Inhomogeneous Cosmological Models: New Exact Solutions
With Dust, Isotropic Radiation
And Electromagnetic Field*

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Abstract

In this paper we introduce isotropic pressure radiation and an electromagnetic field in the irrotational dust models considered by Szekeres's. New classes of inhomogeneous cosmological models are obtained. A particular class of solutions generalizes ones investigated by Doroshkevich, another class is also a special case of Pollock-Caderni models.

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§ (1): Introduction

There is some observational evidence that a uniform magnetic field of strength $H < 10^{-9}$ gauss, probably primordial in origin, exists on a large-scale [1-5]. Such an hypothesis was first proposed by Hoyle [6] and has been considered in order to explain the origin of the galactic magnetic field.

The theory of the "magnetic universes" has been developed by several authors: in particular, Zel'dovich [7] introduced it qualitatively in the context of the anisotropic homogeneous cosmological models, and subsequently Doroshkevich [8] presented a number of exact solutions of these models. These were also reproduced by Shikin [9] and independently by Thorne [10].

In the past few years there has been a growing interest in cosmological models which are not spatially homogeneous. In particular Szekeres [11] examined a class of inhomogeneous dust models which are divided into two classes. The class II solutions are usually considered as generalizations of the Kantowski-Sachs and Friedman-Robertson-Walker solutions and have primarily been studied as cosmological models [12]. Generalizations of Szekeres's solutions to include pressure are also available in the literature [13-15]. Among these is a paper by Pollock and Caderni [15] who introduced isotropic radiation into the spacetime, in addition to the irrotational dust considered by Szekeres's solutions, class II. They argue that the universe contains not only matter, but also radiation, the latter having been dominant in a past epoch.

In the following we consider the next generalization in which,
in addition to the isotropic pressure radiation, we introduce a parallel magnetic and electric field into the space-time as well as irrotational dust considered by Szekeres. For the sake of completeness the electric field was introduced, although only evidence of a magnetic field on the large-scale exists.

§ (2): The Models

The metric in all cases is

$$ds^2 = dt^2 - Q^2 dz^2 - R^2 (dy^2 + h^2 dz^2)$$  \hspace{1cm} (2.1)

where $Q = Q(x,y,z,t)$, $R = R(t)$ and $h = h(y)$ are functions to be determined.

The models are solutions of Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu}$$  \hspace{1cm} (2.2)

$R_{\mu\nu}$ being the Ricci tensor, $T_{\mu\nu}$ is the energy-momentum tensor, here, assumed to represent a non-interacting mixture of isotropic radiation and non-rotational dust, and Maxwell sources, namely

$$T_{\mu\nu} = \left( \rho_d + \frac{4}{3} \rho_r \right) U_\mu U_\nu - \frac{4}{3} \rho_r g_{\mu\nu} + \mathcal{F}_{\mu\nu}$$  \hspace{1cm} (2.3)

where

$$\mathcal{F}_{\mu\nu} = -\frac{4}{4\pi} \left( F_{\mu\sigma} F_{\nu}^{\sigma} - \frac{4}{3} g_{\mu\nu} F_{\sigma\tau} F^{\sigma\tau} \right).$$  \hspace{1cm} (2.4)

$U_\nu$ is the unit four-velocity vector, $\rho_d(x^\alpha)$ and $\rho_r(t)$ are the dust and radiation densities, $\rho_r = \frac{1}{3} \rho^\alpha$ is the radiation pressure and the units are such that both the velocity of light and the constant of gravitation are equal to one. The co-ordi-
The coordinates used are comoving so, \( \mathbf{u} = 0. \)

\( F_{\mu\nu} \) is the Maxwell tensor which satisfies the equations

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} F^{\mu\nu}) = 0
\]

and

\[
F_{[\mu\nu\zeta]} = 0
\]

The magnetic field is given by the component \( F_{32} = H(x,y,z,t) \) and the electric field by the component \( F_{01} = E(x,y,z,t) \). This corresponds to electric and magnetic fields directed along the \( x \)-axis. Therefore, the only non-zero components of \( T_{\mu\nu} \) are

\[
T_{oo} = T_{11} = -T_{22} = -T_{33} = \frac{H}{8\pi k^2 a^2} + \frac{E^2}{8\pi a^2}
\]

The field equations (2.2) for the line element (2.1) are given in the appendix A.

