$\frac{\partial\vec{x}}{\partial t} = - J\left(\frac{\partial\psi}{\partial t}\,\vec{\nabla}\theta \times \vec{\nabla}\phi + \frac{\partial\theta}{\partial t}\,\vec{\nabla}\phi \times \vec{\nabla}\psi + \frac{\partial\phi}{\partial t}\,\vec{\nabla}\psi \times \vec{\nabla}\theta\right) \quad . \tag{26}$$

The canonical form for $\vec{B}$, Eq. (1), then implies

$$\frac{\partial\vec{x}}{\partial t} \times \vec{B} = -\left(\frac{\partial\theta}{\partial t} - \frac{\partial\chi}{\partial\psi}\frac{\partial\phi}{\partial t}\right)\vec{\nabla}\psi + \left(\frac{\partial\psi}{\partial t} + \frac{\partial\chi}{\partial\theta}\frac{\partial\phi}{\partial t}\right)\vec{\nabla}\theta - \left(\frac{\partial\chi}{\partial\psi}\frac{\partial\psi}{\partial t} + \frac{\partial\chi}{\partial\theta}\frac{\partial\theta}{\partial t}\right)\vec{\nabla}\phi \quad . \tag{27}$$

The relation between $(\partial\chi/\partial t)_{\vec{x}}$ and $(\partial\chi/\partial t)_c$,

$$\left(\frac{\partial\chi}{\partial t}\right)_{\vec{x}} = \left(\frac{\partial\chi}{\partial t}\right)_c + \frac{\partial\chi}{\partial\psi}\frac{\partial\psi}{\partial t} + \frac{\partial\chi}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial\chi}{\partial\phi}\frac{\partial\phi}{\partial t} , \tag{28}$$

and Eq. (16) for $\partial\vec{A}/\partial t$ imply

$$\frac{\partial\vec{A}}{\partial t} = - \left(\frac{\partial\chi}{\partial t}\right)_c \vec{\nabla}\phi + \frac{\partial\vec{x}}{\partial t} \times \vec{B} + \vec{\nabla}s \ . \tag{29}$$

This equation will prove to be one of the most useful in the paper. Its first application is the evaluation of $(\partial\chi/\partial t)_c$ by dotting $\partial\vec{A}/\partial t$ with $\vec{B}$,

$$\left(\frac{\partial\chi}{\partial t}\right)_c = \frac{1}{\vec{B}\cdot\vec{\nabla}\phi}\left(\vec{B}\cdot\vec{\nabla}s - \vec{B}\cdot\frac{\partial\vec{A}}{\partial t}\right) \quad . \tag{30}$$

Canonical perturbation theory is based on Eq. (30) for $(\partial\chi/\partial t)_c$ and Eq. (25) for $\partial\vec{x}/\partial t$. A related noncanonical perturbation procedure has been given by Cary and Littlejohn.[8] In perturbation theory, the vector potential is written as $\vec{A}(\vec{x},t) = \vec{A}_o(\vec{x}) + t\,\vec{A}_1(x)$. Let us assume that $|\vec{A}_1| \ll |\vec{A}_o|$ and consider only the first order theory. The perturbation procedure starts with the equation for $(\partial\chi/\partial t)_c$, Eq. (30), which in first order can be written as

$$\chi_1 = \frac{1}{\vec{B}_o \cdot \vec{\nabla}\phi} (\vec{B}_o \cdot \vec{\nabla}s - \vec{B}_o \cdot \vec{A}_1) \tag{31}$$

with χ at $t = 1$ equal to $\chi_o + \chi_1$ and $\vec{B}_o = \vec{\nabla} \times \vec{A}_o$. Assuming the unperturbed field is integrable, χ_o can be taken to be a function of ϕ alone and

$$\chi_1 = \frac{\partial s}{\partial \phi} + ɩ \frac{\partial s}{\partial \theta} - \alpha \tag{32}$$

with $ɩ = d\chi_o/d\phi$, the rotational transform, and $\alpha \equiv \vec{A}_1/\vec{B}_o \cdot \vec{\nabla}\phi$. If χ_1, s, and α are Fourier decomposed,

