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ABSTRACT

We consider the possibility of describing fermions by inhomogeneous differential forms within the framework of the Ivanenko-Landau-Kahler theory. Invariance properties of the theory in flat and in curved space-time are discussed and the canonical quantization is studied.

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1. INTRODUCTION

Recently much attention has been paid to the investigation of non-standard descriptions of the lower spin fields. An important example is presented by the possibility of representing the massless bosons by the gauge antisymmetric tensor fields (ATF) [1]. However as long ago as the Dirac equation exists, it was recognized that fermions can also be described by the system of ATF. In 1928 Ivanenko and Landau [2] have suggested a relativistic wave equation in terms of ATF for the particle with spin $\frac{1}{2}$ and they have shown that it correctly describes spin effects in an electromagnetic field. Later the mathematician E.Kähler [3] has rediscovered this equation, which is now often called the Dirac-Kähler equation. There exists an extensive literature, devoted to this subject (see refs [4-16] and the bibliography therein).

The aim of this paper is to give some new arguments in favour of the fermion interpretation of the mentioned equation. For the sake of completeness we also rederive some known results and present a critical review of representations, used by different authors. The main purpose for this is to draw attention to certain contradicting conclusions of the previous investigators [5-12]. In other words, we want to clear up the main point: who is right - Plebansky [12], Leonovich [10] and some other Soviet authors, saying that this equation describes a set of bosons, or e.g. Benn and Tucker [6], who insist on the fermion interpretation? In our opinion the second point of view is correct and we make an attempt to reveal the mechanism of such a non-trivial "transmutation" of tensors (i.e. system of ATF) into spinors.

2. CLASSICAL REPRESENTATIONS OF THE IVANENKO-LANDAU-KAHLER EQUATION

(a) Representation by differential forms (or ATF)

Let Φ be the complex inhomogeneous differential form on the four-dimensional flat differentiable manifold,

$$\Phi = \sum_{k=0}^{4} \frac{1}{k!} \varphi_{\mu_1 \ldots \mu_k} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} . \tag{1}$$

The Ivanenko-Landau-Kähler (ILK) equation is

$$\{ i(d-\delta) - m \} \Phi = 0 , \tag{2}$$

where d and δ are, respectively, the exterior differential and co-differential: in the local coordinates we have for the k-form ψ,

$$\psi = \frac{1}{k!} \psi_{\mu_1 \ldots \mu_k} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} ,$$

$$d\psi = \frac{1}{k!} (\partial_{\mu_1} \psi_{\mu_2 \ldots \mu_{k+1}}) dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_{k+1}}$$

$$\delta\psi = -\frac{1}{(k-1)!} (\partial^{\alpha} \psi_{\alpha\mu_1 \ldots \mu_{k-1}}) dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{k-1}} .$$

Originally [2] the equation (2) was written down in components. The inhomogeneous field Φ is then described by the set of ATF of the rank $0,1,\ldots,4$,

$$\Phi = \{ \varphi_{\mu_1 \ldots \mu_k} , \; k = 0, 1, \ldots 4 \} , \tag{3}$$

and the equation (2) takes the following form:

$$i(k\, \partial_{[\mu_1} \varphi_{\mu_2 \ldots \mu_k]} + \partial^{\alpha} \varphi_{\alpha\mu_1 \ldots \mu_k}) - m \varphi_{\mu_1 \ldots \mu_k} = 0 , \quad k = 0, 1, \ldots 4. \tag{4}$$

The set of equations (4) can be derived from the action principle with the Lagrangian

$$L = \sum_{k=0}^{4} \frac{1}{k!} \Big\{ \frac{i}{2} \bar{\varphi}^{\mu_1 \ldots \mu_k} (k\, \partial_{\mu_1} \varphi_{\mu_2 \ldots \mu_k} + \partial^{\alpha} \varphi_{\alpha\mu_1 \ldots \mu_k}) - \frac{i}{2} (k\, \partial_{\mu_1} \bar{\varphi}_{\mu_2 \ldots \mu_k} + \partial^{\alpha} \bar{\varphi}_{\alpha\mu_1 \ldots \mu_k}) \varphi^{\mu_1 \ldots \mu_k} - m \bar{\varphi}_{\mu_1 \ldots \mu_k} \varphi^{\mu_1 \ldots \mu_k} \Big\} . \tag{5}$$

(b) Spinor matrix representation

As was shown earlier [5] there exists a transformation from antisymmetric tensors to the Dirac spinors. It can be formulated as follows

$$\varphi_{\mu_1 \ldots \mu_k} = (-1)^{\frac{k(k-1)}{2}} \, Tr(\psi \Gamma_{\mu_1 \ldots \mu_k}) . \tag{6}$$

Here ψ is the Dirac spinor matrix, i.e. it is the complex 4x4 matrix ψ_{ij} with $i,j = 1,2,3,4$ as the spinor indices.

