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ABSTRACT

The hidden symmetries in various integrable models are derived by applying a newly developed method that uses the Riemann-Hilbert transform in a Z_N-reduction of the linearization systems. The method is extended to linearization systems with higher algebras and with supersymmetry.

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I. INTRODUCTION

Hidden symmetries such as the recently discovered [1] Kac-Moody type, in mostly 2-dimensional integrable models, have attracted considerable interest: since they are thought to be responsible for the infinite set of of conserved currents in these models. The affine Kac-Moody algebra is defined as

$$[T_a^{(n)}\ T_b^{(m)}] = C_{abc}\ T_c^{(n+m)} \quad ,$$

where $T_a^{(n)} = T_a \otimes t^n$, $(n \in \mathbb{Z})$ are the generators for the loop group $G \otimes C[-t,t]$

In this paper we shall study a method [2] for deriving these hidden symmetries from the linearization system of various integrable models. The method will here be extended to systems with more complex algebraic structure and to supersymmetric cases. Such extensions therefore make it plausible that 2-dimensional integrable models in general have associated an infinite parameter hidden symmetry.

The method recently developed for finding the hidden symmetries will be reviewed in the next section. It goes briefly as follows: on starts with the linearization system (i.e. the Lax pair) of the model under investigation. Next one finds a reduction system that together with the Frobenius consistency condition reproduce the equation of motion of the model. The global version of the Kac-Moody generators are the Riemann-Hilbert transformations [3] and once they are guaranteed to satisfy the reduction system they should leave the linearizarion system invariant and their infinitesimal limit yields therefore the corresponding Kac-Moody algebra.

In the following section we shall apply this technic to the Dodd-Bullough [4] equation. Originally the technic was applied to the Sine-Gordon and the Liouville models. Both are 2-dimensional, integrable SU(2) systems. By extending the method to the Dodd-Bullough system which possesses an SU(3) symmetry and is a mixture of the two former models it then becomes clear how to extend the technics to an arbitrary SU(N) symmetry. Furthermore, we shall supersymmetrize the model in order to see how the method works in superspace and with a graded algebra structure. It might even be possible to extend the technics to higher dimensions as for example the case of the non-relativistic 3-dimensional Kodomstov-Petviasvili equation.

II. A METHOD OF DERIVING THE HIDDEN SYMMETRIES FROM LINEARIZATION SYSTEMS

In this section we shall review a method [2] proposed by one of the authors, M.L. Ge and I. Volovich, by which hidden symmetries can be derived from Lax-pair of integrable systems. The method was particularly intended for the simple 2-dimensional models such as the Sine-Gordon and the Liouville model. Before generalizing the method to more complicated algebraic structures we shall first introduce the method in the simple case of Sine-Gordon equation.

The strategy is the following. Start with a general Lax-pair that can represent a class of 2-dimensional integrable models and then find a reduction system that together with the usual Frobenius consistency conditions can reproduce the particular model we want. The reduction system is therefore applied to the linear wave-function to yield the particular non-linear equation we want. The Riemann-Hilbert transform gives an analytic continuation of the linear wave-function in the λ-plane (λ is the spectral parameter of Lax-pair). If we therefore apply the reduction system to the Riemann-Hilbert transform this should correspond to our particular model, i.e. the Riemann-Hilbert transform that obeys the reduction system should leave the particular Lax-pair invariant. Finally the infinitesimal limit of the Riemann-Hilbert transform gives us the corresponding Kac-Moody algebra as a hidden symmetry.

Let us illustrate this procedure by the Sine-Gordon case. We start with a general linearization system (or Lax-pair):

$$\begin{aligned} \partial_x\psi &\equiv U\psi = [U_0 + U_1\lambda]\,\psi \\ \partial_y\psi &\equiv V\psi = [\tfrac{1}{\lambda}V_1]\,\psi \quad . \end{aligned} \tag{1}$$

The Frobenius consistency condition is obtained by differentiating the first equation by y and the last by x and using compatibility ($\partial_x\partial_y\psi = \partial_y\partial_x\chi$):

$$\partial_y U_1 = 0, \quad \partial_y U_0 = [V_1, U_1], \quad \partial_x V_1 = [U_0, V_1] \quad . \tag{2}$$

The reduction system we now choose is the following

$$\begin{aligned} &(a) \qquad \psi \in \Omega(2,\mathbb{C}) \\ &(b) \qquad (\psi(\lambda))^* = \psi(-\bar\lambda)^{-1} \quad . \\ &(c) \qquad \delta_3\psi(\lambda)\delta_3^{-1} = \psi(-\lambda) \end{aligned} \tag{3}$$

(In the next section we shall apply a more general scheme of reduction systems (see also Ref.5), the so-called Z_N reduction groups.)