$$\chi_1 = \sum \chi_{nm} \exp[i(n\phi - m\theta)]$$

$$s = i \sum s_{nm} \exp[i(n\phi - m\theta)] \tag{33}$$

$$\alpha = \sum \alpha_{nm} \exp[i(n\phi - m\theta)] \quad ,$$

then the Fourier components are related by simple algebraic equations

$$\chi_{nm} + (n - ɩm)s_{nm} = \alpha_{nm} \quad . \tag{34}$$

In these equations, the Fourier components $\alpha_{nm}(\phi)$ are assumed known as is $ɩ(\phi)$, but the $\chi_{nm}(\phi)$ and the $s_{nm}(\phi)$ components are to be determined. There is considerable freedom in the choice of the χ_{nm} and the s_{nm}, the freedom of canonical transformations. For example, the χ_{nm} can always be chosen so that the s_{nm} are zero, but this choice is not convenient for studying topological changes. Ideally, one would like to make all the χ_{nm} zero so that the field line integrations would be trivial, $d\theta/d\phi = \iota(\psi)$ and $d\psi/d\phi = 0$. However, this cannot be done if for some value of ψ the rotational transform is a rational number, $\iota = N/M$, with α_{NM} nonzero. Clearly, near $\iota = N/M$ the Fourier component χ_{NM} must be chosen to be equal to α_{NM}. That is, the perturbed Hamiltonian has the form

$$\chi = \chi_o(\psi) + \alpha_{NM} \cos(N\phi - M\theta) \quad . \tag{35}$$

This Hamiltonian is well known from the Hamiltonian mechanics literature.[6,17] The perturbation α_{NM} changes the topology of the trajectories over a region in ψ space of width $4|\alpha_{NM}/\iota'|^{1/2}$ with $\iota' = d\iota/d\psi$ evaluated at the rational surface $\iota = N/M$. Such changes in topology are called magnetic islands.

The condition for a topological change can be expressed in a different way. On a rational surface, $\iota = N/M$, each field line closes on itself. The converse is also true. Every constant ψ surface which contains closed field lines is rational. The topology of a surface, which contains closed field lines, is conserved if and only if the loop integral $\oint \vec{A} \cdot d\vec{\ell}$ is identical on each line of the surface with $d\vec{\ell}$ the differential distance along a field line.

Traditionally, magnetic perturbation theory was based on the resonances in $\vec{B}_1 \cdot \vec{\nabla}\psi/\vec{B}_o \cdot \vec{\nabla}\phi$ instead of resonances in α. To clarify the relationship let $\vec{A}_1 = a_\psi \vec{\nabla}\psi + a_\theta \vec{\nabla}\theta + a_\phi \vec{\nabla}\phi$, then

$$\frac{\vec{B}_1 \cdot \vec{\nabla}\psi}{\vec{B}_o \cdot \vec{\nabla}\phi} = \frac{\partial \alpha}{\partial \theta} - \left(\frac{\partial a_\theta}{\partial \phi} - \iota \frac{\partial a_\theta}{\partial \theta}\right) \quad . \tag{36}$$

The resonant terms in the Fourier decomposition of α, α_{NM} with $\iota = N/M$, are therefore simply related to the resonant terms in the Fourier decomposition of $\vec{B}_1 \cdot \vec{\nabla}\psi/\vec{B}_o \cdot \vec{\nabla}\phi$.

To return to first order perturbation theory, let us assume that the χ_{NM} have been chosen to equal the α_{NM} near the rational surfaces $\iota = N/M$. Equation (34) for the Fourier amplitudes χ_{nm} and s_{nm} can be solved for the remaining Fourier components by letting the χ_{nm} be zero and the $s_{nm} = \alpha_{nm}/(n - \iota m)$. This defines a function $s_o(\psi,\theta,\phi)$ and the Hamiltonian $\chi = \chi_H(\psi,\theta,\phi)$. Equation (25) can now be used to obtain the perturbed transformation equations $\vec{x} = \vec{x}_o + \vec{\xi}$ with $\vec{\xi}$ the displacement, a standard notation of magnetohydrodynamics,

$$\vec{\xi}(\psi,\theta,\phi) = \left(\frac{\partial s}{\partial \theta} - \vec{A}_1 \cdot \frac{\partial \vec{x}}{\partial \theta}\right) \frac{\partial \vec{x}}{\partial \psi} - \left(\frac{\partial s}{\partial \psi} - A_1 \cdot \frac{\partial \vec{x}}{\partial \psi}\right) \frac{\partial \vec{x}}{\partial \theta} + \frac{\partial s}{\partial \chi} \frac{\partial \vec{x}}{\partial \phi} \quad . \tag{37}$$

In the equation for $\vec{\xi}$, s is to be viewed as a function of ψ,θ,ϕ,χ with the derivatives evaluated at $\chi = \chi_H(\psi,\theta,\phi)$. The relation between s and s_o is that $s_o(\psi, \theta, \phi) = s(\psi,\theta,\phi,\chi_H)$. As discussed earlier, the χ dependence of s is the freedom to choose the toroidal angle $\phi(\vec{x},t)$ arbitrarily.