The Dirac spin-tensors

$$\Gamma_{\mu_1 \ldots \mu_k} = \gamma_{[\mu_1} \gamma_{\mu_2} \ldots \gamma_{\mu_k]} , \quad k = 1, \ldots 4,$$

together with the identity matrix, form the basis of the four-dimensional Dirac algebra, defined by the standard relations

$$\gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} = 2 \eta_{\mu\nu} I ,$$

where I is the 4x4 identity matrix and the Minkowski metric is $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$. For the γ-matrices we use the

Bogolubov-Shirkov [17] conventions:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \; i=1,2,3, \quad \gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

σ^i – are the 2x2 Pauli matrices.

The complex conjugation of (6) defines the conjugated spinor matrix via

$$\bar\varphi_{\mu_1 \dots \mu_k} = Tr\left(\bar\psi \Gamma_{\mu_1 \dots \mu_k}\right), \tag{7}$$

where
$$\bar\psi = \gamma^0 \psi^+ \gamma^0 .$$

Substituting (6)-(7) into (5) we find for the Lagrangian

$$L = 4 Tr\left\{\frac{i}{2}\left(\bar\psi \gamma^\mu \partial_\mu \psi - \partial_\mu \bar\psi \gamma^\mu \psi\right) - m \bar\psi \psi\right\}. \tag{8}$$

This result is easily obtained with the help of the main Fierz identity

$$\delta_{im}\delta_{jn} = \frac{1}{4}\sum_{k=0}^{4} \frac{1}{k!}(-1)^{\frac{k(k-1)}{2}} \left(\Gamma_{\mu_1 \dots \mu_k}\right)_{nm} \left(\Gamma^{\mu_1 \dots \mu_k}\right)_{ij} .$$

In what follows we also use the identities for the products of the Dirac matrices

$$\gamma^\nu \Gamma^{\mu_1 \dots \mu_k} = \Gamma^{\nu \mu_1 \dots \mu_k} + k\, \eta^{\nu[\mu_1} \Gamma^{\mu_2 \dots \mu_k]}, \tag{9 a}$$

$$\Gamma^{\mu_1 \dots \mu_k} \gamma^\nu = \Gamma^{\mu_1 \dots \mu_k \nu} + k\, \Gamma^{[\mu_1 \dots \mu_{k-1}} \eta^{\mu_k]\nu}. \tag{9 b}$$

The corresponding matrix version of the ILK equation is easily derived from (8),

$$i \gamma^\mu \partial_\mu \psi - m \psi = 0 . \tag{10}$$

(c) "Coloured" Dirac spinor representation

Let $E_i^{(\alpha)}$, $\alpha = 1,2,3,4$ be any four linearly independent constant Dirac spinors. Evidently they form the basis for the 4x4 spinor matrices, and hence ψ_{ij} can be decomposed with respect to this basis,

$$\psi_{ij}(x) = \sum_{\alpha=1}^{4} \psi_i^{(\alpha)}(x) \bar E_j^{(\alpha)} \tag{11}$$

where $\bar E = E^+ \gamma^0$ is the usual Dirac conjugated 4-spinor.

Inserting (11) into (8) and (10) we get

$$L = 4 g_{\alpha\beta}\left\{\frac{i}{2}\left(\bar\psi^{(\alpha)}\gamma^\mu \partial_\mu \psi^{(\beta)} - \partial_\mu \bar\psi^{(\alpha)} \gamma^\mu \psi^{(\beta)}\right) - m \bar\psi^{(\alpha)}\psi^{(\beta)}\right\} \tag{12}$$

for the Lagrangian, and

$$i \gamma^\mu \partial_\mu \psi^{(\alpha)} - m \psi^{(\alpha)} = 0 \tag{13}$$

for the ILK equation. Here again $\bar\psi^{(\alpha)} = \psi^{(\alpha)+}\gamma^0$.