To solve these equations we shall assume that the radiation energy density is given by the relation

\[
\rho_\gamma = \frac{3c^2}{8\pi} R^{-4}, \quad \zeta^2 > 0
\]

The reasons for this choice are two fold. On the one hand \( \rho_\gamma(t) \) and \( R(t) \) are the unique time-dependent functions for the models considered and a functional relation among them must exist. On the other hand it was suggested by simplicity. Thus, the equation for \( R(t) \) takes the same form as the one obtained for Friedmann models with pressure \( p = \frac{1}{3} \rho \) and, with this choice for \( \rho_\gamma \), the models under consideration (quasi-Euclidean case) tend to homogeneity and isotropy as \( t \rightarrow \infty \).

Equations (A.5) to (A.7) are easily integrated and their most general solution for \( Q \) is:
where \( p \) and \( q \) are arbitrary functions of the argument stated.

The integration of (2.5) gives, for the electromagnetic field, the following expressions:

\[
Q = R(t) \left[ h(y) p(x, z) + q(x, y) \right] + S(x, t) \tag{2.9}
\]

The integration of (2.5) gives, for the electromagnetic field, the following expressions:

\[
F^{-23} = \frac{m(l, x)}{\sqrt{-q}} \tag{2.10}
\]

and

\[
F^{-01} = \frac{\eta(y, z)}{\sqrt{-q}} \tag{2.11}
\]

where \( \eta \) and \( \eta \) are arbitrary functions of integration. Equation (2.10) and (2.11) must satisfy also (2.6). This gives for \( Q \) the following forms,

\[
Q = \frac{R^2 \eta(l, x)}{f(y, z)} \tag{2.12a}
\]

and

\[
Q = \frac{\bar{R}(l, x)}{\bar{m}(y, z)} \tag{2.12b}
\]

where \( \bar{\eta} \) and \( \bar{\ell} \) are new arbitrary functions of integration such that

\[
F_{\alpha\beta} = - \bar{\ell} R^{-2} \tag{2.13}
\]

and

\[
F_{23} = \bar{\eta} \, h \tag{2.14}
\]

Equations (2.12a) and (2.12b) must be compatible with (2.9) in order to obtain a unique solution for the metric (2.1). Comparing these equations and taking into account equation (A.1) we arrive at the following result for \( Q \):

a) \( Q = R(t) \) and \( \bar{\ell} = \bar{\ell} \), \( \bar{\ell} \equiv 1 \), \( \bar{\ell} = 1 \) \tag{2.15}
b) $Q = Q(y)$ and $R = 1$

c) $Q = Q(t,x)$.

Case (a) gives the line element of the Friedmann models with Euclidean section and the field equations are incompatible unless the electromagnetic field is absent. In the case (b) the solution is a static one. We abandon it, because we are only interested in time dependent models.

Analyzing the case (c) that is $Q = Q(t,x)$, it is easily seen from equation (2.12a) that $R(t)$ can be absorbed into $\bar{m}(t,x)$ and $f(y,z)$ must be a constant, namely $f(y,z) = H_0$. The same procedure is applied to equation (2.12b) to give $\bar{m} = E_0 H(y)$. $E_0$ and $H_0$ are constants associated with the intensities of the electric and magnetic fields respectively.

We now turn to the Maxwell stress-energy tensor (2.4); this should be expressed in terms of the solutions for $F_{\mu\nu}$ found from equations (2.5) and (2.6) and the resulting equations (2.13), (2.14) and (2.12b). Thus, equation (2.7) can be rewritten as

$$\mathcal{Z}_0 = \mathcal{Z}_t = - \mathcal{Z}_x = - \mathcal{Z}_y = \frac{m^2}{8\pi R^3} \tag{2.16}$$

where $m^2 = \frac{h}{E_0^2}$.