By applying the system (3) on the Lax-pair in (1), together with Eq.(2) we obtain (see Ref.2) uniquely the specific Lax pair for the Sine-Gordon model :

$$\begin{aligned} \partial_x\psi &= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\phi_x + \frac{\lambda}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right]\psi \\ \partial_y\psi &= \frac{1}{\lambda}\cdot\frac{1}{2}\begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0\end{pmatrix}\psi . \end{aligned} \tag{4}$$

Applying again the compatibility condition on Eqs.(4) we obtain the Sine-Gordon equation

$$\partial_x\partial_y\phi = \sin\phi \quad .$$

We now apply the Riemann-Hilbert transform to the reduction system (3):

$$\begin{aligned} &(a) \qquad \chi_\pm \in SL(2,\mathbb{C}) \\ &(b) \qquad (\chi_\pm(\lambda))^* = -\chi_\pm(-\bar\lambda) \\ &(c) \qquad \sigma_3\chi_\pm(\lambda)\sigma_3^{-1} = \chi_\pm(-\lambda) \end{aligned} \tag{5}$$

The R.H. -transform $\chi(x)$ is in general given by:

$$\chi(\lambda) = 1 - \frac{1}{2\pi i}\int\frac{dt}{t-\lambda}\,\chi(t)\left[S(t)S(\lambda)^{-1} - 1\right]; \tag{6}$$

$$\chi : \psi \to \psi'$$

χ_+ and χ_- denote the analytic transformations inside and outside a contour C (in the complex λ-plane). We have introduced S as

$$\chi_+ = \chi_- S \; ; \qquad S(\lambda) = \psi(\lambda)\gamma(\lambda)\psi^{-1}(\lambda) \tag{7}$$

where ψ is the wave-function of λ in Eq.(1) and $\gamma(\lambda)$ a general matrix function.

It can now be proven (see Ref.2) that if the R.H.-transform $\chi_+(\lambda)$ and $\gamma(\lambda)$ obey Eq.(5) then it leaves the linearization system invariant ; which is rather obvious.

From the constraints in Eq.(5) we can now easily derive the basis that spands the subalgebra of the χ-transformations that leave the Sine-Gordon invariant. First we go from the group elements to the algebra which in the infinitesimal limit becomes (θ small)

$$\begin{aligned} \gamma(\lambda) &= \exp(i\theta(\lambda)) \cong 1 + \theta(\lambda) \\ \theta(\lambda) &= \sum_{-\infty}^{\infty} \lambda^n \theta_n \end{aligned} \tag{8}$$

If $\gamma(\lambda)$ should obey Eq.(5) then we obtain for the Laurent series above:

$$\begin{aligned} &(a) \quad \mathrm{Tr}\,\theta(\lambda) = 0 \\ &(b) \quad (\theta(\lambda))^* = -\theta(-\bar{\lambda}) \quad \Rightarrow \quad \theta_n^* = (-1)^{n+1}\theta_n \\ &(c) \quad \sigma_3\theta(\lambda)\sigma_3 = \theta(-\lambda) \qquad\quad \sigma_3\theta_n\sigma_3 = (-1)^n\theta_n \end{aligned} \tag{9}$$

and the grading is obtained as powers of λ.

From Eq.(9) it follows that the θ-transformations can be expanded or the following basis :

$$\theta(\lambda) = \sum C_K i\sigma_3\lambda^{2K} + \sum C_K'\sigma_1\lambda^{2K+1} + \sum C_K''\sigma_2\lambda^{2K+1} \tag{10}$$

It is then clear that the particular Kac-Moody algebra that is infinite parameter for the hidden symmetries of the Sine-Gordon model is the following subalgebra:

$$\begin{aligned} [T_a^m, T_b^n] &= \epsilon_{abc} T_c^{m+n} \\ T_a^m &= \begin{cases} \lambda^m i\sigma_3\,\delta_{a,3} \; ; & m \text{ even} \\ \lambda^m(\sigma_1\delta_{a,1} + \sigma_2\delta_{a,2}) \; ; & m \text{ odd}. \end{cases} \end{aligned} \tag{11}$$

which is a subalgebra of the ordinary Kac-Moody algebra of the chiral model and has a special structure depending on whether the grading is even or odd.