Here $g_{\alpha\beta} = Tr\left(\bar E_{(\alpha)} E_\beta\right)$. This quantity plays the role of the metric in the internal "colour" space, and the index α specifies the type of the Dirac four-spinor $\psi^{(\alpha)}$. The representation (12)-(13) is probably the most natural from the point of view of the gauge theory. Note that though (12),(13) and (8),(10) look very similar (replace $j \leftrightarrow \alpha$) these representations are essentially different: the right index j in ψ_{ij} in (8),(10) is not the internal one, it is the Dirac spinor index. Thus one cannot straightforwardly (as proposed e.g. in [5]) identify $\psi_{ij} = \psi_{i(j)}$ as the "set of four types of the Dirac fields", since these different "types" (specified by (j)) would mix up under the usual Lorentz transformations of ATF (3) (see Eq.(17)

below). On the contrary, the index α is indeed the internal index and the fields $\psi^{(\alpha)}$ and $\psi^{(\alpha')}$ with different "colour" $\alpha \neq \alpha'$ do not mix under the external Lorentz transformation.

(d) 16-component vector representation

Let us introduce the 16-component column ψ_A, $A = 1, \dots, 16$ whose elements are just ψ_{ij} and the index A is understood as the pair (i,j). Define now the 16x16 matrices $\hat{\Gamma}_\mu$, $\check{\Gamma}_\nu$, satisfying the relations

$$\hat{\Gamma}_\mu \hat{\Gamma}_\nu + \hat{\Gamma}_\nu \hat{\Gamma}_\mu = 2\eta_{\mu\nu} \ , \ [\hat{\Gamma}_\mu, \check{\Gamma}_\nu] = 0 \ ,$$

as follows

$$(\hat{\Gamma}_\mu)_{AA'} = (\gamma_\mu)_{ii'} \delta_{jj'} \ , \ (\check{\Gamma}_\mu)_{A,A'} = (\gamma_\mu)_{jj'} \delta_{ii'} \ ,$$

where $A = (i,j), A' = (i',j')$.

Then the ILK equation (10) can be rewritten as the 16-component Dirac equation,

$$i \hat{\Gamma}^\mu_{AB} \partial_\mu \psi^B - m \psi_A = 0 \qquad (14)$$

with the evidently defined reducible representation of the Lorentz group in a 16-dimensional vector space.

This form of the ILK equation was studied by Durand [7] and by the Byelorussian group [8].

The representations (a)-(d), which we have briefly described above, are well known in the literature (except probably for the improved "colour" spinor representation (c)). However the fact that they are equivalent is not so well known, since authors working in different representations gave incompatible (either boson or fermion) interpretations for the field under consideration. In this section we have clearly shown that all the above-mentioned representations are equivalent.

2. SYMMETRIES AND CONSERVED CURRENTS

Let us now consider the invariance properties of the ILK theory. It is convenient to discuss this problem without confining oneself to any particular representation (a)-(d), but instead it is useful to combine results, which are most evident in one of them.

We start from the aTF-representation (2),(4),(5). Obviously the ILK equation is invariant under the group of phase transformations,

$$\left.\begin{aligned} \varphi_{\mu_1 \dots \mu_k} &\to e^{i\alpha} \varphi_{\mu_1 \dots \mu_k} \ , \\ \bar{\varphi}_{\mu_1 \dots \mu_k} &\to e^{-i\alpha} \bar{\varphi}_{\mu_1 \dots \mu_k} \ , \end{aligned}\right\} \qquad (15)$$

under translations,

$$x^\mu \to x^\mu + a^\mu \ ,$$

and under the Lorentz group

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \ ,$$

$$\varphi^{\mu_1 \dots \mu_k} \to \varphi'^{\mu_1 \dots \mu_k} = \Lambda^{\mu_1}{}_{\nu_1} \dots \Lambda^{\mu_k}{}_{\nu_k} \varphi^{\nu_1 \dots \nu_k} \ , \qquad (16)$$

where α, a^μ, $\Lambda^\mu{}_\nu$ — are the constant (i.e. global) transformations.