There are three remaining fields equations which reduce to:

$$-8\pi R^2 Q^{-2} T_{\delta\mu} = (m^2 - \varepsilon^2) R^{-2} = 2R \dddot{R} + \ddot{R}^2 - h_{22} R^{-2} \tag{2.17}$$

$$-8\pi Q R^{-2} \dot{h}_{22} = -8\pi Q R^{-2} h^{-2} T_{22} = -(m^2 + \varepsilon) QR^{-3} \ddot{Q} + \dddot{Q} + \dddot{R} + Q \dot{R} \tag{2.18}$$

$$-8\pi Q R^2 T_{00} = -m^2 Q R^{-2} - 8\pi Q R^2 p_m = -QR^2 - 2QR \dddot{R} + h_{22} R^{-4} Q \tag{2.19}$$
Equation (2.19) simply determines \( \rho_m = \rho_1 + \rho_n \).

It is evident from the above equations, that the electromagnetic field acts like a "positive pressure" in the directions transverse to the fields, and like "negative pressure" along the direction of the field. It can also be seen, from the mathematical point of view, that the field equations are not affected if we consider just one of the fields. So, in the following, only the magnetic field will be considered, since it is the only one which is detected on a large-scale.

Equation (2.17) can be rewritten as

\[
\rho^{-1} h_{22} = 2 R \ddot{R} + \dot{R}^2 - (m^2 - \zeta^2) R^{-2}
\]

(2.20)

Since \( R \) is a function of \( t \) only and \( h \) is a function of \( y \) only, equation (2.20) gives

\[
\rho^{-1} h_{22} = -K
\]

(2.21)

and

\[
2 R \ddot{R} + \dot{R}^2 - (m^2 - \zeta^2) R^{-2} = -K
\]

(2.22)

where \( K = 0 \), or \( \pm 1 \). \( R \) satisfies a Friedmann type equation.

In this paper, just the quasi-Euclidean case \( (K = 0) \) will be considered because of its simplicity. In this case the most general solution of (2.21) is

\[ h = ay + b \]

where \( a \) and \( b \) are constants.

It is easy to show that the co-ordinate transformations

\[ y' = (y + b/a) \cos \alpha \]

and

\[ z' = (y + b/a) \sin \alpha \]

will yield \( h = 1 \).

Equation (2.22) is easily integrated and its most general solution is
\[ R(\eta) = \frac{l}{4} \eta^2 + \frac{m^2 - \xi^2}{l} \]  

(2.23)

\[ t(\eta) = \frac{l}{12} \eta^3 + \frac{m^2 - \xi^2}{l} \eta \]  

(2.24)

The integration constants of \( R \) and \( t \) are chosen such that for \( \eta = 0 \), \( t = 0 \) and \( R \) is minimum.

We now turn to the last field equation in the form (2.18). Substituting (2.23), (2.24) in (2.18), we find that the field equation for \( Q \) is

\[ Q'' - Q \left[ \frac{\frac{1}{4} \eta^2 - \frac{1}{2} (3m^2 + \xi^2)}{\frac{1}{4} \eta^2 + (m^2 - \xi^2)} \right] = 0 ; \quad Q'' = \frac{d^2 Q}{d\eta^2} \]  

(2.25)

Depending on the magnitude of the ratio \( \frac{m^2}{\xi^2} \), distinct classes of solutions are obtained. We begin with the case \( \frac{m^2}{\xi^2} \neq 1 \).

With the introduction of a new variable \( \gamma \) defined by

\[ \gamma \equiv R = \frac{l}{4} \eta^2 + \frac{m^2 - \xi^2}{l} \]

equation (2.25) takes the standard form, namely

\[ \gamma^2 \frac{d}{d\gamma} \left[ \frac{1}{2} \gamma^2 Q_{\gamma} - Q \left[ \frac{1}{2} (\gamma - d) - \frac{(3m^2 + \xi^2)}{d} \right] \right] = 0 \]

(2.26)

where \( \alpha = \frac{l}{4} \), \( d = \frac{m^2 - \xi^2}{l} \) and \( Q_{\gamma} = \frac{dQ}{d\gamma} \).