III. EXTENSION TO HIGHER ORDER LAX PAIR

After the above introduction to the procedure of obtaining the hidden symmetries we here proceed to show how the same procedure can be effectively used to deduce the hidden symmetries in the case of nonlinear equations, associated with a NxN matrix Lax pair with N > 2. It has already been observed that it is possible to generate a family of nonlinear equation whose members are the Sine-Gordon and Liouville equations. Such a system of equation is known as generalized Toda system.[5] The equation belonging to this heirarchy next to Sine-Gordon is the Dodd- Bullough equation [4] which is known to be associated with a 3x3 Lax pair. In the following we show how our method is useful to extract the stucture of Kac-Moody algebra associated with such 3x3 system, with the help of reduction technique . Finally in the next section, we also consider the corresponding grade extension for the supersymmetric case.

Let us consider the Lax pair of the form ;

$$\begin{aligned} L_1\psi &= (\partial_x - U_0 - \lambda U_1)\psi = 0 \\ L_2\psi &= (\partial_t - V_0 - \lambda^{-1}V_1)\psi = 0 . \end{aligned} \tag{12}$$

The compatibility condition yields

$$\begin{aligned} U_{0t} - V_{0x} + [U_0, V_0] + [U_1, V_1] &= 0 \\ U_{1t} + [U_1, V_0] &= 0 \\ V_{1x} + [V_1, U_0] &= 0 \end{aligned} \tag{13}$$

In Eqs. (12) and (13) U_0, U_1, and V_0, V_1 all are assumed to be 3x3 matrices.

Our Initital problem is to deduce the Dodd-Bullough equation by imposing the reduction condition on the scattering problem (12). The general ansatz as elaborated in reference (5) reads un our case

$$Q^{-1} L_1(\lambda) Q = L_1(\lambda q)$$
$$Q^{-1} L_2(\lambda) Q = L_2(\lambda q) \qquad (14)$$

and $$t^{-1} [L_{1,2}^{tr}(-\lambda)] t = L_{1,2}(\lambda)$$

with $$Q_{\alpha\beta} = \delta_{\alpha\beta} q^{\alpha} ; \quad q = e^{\frac{2\pi i}{N}}$$

Eq.(14) yields
$$Q^{-1} U_0 Q = U_0$$
$$Q^{-1} V_0 Q = V_0 \qquad (15a)$$

and
$$Q^{-1} U_1 Q = q U_1$$
$$Q^{-1} V_1 Q = q^{-1} V_1 \qquad (15b)$$

The second part of the reduction condition yields

$$t U_0 t^{-1} = - U_0^{tr} \quad , \quad t V_0 t^{-1} = - V_0^{tr}$$
$$t U_1 t^{-1} = U_1^{tr} \quad , \quad t V_1 t^{-1} = V_1^{tr}. \qquad (16)$$

These quations lead to the following form of the matrices U and V ;

$$U_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad V_0 = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & -a_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (17)$$

$$U_1 = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \beta & 0 & 0 \end{pmatrix} ; \quad V_1 = \begin{pmatrix} 0 & 0 & h_1 \\ d_1 & 0 & 0 \\ 0 & h_1 & 0 \end{pmatrix} \qquad (17)$$

when used in the compatibility Eqs.(13) these matrices lead to the equation

$$\theta_{1xt} = e^{-\theta} - e^{2\theta} \qquad (18)$$

if we choose $$a_1 = \frac{\partial \theta}{\partial t} .$$

Finally the linear problem takes the form:

$$\Psi_x = \lambda \begin{pmatrix} 0 & e^{2\theta} & 0 \\ 0 & 0 & e^{-\theta} \\ e^{-\theta} & 0 & 0 \end{pmatrix} \Psi$$
$$\Psi_t = \left[\begin{pmatrix} \theta t & 0 & 0 \\ 0 & -\theta t & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right] \Psi \qquad (19)$$

The next step to the Riemann-Hilbert transform is to define the wave function $\chi(x)$ as in Equation (6) and its analytic conditions χ_+ and χ_-, related through (7). Following the same line of reasoning as in Ref.2 it can be proved that the Riemann-Hilbert transform keeps the linear equation unaltered. So that we can now proceed to set up the conditions to be satisfied by the numerical matrix $\gamma(\lambda)$ or its infinitesimal form $\theta(\lambda)$ and the wave function χ.