The corresponding Noether conserved currents are most compactly expressed in the spinor matrix notation. One can easily see that the phase group (15) acts on ψ_{ij} in the usual manner $\psi \to \psi e^{i\alpha}$, $\bar\psi \to \bar\psi e^{-i\alpha}$, while the Lorentz transformaton (16) of the spinor matrix is given by

$$\psi \to \psi' = S\psi S^{-1} \quad . \tag{17}$$

where the matrix S realizes the spinor representation of the Lorentz group and is connected with $\Lambda^\mu{}_\nu$ via

$$S^{-1}\gamma^\mu S = \Lambda^\mu_{\cdot\nu}\gamma^\nu \quad . \tag{18}$$

At the same time in the "colour" representation the Lorentz group (16) is described by

$$\psi^{(\alpha)} \to S\psi^{(\alpha)} \;, \quad g_{\alpha\beta} = inv \quad . \tag{19}$$

The invariance of the action of the theory under (15)-(16) yields the conservation of the electric current

$$J^\mu = 4\,Tr(\bar\psi\gamma^\mu\psi) = 4g_{\alpha\beta}(\bar\psi^{(\alpha)}\gamma^\mu\psi^{(\beta)}) , \tag{20}$$

the canonical energy-momentum tensor

$$T^\mu_{\cdot\nu} = 4\cdot\frac{i}{2}Tr\{\bar\psi\gamma^\mu\partial_\nu\psi - \partial_\nu\bar\psi\gamma^\mu\psi\} , \tag{21}$$

and the tensor of the total angular momentum

$$M^\mu_{\cdot\alpha\beta} = T^\mu_{\cdot\alpha}x_\beta - T^\mu_{\cdot\beta}x_\alpha + S^\mu_{\cdot\alpha\beta} \;, \tag{22}$$

which includes the spin, defined in the spinor matrix representation by the formulas

$$S^\mu_{\cdot\alpha\beta} = S^\mu_{(1)\alpha\beta} + S^\mu_{(2)\alpha\beta} \;, \tag{23}$$

$$S_{(1)}{}^\mu{}_{\alpha\beta} = i\,Tr(\bar\psi\{\Gamma_{\alpha\beta}\gamma^\mu + \gamma^\mu\Gamma_{\alpha\beta}\}\psi) \;, \tag{24}$$

$$S_{(2)}{}^\mu{}_{\alpha\beta} = -2i\,Tr(\Gamma_{\alpha\beta}\bar\psi\gamma^\mu\psi) \quad . \tag{25}$$

(here as usual $\Gamma_{\alpha\beta} = \gamma_{[\alpha}\gamma_{\beta]}$).

The first term in (23) has formally the standard structure of the Dirac spin, and the second contribution comes from the fact that ψ_{ij} transforms like the matrix according to (17) and not like the four-spinor $(\psi' = S\psi)$.

It is evident from (8) that the ILK equation possesses an additional internal symmetry group, which is called the "right symmetry" in [6]. Indeed, the transformations

$$\psi \to \psi' = \psi M \,, \quad \bar\psi \to \bar\psi' = M^{-1}\bar\psi \tag{26}$$

leave (8) invariant. In order for the first and the second equations in (26) to be compatible, the matrix M should satisfy

$$\gamma^0 M^+ \gamma^0 = M^{-1} \tag{27}$$

and this means that the matrix M realizes the representation of the conformal group SO(2,4). This is clearly seen from the infinitesimal transformations,

$$M \simeq 1\!\!1 + i\hat K \;, \qquad \delta\psi = i\psi\hat K \;, \tag{28}$$

where from (27) we get explicitly

$$\hat K = \frac{1}{4}\sum_{k=0}^{4}\frac{1}{k!}(-i)^{\frac{k(k-1)}{2}}\alpha_{\mu_1\ldots\mu_k}\Gamma^{\mu_1\ldots\mu_k} \tag{29}$$

and thus the generators are

$$1, \;\; i\gamma_5, \;\; \gamma^\mu\gamma_5, \;\; \gamma^\mu, \;\; i\sigma^{\mu\nu} = i\gamma^{[\mu}\gamma^{\nu]} .$$

$\alpha_{\mu_1 \ldots \mu_k}$ being the constant group parameters.