Equation (2.26) has two linearly independent solutions that can be expressed in terms of the Gauss's hypergeometric functions. The solutions depending on the parameter \( \alpha \) defined by

\[ \alpha = \left( 1 + \frac{3m^2}{m^2 - \xi^2} \right)^{\frac{1}{2}} \]

which is known from the given value of \( \frac{m^2}{\xi^2} \). The parameter \( \alpha \) assumes positive values and it can be pure imaginary for
\[ m^2 < c^2 < 1 \text{m}^2 \]. For \( m^2 = 0 \) (Solutions without magnetic field), \( n = 1 \). For \( c^2 = 0 \) (Solutions without pressure), \( \alpha = 3 \).

As \( \frac{m^2}{c^2} > 0 \), \( \alpha \) must be excluded from the interval \( 1 < \alpha < 3 \).

We now consider the case \( \alpha \notin \mathbb{Z}^+, \alpha \notin \{1, 3\} \). The two solutions of equation (2.25) are

\[ Q_{A} = A(x) \binom{(4+n)}{2} F \left( 1 + \frac{\alpha}{2}, -\frac{1}{2} + \frac{\alpha}{2}; 1 + \alpha; \frac{1}{4} \right) \quad (2.27) \]

and

\[ Q_{B} = B(x) \binom{-2-\alpha}{2} F \left( 1 - \frac{\alpha}{2}, -\frac{1}{2} - \frac{\alpha}{2}; 1 - \alpha; \frac{1}{4} \right) \quad (2.28) \]

The above hypergeometric functions may be reduced to the following elementary functions of \( \eta \):

\[ Q_{A} = A(x) \left( \frac{6}{4} \eta^2 + \frac{6 \text{m}^2}{l(\alpha^2-1)} \right) \binom{(4+n)}{2} \left\{ \left( \frac{4(4-n^2)}{4 \sqrt{2} m} \eta \right) (1 + \frac{4(4-n^2)}{4 \sqrt{2} m} \eta)^{\alpha} \right\} + \frac{n}{2} \left[ \frac{4(4-n^2)}{32 \text{m}^2} \eta^2 + 1 \right] (1 + \frac{4(4-n^2)}{4 \sqrt{2} m} \eta)^{-\alpha-1} \right\} \quad (2.29) \]

\[ Q_{B} = B(x) \left( \frac{6}{4} \eta^2 + \frac{8 \text{m}^2}{l(\alpha^2-1)} \right) \binom{(4-n)}{2} \left\{ \left( \frac{4(4-n^2)}{4 \sqrt{2} m} \eta \right) (1 + \frac{4(4-n^2)}{4 \sqrt{2} m} \eta)^{-\alpha} \right\} - \frac{n}{2} \left[ \frac{4(4-n^2)}{32 \text{m}^2} \eta^2 + 1 \right] (1 + \frac{4(4-n^2)}{4 \sqrt{2} m} \eta)^{\alpha-1} \right\} \quad (2.30) \]

We can see from the above expressions that for \( \alpha > 1 \) the solutions are complex functions. In this case we must consider only the real part of them.

Next for \( \alpha = 0, 1, 3, 4, \ldots \) the hypergeometric functions are degenerate and must be analyzed separately for each case.

a) \( \alpha = 0 \), \( (c^2 = 9 \text{m}^2) \).
We have
\[ Q_A = A(x)^{-\frac{i}{3}} \left( 1 - \frac{3}{d} \right)^{\frac{i}{3}} F \left( 0, \frac{3}{2}; 1; \frac{7}{d} \right) \] (2.31)

and
\[ Q_B = B(x)^{-\frac{i}{3}} F \left( 1, 1; \frac{5}{2}; \frac{1}{d} \right) \] (2.32)

The elementary expressions for the above hypergeometric functions are:
\[ Q_A = A(x) \eta \left( a^2 \eta^2 - 1 \right)^{\frac{i}{2}} \] (2.33)
\[ Q_B = B(x) \left[ a \eta \left( a^2 \eta^2 - 1 \right)^{\frac{i}{2}} \arcsinh \left( a^2 \eta^2 - 1 \right)^{\frac{i}{2}} - \left( a^2 \eta^2 - 1 \right)^{\frac{i}{2}} \right] \] (2.34)

where \[ \eta > \frac{\sqrt{32} \cdot m}{l} \]