As a closing note, consider the relation between the gauge function G and the generating function s. The fact that an arbitrary infinitesimal canonical transformation defines the gauge function G is demonstrated by Eq. (17) which relates s and $(\partial G/\partial t)_{\vec{x}}$. This equation can be written in a simpler form using Eq. (25) for the velocity of the canonical coordinates $\partial \vec{x}/\partial t$. This form is

$$\left(\frac{\partial G}{\partial t}\right)_c = s + \vec{A} \cdot \frac{\partial \vec{x}}{\partial t} \quad . \tag{38}$$

IV. TIME EVOLUTION

The time evolution of a magnetic field is controlled by Faraday's law

$$\frac{\partial \vec{B}}{\partial t} = - c\, \vec{\nabla} \times \vec{E} , \tag{39}$$

and if it is embedded in a plasma, by Ohm's law

$$\vec{E} + \frac{\vec{v}}{c} \times \vec{B} = \vec{R} . \tag{40}$$

The velocity of the plasma is $\vec{v}$, and $\vec{R}$ is the electric field in the rest frame of the plasma. It is customary to approximate $\vec{R}$ by $\eta\vec{j}$ with η the plasma resistivity and $\vec{j}$ the current density. The major result of this section is that Faraday's law and Ohm's law determine the time evolution of the field line Hamiltonian $\chi(\psi, \theta, \phi, t)$, but they do not constrain the time evolution of the transformation equations $\vec{x}(\psi\ \theta\ \phi,\ t)$. The time evolution of the transformation equations is determined by force balance and will be discussed in Sec. V.

Faraday's law can be written in a form which is more easily applied,

$$\frac{\partial \vec{A}}{\partial t} = - c\left(\vec{E} + \vec{\nabla}\Phi\right) \tag{41}$$

with Φ the single-valued electric potential. An immediate consequence is the evolution equation for the poloidal flux function of magnetic fields which remain integrable. Using Eq. (29) for $\partial\vec{A}/\partial t$ and the fact that the volume element $d^3x = (1/\vec{B} \cdot \vec{\nabla}\phi)\, d\psi\, d\theta\, d\phi$, one finds

$$\frac{\partial \chi(\psi,t)}{\partial t} = \frac{c}{(2\pi)^2} \frac{\partial}{\partial \psi} \left(\int \vec{E} \cdot \vec{B}\, d^3x\right) \tag{42}$$

with the volume integral covering the region bounded by a constant ψ surface. This equation coupled with the canonical transformation to helical coordinates, Eq. (14), has been used to study the growth of magnetic islands.[18]

The use of $\partial\vec{A}/\partial t$ from the last section, Eq. (29), Faraday's law, Eq. (41), and Ohm's law, Eq. (40) gives the equation

$$\vec{R} = \frac{1}{c}\left(\frac{\partial\chi}{\partial t}\right)_c \vec{\nabla}\phi + \frac{1}{c}\left(\vec{v} - \frac{\partial\vec{x}}{\partial t}\right) \times \vec{B} - \vec{\nabla}\Phi_c \qquad (43)$$

with $\Phi_c = \Phi + s/c$ the electric potential in the frame of reference of the canonical coordinates. The parallel component

$$\left(\frac{\partial\chi}{\partial t}\right)_c = \frac{c}{\vec{B}\cdot\vec{\nabla}\phi}\left(\vec{B}\cdot\vec{R} - \vec{B}\cdot\vec{\nabla}\,\Phi_c\right) \qquad (44)$$

obviously has the same properties as Eq. (30). For example, if $\vec{B}\cdot\vec{R}$ is zero, or more generally if it is derivable from a single-valued potential, then χ need not evolve as a function of the canonical coordinates. This would mean, of course, that the field line topology would be conserved. Equation (43) has another consequence. Since only the relative velocity, $\vec{v} - \partial\vec{x}/\partial t$, between the plasma and the canonical coordinates is determined, Ohm's law and Faraday's law do not restrict the time evolution of the transformation equations $\vec{x}(\psi, \theta, \phi, t)$. To determine the evolution of the transformation equations, one must add additional physics, namely force balance.