The Noether theorem tells us that the invariance under the right symmetry group (26)-(29) yields the conservation of additional currents:

a) right γ_5-current

$$ {}_R J_5^{\mu} = 4i \, Tr(\gamma_5 \bar{\Psi} \gamma^{\mu} \Psi) \qquad (30) $$

b) right tensor and axial tensor currents

$$ J^{\mu}_{\cdot \nu} = 4 \, Tr(\gamma_\nu \bar{\Psi} \gamma^{\mu} \Psi), \qquad (31) $$

$$ J_5{}^{\mu}_{\cdot \nu} = 4 \, Tr(\gamma_\nu \gamma_5 \bar{\Psi} \gamma^{\mu} \Psi) \qquad (32) $$

c) right Lorentz current, which coincides with the second part of the total spin (25),

$$ J^{\mu}_{\cdot \alpha\beta} = S^{\mu}_{(2)\alpha\beta} \qquad (33) $$

It is the conservation of the right spin (33), which provides the conservation of the usual Dirac (left) spin (24) and thus makes it possible to interpret the ILK field as the spin $\frac{1}{2}$ fermion.

For completeness we point out that the right symmetry is also present in different representations. In the "Colour" representation it is realized as the standard internal symmetry group:

$$ \Psi'^{(\alpha)} = A^{\alpha}_{\beta} \Psi^{(\beta)}, \qquad (34) $$

where $\quad A^{\alpha}_{\alpha'} A^{\beta}_{\beta'} g_{\alpha\beta} = g_{\alpha'\beta'} \; (= Tr \bar{E}_{\alpha'} E_{\beta'}) .$

For the simplest choice of the basis $E_i^{(\alpha)} = \delta_i^{\alpha}$ we get $g_{\alpha\beta} = \mathrm{diag}(+1,+1,-1,-1)$ and clearly (34) is again a representation of the conformal group.

In the initial ATF representation the right transformations do not have such a simple form. For example the right Lorentz transformations, defined by the parameters $\alpha^{\mu\nu} = -\alpha^{\nu\mu}$, are

$$ \delta\varphi_{\mu_1 \ldots \mu_k} = -\frac{1}{2} \Big[-\alpha^{\rho\sigma} \varphi_{\rho\sigma\mu_1 \ldots \mu_k} + k(k-1)\, \alpha_{[\mu_1 \mu_2} \varphi_{\mu_3 \ldots \mu_k]} + 2\, \alpha_{\rho[\mu_1} \varphi^{\rho}_{\cdot \mu_2 \ldots \mu_k]} \Big], \qquad (35) $$

i.e. k-tensor transforms into the mixture of k- and k$\pm$2-tensors.

3. CANONICAL QUANTIZATION

According to the standard procedure [17] let us define the decomposition of the ILK field into the positive and negative frequency parts.

This is most easily performed in the spinor matrix representation, where 16 linearly independent plane wave solutions of the ILK equation (10) are given by

$$ \psi^{(A)}_{ij} = C^{A}_{ij}(\vec{k})\, e^{-i k^{\mu} x_{\mu}}, \quad A = 1, \ldots, 16, \qquad (36) $$

with $C^{A}_{ij} = \frac{1}{2} v_i^{\nu, \pm}(\vec{k})\, \delta^{\ell}_{j}$, where the index A comprises 16 different combinations of $\ell = 1,2,3,4$, $\nu = 1,2$ and $\pm$. Here $v_i^{\nu,\pm}$ are the usual orthonormalised basis solutions of the standard Dirac equation in the notations of [17], and $\pm$ corresponds to the sign of k^0 and $\nu = 1,2$ describes two spin polarizations.

Now an arbitrary solution of the equation (10) can be decomposed into positive and negative frequency parts,

$$\psi(x) = \psi^+(x) + \psi^-(x) =$$

$$= \frac{1}{2(2\pi)^{3/2}} \int d^3\vec{k} \Big[e^{ikx} \sum_{\nu=1,2} v^{\nu,+}(\vec{k}) \bar{a}_{\nu,}{}^{+}(\vec{k}) + + e^{-ikx} \sum_{\nu=1,2} v^{\nu,-}(\vec{k}) \bar{a}_{\nu,}{}^{-}(\vec{k}) \Big] , \tag{37}$$

where the amplitudes $a_{\nu,}{}^{\pm}(\vec{k})$ are the Dirac spinors (that is different from the standard spinor theory, where the amplitudes are complex numbers). This quantity arises from the decomposition of $\psi^{\pm}_{ij}(\vec{k})$ with respect to the basis (36), i.e.