We must point out why \[ \eta > \frac{\sqrt{32} \cdot m}{l} \]. If \[ \eta \] were less than this value no solution in the real domain can be obtained, and if \[ \eta = \frac{\sqrt{32} \cdot m}{l} \] then, \[ R = 0 \] and the energy density diverges.

b) \[ \alpha = 1, \ \ (m^2 = 0) \]

The solutions in this case are:
\[ Q_A = A(x)^{-1} F \left( 0, \frac{3}{2}; 1; \frac{7}{d} \right) \] (2.35)

and
\[ Q_B = B(x)^{-1} \left( 1 - \frac{3}{d} \right)^{\frac{1}{2}} F \left( 1, 2; \frac{5}{2}; \frac{1}{d} \right) \] (2.36)

which may be written in terms of the elementary functions as
\[ Q_A = A(x) \left( \frac{\ell^2}{4 \xi^2} \eta^2 - 1 \right) \] (2.37)

and
\[ Q_B = B(x) \left[ \left( \frac{\ell^2}{4 \xi^2} \eta^2 - 1 \right) \arcsinh \left( \frac{\ell^2}{4 \xi^2} \eta^2 - 1 \right)^{\frac{i}{2}} - \frac{6}{2 \eta} \eta \right] \] (2.38)

where \[ \eta > \frac{2 \ell}{\xi} \].

The reasons that \[ \eta \] must be taken greater than \[ \frac{2 \ell}{\xi} \] are the same mentioned for the case (a).
We list in the appendix B the solutions for the cases when 
\( \alpha = 4, 6, 8 \), and \( \alpha = 3, 5, \ldots \).

Of particular interest are the solutions for \( \alpha = 3 \). In this case we obtain a simple inhomogeneous generalization of one class of solutions obtained for the first time by Doroshkevich. The solutions, then, are:

\[
Q_A = A(\alpha) \gamma^{-\alpha} F(-2, -\frac{\alpha}{2}; \alpha; \gamma/\alpha) \tag{2.39}
\]

and

\[
Q_B = B(\alpha) \gamma^{-\alpha} \left( 1 - \frac{\alpha}{\gamma} \right)^{\frac{\alpha}{2}} F(-\frac{\alpha}{2}, 0; \frac{\alpha}{2}; (1 - \frac{\alpha}{\gamma})^2) \tag{2.40}
\]

which, in terms of the elementary functions, reduces to

\[
Q(\gamma, \alpha) = A(\alpha) \frac{\eta^{\alpha} + 2 \eta^2 \gamma^2 - 4 \eta^4}{\eta^{\alpha} + 4 \eta^2} + B(\alpha) \frac{\eta^2}{\eta^{\alpha} + 4 \eta^2} \tag{2.41}
\]

and the parametric equations for \( t(\eta) \) and \( R(\eta) \) with \( \eta^2 = 0 \), are:

\[
R(\eta) = \frac{1}{4} \eta^2 + \frac{m^2}{2} \tag{2.42}
\]

and

\[
t(\eta) = \frac{1}{72} \eta^2 + \frac{m^2}{2} \eta \tag{2.43}
\]

Choosing \( A(x) \) and \( B(x) \) as constants, one obtained the spacial homogeneous models of Doroshkevich (quasi-Euclidean case). For \( A(x) = 0 \) and \( B(x) = \text{cte.} \) one finds a particular case of Rosen's models \([16]\). Also for the particular case \( m^2 = 0 \) we have plane-symmetric models for dust; such models were studied by Tomimura \([17]\).

Finally, we present the solution for the case \( \frac{m^2}{\eta^2} = 1 \).