V. ENERGY PRINCIPLE

Plasma equilibrium and stability can be studied by the well-known energy principle.[2,10,11] The energy of the plasma and the magnetic field is

$$W = \int\left(\frac{p}{\gamma-1} + \frac{B^2}{8\pi}\right)d^3x \tag{45}$$

with p the plasma pressure and $\gamma = 5/3$, the adiabatic index. The energy principle states that equilibria are stationary points of properly constrained variations of the energy and ideally stable plasmas are minima of the energy. Here we will show that the ideal energy principle corresponds to finding extrema of the energy by varying the transformation equations $\vec{x}(\psi, \theta, \phi)$ while holding the Hamiltonian $\chi(\psi, \theta, \phi)$ fixed. The ordinary energy principle can be extended to cover tearing modes, Furth's energy principle,[19] by considering variations in the Hamiltonian χ. To evaluate variations in the energy, we will calculate time derivatives. In this context, one should consider the time t an arbitrary parameter as it was viewed in Sec. III.

The most obvious constraint required by the energy principle is that no energy cross the surface which bounds the integration volume. Although the surface terms need to be examined carefully in practical calculations, we will ignore such terms for the sake of brevity.

First, consider the time derivative of the magnetic energy. Using an integration by parts and Ampere's law, $\vec{\nabla} \times \vec{B} = 4\pi\vec{j}/c$, one has

$$\frac{d}{dt} \int \frac{B^2}{8\pi}\, d^3x = \frac{1}{c} \int \vec{j} \cdot \frac{\partial \vec{A}}{\partial t}\, d^3x \quad . \tag{46}$$

The use of Eq. (29) for $\partial\vec{A}/\partial t$ gives the desired result

$$\frac{d}{dt} \int \frac{B^2}{8\pi}\, d^3x = -\int \frac{\partial \vec{x}}{\partial t} \cdot \left(\frac{\vec{j}}{c} \times \vec{B}\right)d^3x - \frac{1}{c} \int \left(\frac{\partial \chi}{\partial t}\right)_c \vec{j} \cdot \vec{\nabla}\phi\, d^3x \quad . \tag{47}$$

The calculation of the change in the plasma energy requires somewhat more effort. Let the plasma pressure p be a function of the density n and the entropy per particle, then

$$\frac{\partial p(n,S)}{\partial n} = \frac{\gamma p}{n} \quad \text{and} \quad \frac{\partial p(n,S)}{\partial S} = (\gamma - 1)p \quad . \tag{48}$$

The change in the density, $(\partial n/\partial t)c$, consists of two parts. The change in the number of particles in an element of canonical coordinate space, $[\partial(nJ)/\partial t)]_c/J$, is one part and the change in the Jacobian

$$\left(\frac{\partial J}{\partial t}\right)_c = J\vec{\nabla} \cdot \frac{\partial \vec{x}}{\partial t} \tag{49}$$

is the other. Consequently, we use the expression

$$\left(\frac{\partial n}{\partial t}\right)_c = \frac{1}{J}\left(\frac{\partial nJ}{\partial t}\right)_c - n\vec{\nabla} \cdot \frac{\partial \vec{x}}{\partial t} \quad . \tag{50}$$

The time derivative of the pressure in canonical coordinate space is therefore

$$\left(\frac{\partial p}{\partial t}\right)_c = -\gamma p\vec{\nabla} \cdot \frac{\partial \vec{x}}{\partial t} + (\gamma - 1)p\left(\frac{\partial \sigma}{\partial t}\right)_c \tag{51}$$

with $(\partial\sigma/\partial t)_c$ defined by

$$\left(\frac{\partial \sigma}{\partial t}\right)_c = \frac{\gamma}{\gamma-1}\frac{1}{nJ}\left(\frac{\partial nJ}{\partial t}\right)_c + \left(\frac{\partial S}{\partial t}\right)_c \quad . \tag{52}$$