$$\psi^{\pm}_{ij}(\vec{k}) = \sum_{A=1}^{16} \varphi_A(\vec{k}) C^A_{ij}(\vec{k}) =$$

$$= \sum_{\nu=1,2} \varphi_{\ell,\nu}{}^{\pm}(\vec{k}) v_{i,}{}^{\nu\pm} \delta^{\ell}_{j} = \sum_{\nu=1,2} v_i{}^{\nu,\pm} \varphi_{j,\nu}{}^{\pm}$$

and for convenience we have denoted $\varphi_{j,\nu}{}^{\pm} = \bar{a}_{\nu,}{}^{\pm}$ as the Dirac conjugated spinor.

Hence for the conjugated spinor matrix $\bar{\psi}_{ij}$ we get

$$\bar{\psi} = \bar{\psi}^+ + \bar{\psi}^- =$$

$$= \frac{1}{2(2\pi)^{3/2}} \int d^3\vec{k} \Big[e^{ikx} \sum_{\nu=1,2} a_\nu^+(\vec{k}) \bar{v}^{\nu,+}(\vec{k}) + + e^{-ikx} \sum_{\nu=1,2} a_\nu^-(\vec{k}) \bar{v}^{\nu,-}(\vec{k}) \Big] , \tag{38}$$

and evidently $\bar{a}_\nu{}^{\pm} = \overline{(a_\nu{}^{\mp})}$.

Consequently we can now calculate any dynamical invariant in terms of amplitudes. These are as follows:

a) Energy-momentum vector

$$P_\mu = \int d^3x\, T^o_\mu(x) = \int d^3\vec{k}\, k_\mu \mathrm{Tr}\{ -a_\nu^- \bar{a}_\nu^+ + a_\nu^+ \bar{a}_\nu^- \} \tag{39}$$

b) Projection of the total spin on the axis x^3

$$S^3 = S^3_{(1)} + S^3_{(2)} = \int d^3x\, S^o{}_{12} =$$

$$= \frac{1}{2} \mathrm{Tr} \int d^3\vec{k} \{ a_1^- \bar{a}_1^+ - a_2^- \bar{a}_2^+ + a_1^+ \bar{a}_1^- - a_2^+ \bar{a}_2^- - - \hat{\sigma}_3 (a_1^- \bar{a}_1^+ + a_2^- \bar{a}_2^+ + a_1^+ \bar{a}_1^- + a_2^+ \bar{a}_2^-) \} , \tag{40}$$

where $\hat{\sigma}_3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$ and the first line in (40) represents the left spin, while the second line is the right spin contribution.

c) Electric charge

$$Q = \int d^3x\, J^o = \sum_\nu \int d^3\vec{k}\, \mathrm{Tr}\{ a_\nu^- \bar{a}_\nu^+ + a_\nu^+ \bar{a}_\nu^- \} \tag{41}$$

d) Right γ_5 charge

$$Q_5 = \int d^3x\, {}_R J^o_5 = i \sum_\nu \int d^3\vec{k}\, \mathrm{Tr}\{ (a_\nu^- \bar{a}_\nu^+ + a_\nu^+ \bar{a}_\nu^-) \gamma_5 \} \tag{42}$$

e) Right "vector" charges

$$R_\mu = \int d^3x\, J^o_\mu = \sum_\nu \int d^3\vec{k}\, \mathrm{Tr}\{ \gamma_\mu (a_\nu^- \bar{a}_\nu^+ + a_\nu^+ \bar{a}_\nu^-) \} \tag{43}$$

$$R^5_\mu = \int d^3x\, J^o_{5\mu} = \sum_\nu \int d^3\vec{k}\, \mathrm{Tr}\{ \gamma_\mu \gamma_5 (a_\nu^- \bar{a}_\nu^+ + a_\nu^+ \bar{a}_\nu^-) \} \tag{44}$$