Equation (2.44) is written, then, as

\[
\eta^2 Q'' + \left( \frac{32}{k^2} \eta^{-2} - 2 \right) Q = 0 \tag{2.44}
\]
The solution of this equation is

$$Q(\eta, x) = A(\xi) [a_1 \cos(\alpha \eta) - \eta^2 \sin(\alpha \eta)] + B(\xi) [a_2 \sin(\alpha \eta) + \eta^2 \cos(\alpha \eta)],$$

where \( a = \frac{32 m^2}{A} \). \hspace{1cm} (2.45)

There is no loss of generality if the constant of integration \( A \) is chosen to be \( 4/9 \). For this choice \( R \) can be expressed easily in terms of \( t \),

$$R = t^{2/3}$$

and \( Q(n, x) \) reduces to the following expression in terms of \( t \).

$$Q(t, x) = A(x) t^{\frac{2}{3}} \left[ a_1 t^{-\frac{1}{3}} \cos \left( \frac{2}{3} t^{\frac{1}{3}} \right) - 3 \sin \left( \frac{4}{3} t^{\frac{1}{3}} \right) \right] +$$

$$+ B(x) t^{\frac{2}{3}} \left[ a_2 t^{-\frac{1}{3}} \sin \left( \frac{2}{3} t^{\frac{1}{3}} \right) + 3 \cos \left( \frac{4}{3} t^{\frac{1}{3}} \right) \right].$$ \hspace{1cm} (2.47)

The expression for the density \( \rho_m = \rho_d + \rho_m \), given by (2.19), can be rewritten in terms of the variable \( \eta \) as

$$8\pi\rho_m = \frac{R'}{R^2} + \frac{2Q'R'}{Q'R^2} - \frac{m^2}{R^4},$$ \hspace{1cm} (2.48)

where the prime denotes derivatives with respect to \( \eta \).

§ (3): General Features of the Models

The models studied so far are those of Szekeres's class II. The energy-momentum tensor is that of a perfect fluid consisting of a mixture of dust and an isotropic radiation in the presence of an electromagnetic field parallel to the \( x \)-axis. The components of the fluid do not interact.

A non stationary solution is obtained only if \( Q = Q(t, x) \), which means that the space-time has gained symmetry. Indeed, the Killing equations were integrated to show that the space-time
presents a 3-parametric group of isometries whose orbits are the bidimensional space-like surfaces \( \{ \tau = \text{cte.}, \alpha = \text{cte.} \} \). The solutions under study exhibit local rotational symmetry and are of type IIa in the Stewart-Ellis classification [18].

The general solution for the case \( h = 1 \) (or \( K = 0 \)) depends on one arbitrary constant, one parameter \( (\gamma^2 / \zeta^2) \) that determines the classes of solutions obtained and two arbitrary functions of one variable. By a transformation in \( x \), one of the two functions, \( A(x) \) or \( B(x) \), can be given some suitable assigned value, say \( \pm 1 \), leaving only one of them with physical significance.

One class of solutions found is a simple generalization of the ones investigated by Doroslkevich, another class is also a special case of Pollock-Caderni models. The temporal evolution of the models is studied to find that, for \( t \) tends to infinity, the models become homogeneous and isotropic. In particular, we study the asymptotic behaviour of the solutions near the singularity to conclude that the magnetic field, in fact, alters the type of singularity, and for the case studied it diminishes the anisotropy of the model at the initial stage. The solutions are geodetics, rotation free and have Petrov classification type D.
Appendix A

The field equations (2.2) for the Szekeres's line element (2.1) with $T^\mu_\nu$ given by (2.3) are:

- $-8\pi Q^{-2} R^2 T_{11} = 2Q \ddot{R} + \dddot{R} - h_{22} \dot{h}^{-1}$, \hspace{1cm} (A.1)

- $-8\pi Q R^{-1} T_{22} = Q \ddot{R} + \dot{Q} \ddot{R} + Q \dddot{R} - h^{-2} R^{-1} (Q_{33} + h_{22} Q) + Q_{22}$, \hspace{1cm} (A.2)

- $-8\pi Q h^{-2} R^{-1} T_{33} = Q \ddot{R} + \dot{Q} \ddot{R} + Q \dddot{R} - R^{-1} Q_{22}$, \hspace{1cm} (A.3)

- $-8\pi Q R^2 T_{00} = - Q \ddot{R}^2 - 2 \dddot{Q} \dot{R} + Q_{22} + h^{-2} (Q_{33} + h_{22} Q^2 + 2 Q_{23})$, \hspace{1cm} (A.4)

- $-8\pi Q T_{23} = Q_{23} - h^{-1} h_{2} Q_{3} = 0$, \hspace{1cm} (A.5)

- $-8\pi Q T_{20} = \dot{Q}_{2} - Q_{2} R^{-1} \ddot{R} = 0$, \hspace{1cm} (A.6)

- $-8\pi Q T_{30} = \dot{Q}_{3} - Q_{3} R^{-1} \ddot{R} = 0$, \hspace{1cm} (A.7)

Differentiation with respect to time is denoted by a dot: \hspace{1cm} \dot{R} = \frac{\partial R}{\partial t}.