The equation

$$\left(\frac{\partial p}{\partial t}\right)_c = \frac{\partial \vec{x}}{\partial t} \cdot \vec{\nabla}p + \left(\frac{\partial p}{\partial t}\right)_{\vec{x}} \tag{53}$$

can be used to determine $(\partial p/\partial t)_{\vec{x}}$. The change in the total energy W is then

$$\frac{dW}{dt} = \int \frac{\partial \vec{x}}{\partial t} \cdot (\vec{\nabla} p - \frac{\vec{j}}{c} \times \vec{B}) d^3x + \int [p(\frac{\partial \sigma}{\partial t})_c - \frac{1}{c}(\frac{\partial \chi}{\partial t})_c \vec{j} \cdot \vec{\nabla}\phi] d^3x \quad . \quad (54)$$

Equation (54) for dW/dt is a much more general variation of the energy than that allowed by the usual ideal energy principle. The ideal energy principle is recovered if the second integral in Eq. (54) for dW/dt vanishes. That is, ideal MHD (magnetohydrodynamics) is recovered if the Hamiltonian χ, the entropy per particle S, and the number of particles, nJ, in a canonical volume element are all fixed functions of the canonical coordinates. In ideal MHD, equilibria correspond to extrema of the energy W under *variation of the transformation equations*. Although this result is expected, it does have an important implication. One can pick an arbitrary Hamiltonian $\chi(\psi, \theta, \phi)$, which may or may not be integrable, and an arbitrary pressure profile $p(\psi, \theta, \phi)$, which may be unrelated to the field line trajectories, and find an equilibrium by varying the transformation equations. By using the constraints of ideal MHD, one can also obtain the standard differential equation for the displacement $\vec{\xi}$, which is used in stability calculations. The displacement $\vec{\xi}$, as mentioned earlier is to be interpreted as $(\partial \vec{x}/\partial t)\delta t$.

The second integral in Eq. (54) for dW/dt also has important implications. The term involving $(\partial\chi/\partial t)_c$ is apparently more important than the term involving $(\partial\sigma/\partial t)_c$. The reason is that the current density $\vec{j}$ can be singular but the pressure cannot. When the energy is minimized with the ideal contraints to find stable equilibria, the current density $\vec{j}$ will contain delta functions on the rational surfaces $\iota = N/M$ except in cases of high symmetry. The presence of these delta functions implies that additional localized resonant terms in the Hamiltonian would lower the energy. This is just a statement that the plasma equilibrium is very sensitive to the weakening of the ideal MHD constraint $(\partial\chi/\partial t)_c = 0$. The presumption is that the plasma will evolve rapidly to a lower energy configuration that will contain magnetic islands.

A three-dimensional equilibrium solver could be based on the canonical coordinates as a generalization of codes (like the Bauer, Betancourt, Garabadian code[20]) that use magnetic coordinates. The advantage of this generalization would be the addition of a capability to study the opening of magnetic islands. Schematically, a canonical coordinate code could start with the pressure and the Hamiltonian as given functions of ψ, $p(\psi)$, and $\chi(\psi)$. The energy would first be minimized preserving the ideal MHD constraints. On ψ surfaces, which are nearly rational and have a large current density, the Hamiltonian χ should then be modified by a small resonant Fourier term, which would reduce the energy and open a magnetic island. The energy should then be reminimized preserving the ideal MHD constraints. The iteration, between minimizing the energy by first varying the transformation equations $\vec{x}(\psi,\theta,\phi)$ using the ideal MHD constraints and then changing a small resonant term in χ, should be continued until a smooth current profile is obtained.

There is a close relation between the opening of islands in three-dimensional equilibria and tearing modes[12,19] in a tokamak. Consider a large aspect ratio tokamak with circular magnetic surfaces which have a minor radius r and a major radius R, $r/R \ll 1$. Suppose that the boundary conditions on this tokamak are perturbed by a magnetic field which has a radial component

$$b_r = \bar{b}_r(r,t)\cos(N\phi - M\theta) \quad . \tag{55}$$

A surface current will be induced at the resonant rational surface $\iota(r_o) = N/M$,

$$\vec{j} \cdot \vec{\nabla}\phi = \frac{c}{4\pi} \frac{r_o}{RM} \left[\frac{\partial \bar{b}_r}{\partial r}\right] \delta(r - r_o) \sin(N\phi - M\theta) \quad , \tag{56}$$

with $[\partial \bar{b}_r/\partial r]$ the jump in the radial derivative of $\bar{b}_r$ across the resonant rational surface. This equation for $\vec{j} \cdot \vec{\nabla}\phi$ is easily derived from the equations for a surface current. The localized current is induced by the change in the Hamiltonian χ which is produced by the resonant radial field b_r. Using Eq. (36), one can show that

$$\left(\frac{\partial \chi}{\partial t}\right)_c = \frac{Rr_o}{M} \frac{\partial \bar{b}_r(r_o,t)}{\partial t} \sin(N\phi - M\theta) \quad . \tag{57}$$