As we see, the sign of the energy P_0 is indefinite, and thus one should quantize the theory according to the Fermi-Dirac rules. So let us impose the anticommutation relations

$$\left. \begin{aligned} \{ \overset{*}{a}_{i,\nu}{}^-(\vec{k}), a_{j,\mu}{}^+(\vec{k}') \}_+ &= \delta_{ij} \delta_{\mu\nu} \delta(\vec{k}-\vec{k}') , \\ \{ a_{i,\nu}{}^-(\vec{k}), \overset{*}{a}_{j,\mu}{}^+(\vec{k}') \}_+ &= \delta_{ij} \delta_{\mu\nu} \delta(\vec{k}-\vec{k}') \end{aligned} \right\} \tag{45}$$

Hence

$$\left.\begin{aligned} \{\bar{a}_{i,\nu}^{-}(\vec{k}), a_{j,\mu}^{+}(\vec{k}')\}_+ &= (\gamma_0)_{ij}\,\delta_{\mu\nu}\,\delta(\vec{k}-\vec{k}') \\ \{a_{i,\nu}^{-}(\vec{k}), \bar{a}_{j,\mu}^{+}(\vec{k}')\}_+ &= (\gamma_0)_{ij}\,\delta_{\mu\nu}\,\delta(\vec{k}-\vec{k}') \end{aligned}\right\} \tag{46}$$

and we get for the commutator function of the ILK field

$$\{\Psi_{ij}(x), \bar{\Psi}_{kl}(y)\}_+ = \frac{1}{2}(\gamma_0)_{jk}\left(i\gamma^\mu\partial_\mu^{(x)}+m\right)D(x-y) \tag{47}$$

where $D(x-y)$ is the Pauli-Jordan scalar function.

One important remark is in order. As we see, for the case $j=k=3,4$ the sign of the commutator function is the wrong one. The same is easily seen from the "colour" representation (12), where formally we have the sum of four types of Dirac fields, two of which ($\alpha=1,2$) enter with the proper sign and two others ($\alpha=3,4$) with an opposite sign (like in the old model of Yokoyama [18]). Hence even the Fermi-Dirac quantization is not sufficient to provide the positive definiteness of the energy. The situation is analogous to that of quantum electrodynamics, where the commutator function for A_ν components has the wrong sign and the longitudinal and time photons give negative contributions to the energy. The general recipe to overcome this difficulty is to use the indefinite metric quantization. We will discuss this problem in a separate publication. Here we only want to point out the source of this phenomenon: the indefiniteness of the energy and the necessity of the indefinite metric quantization are the consequences of the invariance of this theory under the non-compact symmetry group (right conformal group is evidently non-compact). Analogous problems were discussed in the literature (see e.g. [19] and refs. therein), devoted to the quantization

of the gauge fields with non-compact gauge groups. However we see that not only gauge fields, but their spinorial sources (and the ILK field can be the source of the SO(2,4) gauge field after the localisation of the right group) also suffer from the same difficulty, connected with the non-compactness of the symmetry group.

Finally we want to stress that the Bose-Einstein quantization of the ILK field is completely incorrect. Indeed, if we impose, instead of (45), the commutator $[\ ,\ ]_-$ relations on a, a^*, this certainly would not help us to make the energy positive definite, but instead would cause the violation of macroscopic causality since in this case the commutator function of Ψ and $\bar{\Psi}$ would not vanish for $(x-y)^2<0$ as one needs in the local theory.

The last remark about the spin of the ILK field. We see from (40) that the bilinear combinations of creation and annihilation operators contribute into S^3 only with integral coefficients (0, ±1), and hence the total spin is always integral, what is compatible with the tensorial character of ATF. However, the right spin and the left spin are half-integral and they conserve separately. Hence the standard minimal coupling of ATF with the electromagnetic field in the flat Minkowski space-time makes only left spin effects observable. This fact was noticed by many authors starting from Kähler and it is this observation that underlies the fermion interpretation of the ILK field.

4. GRAVITATIONAL INTERACTION

It is generally believed (see e.g. [9], [14]) that the formulation of the ILK theory in a curved space-time does ~~not~~ describe a Dirac-type fermions but a different physical reality. The main argument for this is the supposition that the right symmetry is lost in the presence of the gravitational field. In our opinion this is very strange (even taking into account many other unusual properties of the ILK theory) since the right symmetry is an internal one and it certainly should not depend on the geometry of space-time. In this section we prove that the global right invariance is present also in the curved space-time.