Differentiation with respect to the parameters $y$ and $z$ are denoted by $Q_{2}$, $Q_{3}$ respectively.
Appendix B

For $a = 4, 6, 8, \ldots$, we have

$$Q_a = A^a \left( \frac{1-a}{2} \right) F \left( \frac{1}{2} - \frac{a}{2}, -\frac{a}{2}, 1; \gamma, \gamma \right)$$  \hspace{1cm} (B.1)

and

$$Q_b = B^a \left( 1 - \frac{1-a}{2} \right) \frac{1}{\gamma} F \left( -\frac{1}{2} - \frac{a}{2}, -\frac{a}{2}, \frac{1}{2}, \frac{a}{2} \right)^{1/2}.$$  \hspace{1cm} (B.2)

The hypergeometric function in equation (B.1) can be reduced if we use the following relation [19]

$$F(-m, b; -m - \ell; 2) = \sum_{\eta=0}^{\infty} \frac{(-m)_m (b)_m}{(-m - \ell)_m \eta!} \frac{2^n}{n},$$  \hspace{1cm} (B.3)

where $m$ and $\ell$ are non-negative integers, $(b)_m = \frac{\Gamma(b + m)}{\Gamma(b)}$ and $\Gamma(z)$ are the gamma functions.

It is easily shown, with the aid of the formulas [19]

$$F(a, a + \frac{1}{2}, \frac{1}{2}; 2) = \frac{1}{2} \left( 1 + 2^{1/2} \right)^{-2a} + \frac{1}{2} \left( 1 - 2^{1/2} \right)^{-2a},$$  \hspace{1cm} (B.4)

and

$$\frac{\partial}{\partial z} \left[ 2^{c-1} F(a, b; c; z) \right] = (c-\gamma) \frac{2^{c-\gamma-1}}{c-\gamma} F(a, b; c-\gamma, z),$$  \hspace{1cm} (B.5)

that

$$F(0, -\frac{1}{2}; -\frac{1}{2}; 2) = 1.$$  \hspace{1cm} (B.6)

Using the above relation and applying twice the following differentiation formula [19]

$$\frac{\partial}{\partial z} \left[ 2^{c-a-n-1} (1-2)^{a+b-c} F(a, b; c; z) \right] =$$

$$= (c-a) \frac{2^{c-a-1}}{c-a} (1-2)^{a+b-c-n} F(a-n, b; c; z),$$  \hspace{1cm} (B.7)
one can reduce to elementary functions the hypergeometric in (B.2)

Now for \( a = 3, 5, 7, \ldots \) we have

\[
Q_A = A(n) \left( 1 - \frac{x}{4} \right)^{\frac{n-1}{2}} F\left(1 - \frac{x}{2}, -\frac{1}{2} - \frac{x}{2}; 1 - a; \gamma/d \right) \tag{B.8}
\]

and

\[
Q_B = B(n) \left( 1 - \frac{x}{4} \right)^{\frac{n}{2} - 1} F\left(1 - \frac{x}{2}, \frac{3}{2} - \frac{x}{2}; \frac{5}{2}; (1 - \frac{x}{4})^{-1} \right). \tag{B.9}
\]

Equation (B.8) can be reduced also to elementary functions making use of (B.3). In (B.9) the hypergeometric function can be reduced with the aid of (B.4), (B.7) and the differentiation formula [19]

\[
\frac{d^n}{dz^n} \left[ (1-z)^{a+b-c} F(a, b; c; z) \right] = \frac{(c-a)_n (c-b)_n}{(c)_n} (1-z)^{a+b-c-n} F(a, b; c+n; z) \tag{B.10}
\]

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References