Equation (54) for dW/dt then implies that if an island opens in an equilibrium plasma then

$$\frac{dW}{dt} = -\frac{r_o}{4\pi} \frac{V}{M^2} \left[\frac{\partial \bar{b}_r}{\partial r}\right] \frac{\partial \bar{b}_r}{\partial t} \tag{58}$$

with V the volume inside the resonant rational surface. If one assumes that

$$\Delta' \equiv \frac{1}{\bar{b}_r} \left[\frac{\partial \bar{b}_r}{\partial r}\right] \tag{59}$$

is a constant, which is a good approximation for many tearing modes,

$$\frac{dW}{dt} = -r_o \Delta' \frac{V}{M^2} \frac{\partial}{\partial t} \frac{\bar{b}_r^2(r_o,t)}{8\pi} \quad , \tag{60}$$

which is Furth's energy principle.[19] The energy W is reduced by a tearing mode if Δ' is positive and increased if Δ' is negative. Consequently, the sign of Δ' determines the stability of the plasma.

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FIGURE CAPTIONS

FIG. 1. Canonical Coordinates: The poloidal magnetic flux outside a constant χ surface is $2\pi\chi$. The toroidal magnetic flux inside a constant Φ surface is $2\pi\phi$. However, the constant ϕ and the constant χ surfaces need not be identical. The poloidal angle is θ and the toroidal angle is ϕ.

FIG. 2. Cylindrical Coordinates: The use of cylindrical coordinates for describing a toroidal configuration is illustrated.