Let us now introduce the interaction of the classical ILK field with the gravitational one. As usual (see e.g. [20]) we assume that the minimal coupling recipe is valid, according to which the Minkowski metric is substituted by the Riemannian one and all the partial derivatives are replaced by the covariant ones,

$$\partial_{[\mu_1}\varphi_{\mu_2\ldots\mu_k]} \to \nabla_{[\mu_1}\varphi_{\mu_2\ldots\mu_k]}\,, \quad \partial^\alpha\varphi_{\alpha\mu_1\ldots\mu_k} \to \nabla^\alpha\varphi_{\alpha\mu_1\ldots\mu_k}\,.$$

Hence in the spinor matrix variables we find

$$\nabla_{[\mu_1}\varphi_{\mu_2\ldots\mu_k]} = (-1)^{\frac{(k-1)(k-2)}{2}}\,Tr\left(\nabla_{[\mu_1}\psi\,\Gamma_{\mu_2\ldots\mu_k]}\right) =$$

$$= (-1)^{\frac{k(k-1)}{2}}\,Tr\left(\Gamma_{[\mu_1\ldots\mu_{k-1}}\nabla_{\mu_k]}\psi\right)\,, \tag{48}$$

where

$$\nabla_\mu\psi = \partial_\mu\psi + [\overset{s}{\Gamma}_\mu, \psi]$$

is the spinor covariant derivative, defined by the spinor connection

$$\overset{s}{\Gamma}_\mu = -\frac{1}{4}\gamma^a\gamma_b\,\Gamma^b_{a\mu}\,.$$

As usual the latter is defined so that the curved Dirac matrices are covariantly constant, $\nabla_\mu\gamma^\nu = 0$, and the local Lorentz connection is defined by the Christoffel symbols $\left\{ {\alpha \atop \beta\mu} \right\}$ as follows.

Analogously

$$\nabla^\alpha\varphi_{\alpha\mu_1\ldots\mu_k} = (-1)^{\frac{k(k+1)}{2}}\,Tr\left(\nabla^\alpha\psi\,\Gamma_{\alpha\mu_1\ldots\mu_k}\right) =$$

$$= (-1)^{\frac{k(k-1)}{2}}\,Tr\left\{\Gamma_{\mu_1\ldots\mu_k\alpha}\nabla^\alpha\psi\right\}\,, \tag{49}$$

and thus finally, making use of (9) , we get

$$k\nabla_{[\mu_1}\varphi_{\mu_2\ldots\mu_k]} + \nabla^\alpha\varphi_{\alpha\mu_1\ldots\mu_k} = (-1)^{\frac{k(k-1)}{2}}\,Tr\left(\Gamma_{\mu_1\ldots\mu_k}\gamma^\alpha\nabla_\alpha\psi\right) \tag{50}$$

Thus the Lagrangian of the ILK field in curved space-time is

$$L = 4Tr\left\{\frac{i}{2}\left(\bar\psi\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^\mu\psi\right) - m\bar\psi\psi\right\} \tag{51}$$

The field equations are then

$$\left\{i\gamma^\mu\nabla_\mu - m\right\}\psi = 0\,. \tag{52}$$

Now one can easily see that the right currents (30-33) are covariantly conserved. Indeed, they have the general form

$$J^\mu_{\mu_1\ldots\mu_k} = 4Tr\left(\Gamma_{\mu_1\ldots\mu_k}\bar\psi\gamma^\mu\psi\right),$$

and hence $\nabla_\mu J^\mu_{\mu_1\ldots\mu_k} = 0$ due to the field equations (52) and covariant constancy of the Dirac matrices. These conservations laws are the consequence of the right symmetry (28), (29), where one should only remember that $\alpha_{\mu_1\ldots\mu_k}$ are the constant scalars, labelled by the indices μ_i (and not tensors). Hence then the matrix $\mathcal{M}$ in (28) is covariantly constant and thus the right group commutes with the Dirac operator in the curved space-time.

5. CONCLUSION

In this paper we have discussed the description of fermions by the system of ATF within the framework of the Ivanenko-Landau-Kähler theory. We have reviewed the alternative representations of this theory and have shown that it is invariant under the right conformal group both in flat and in curved space-time. The canonical quantization was considered and we argue that the Dirac-Fermi quantization is correct, though one should use the indefinite metric formulation in view of non-compactness of the right symmetry group.

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