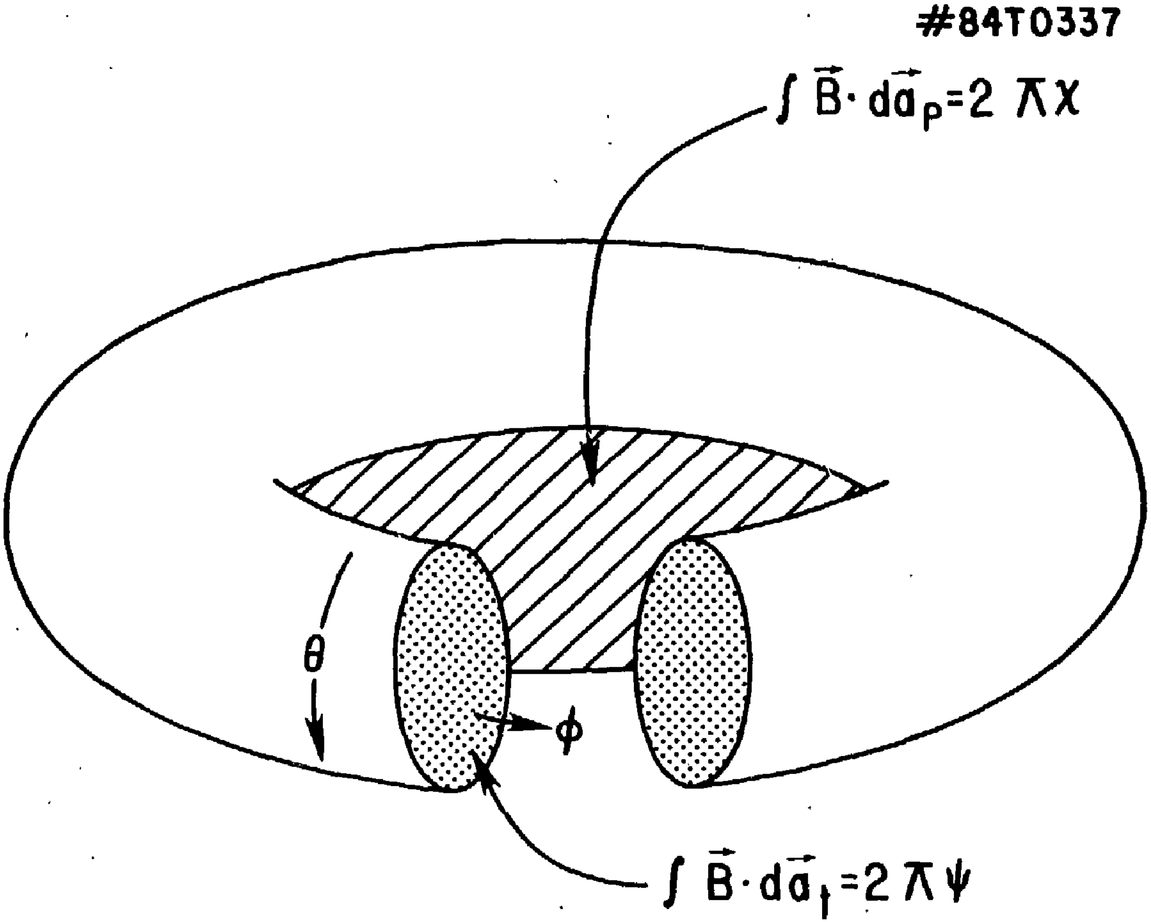

Fig. 1

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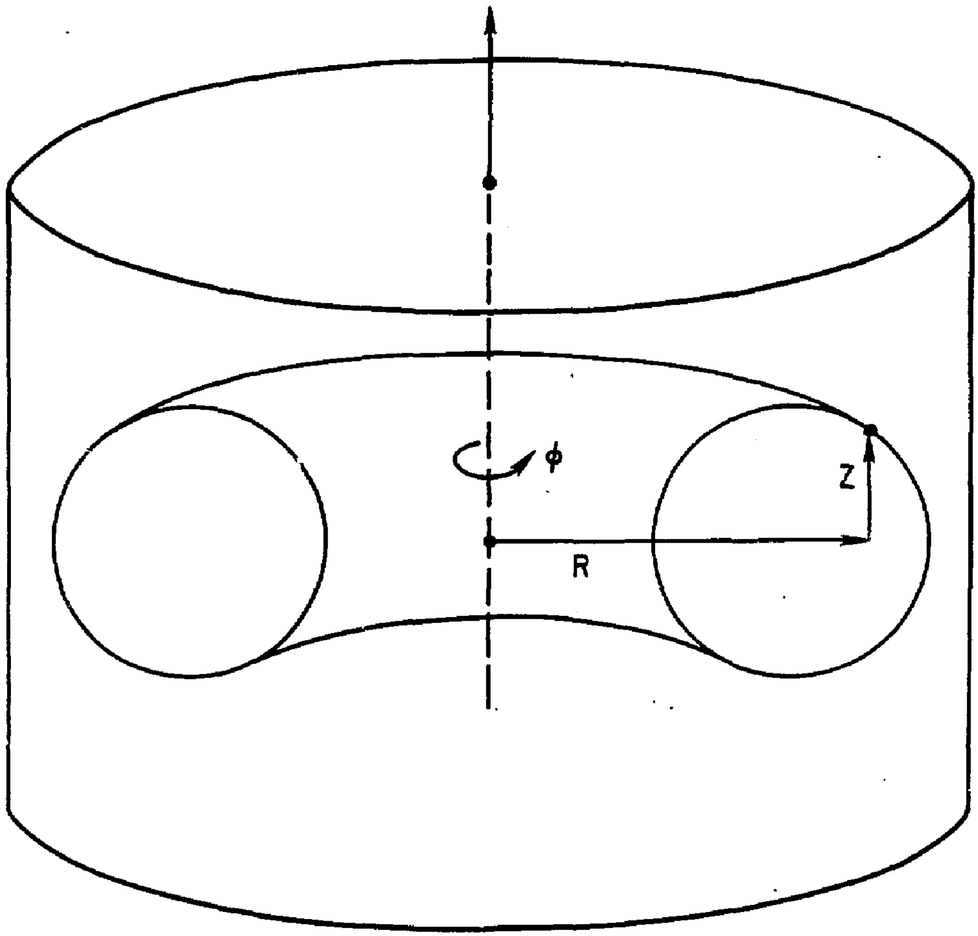

Fig. 2

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