The conditons read

$$\left.\begin{array}{l} \chi(\lambda) \in SL(3,R) \\ Q\, \gamma(\lambda)\, Q^{-1} = \gamma(q\lambda) \\ t\, \gamma(\lambda)\, t^{-1} = -\gamma^{tr}(-\lambda) \end{array}\right\} \qquad (20)$$

Expanding the infinitesimal form of $\gamma(\lambda)$, i.e. $\theta(\lambda)$ as

$$\theta(\lambda) = \sum_{\lambda=-\infty}^{\infty} \lambda^n \theta_n$$

$$\begin{aligned} Q^{-1}\theta_n Q &= q^n \theta_n \\ t^{-1}\theta_n t &= \theta_n^{tr} \end{aligned} \quad . \tag{21}$$

The first condition of (21) yields

$$\left.\begin{aligned} Q^{-1}\theta_{3K} Q &= \theta_{3K} \\ Q^{-1}\theta_{3K+1} Q &= q\,\theta_{3K+1} \\ Q^{-1}\theta_{3K+2} Q &= \bar{q}\,\theta_{3K+2} \end{aligned}\right\} , \tag{22}$$

where K is an integer. With the form of Q given before we can actually solve the set (22) and furthermore, the imposition of the second condition yields the solutions in the following form;

$$\begin{aligned} \theta_{3K} &= p.E_{+1} + \gamma\left(E_{-2} + E_{+3}\right) \\ \theta_{3K+1} &= d\,E_{-1} + h\left(E_{+2} + E_{-3}\right) , \\ \theta_{3K+2} &= \alpha\,\lambda_3 \end{aligned} \tag{23}$$

where p,γ,d,h,α are arbitrary constants and $E_{\pm1} = \lambda_1 \pm i\lambda_2$; $E_{\pm2} = \lambda_4 \pm i\lambda_5$, $E_{\pm3} = \lambda_6 \pm i\lambda_7$ and λ's are the Gell-Mann $SU(3)$ matrices. So the final form of the Kac-Moody algebra can be presented in the form

$$\theta(\lambda) = \sum E_n \lambda^n, \quad \begin{aligned} &E_n \in \{E_{-1}, E_{+2}, E_{-3}\} \text{ for } n = 2 \bmod 3 \\ &E_n \in \{E_{+1}, E_{-2}, E_{+3}\} \text{ for } n = 1 \bmod 3 \\ &E_n \in \{\lambda_3, \lambda_8\} \quad \text{ for } n = 0 \bmod 3 . \end{aligned}$$

IV. SUPERSYMMETRIC EXTENSIONS

Our next motivation is to obtain the Kac-Moody algebra for the hidden symmetries in the supersymmetric version of the Dodd-Bullough equation. We have chosen this particular case as it is sufficiently general to unfold the intricacies involved. The supersymmetric extensions of the generalized Toda system have been discussed in detail by Olshanetsky [6]. Here it has been observed that if one has the following Lax pair:

$$\begin{aligned} D_1 \chi &= U\chi \\ D_2 \chi &= V\chi \end{aligned} \tag{24}$$

where

$$\left.\begin{aligned} D_1 &= -\partial_{\theta_2} + i\theta_2\partial_t \\ D_2 &= \partial_{\theta_1} + i\theta_1\partial_x \end{aligned}\right\} \qquad \left.\begin{aligned} U &= U_0 + \lambda U_1 \\ V &= \lambda^{-1} V_1 \end{aligned}\right\} \tag{25}$$

are super-derivatives and θ_1, θ_2 are the anticommuting coordinates. χ is the super-wave function, with U and V also matrices depending on the super-coordinates, fields and the eigenvalue λ.

The compatibility now reads

$$D_2 U + D_1 V = \{U, V\} , \tag{26}$$

where $\{A,B\}$ denotes anticommutator of two arbitrary matrices. It is easily seen that (25) leads to

$$\begin{aligned} D_1 V_1 &= \{U_0, V_1\} \\ D_2 U_0 &= \{V_1, U_1\} \end{aligned} \quad . \tag{27}$$

Now to reproduce a particular nonlinear equation from (27) one takes recourse to particular super-Lie algebra in which we assume the U and V to belong. For the case under consideration the Lie algebra in Kac's classification is $A^{(4)}(0,2)$ which have the generators E_i^+, E_i^-, H_i following the following set of commutation rules:

$$\begin{aligned} \{E_1^+, E_1^-\} &= H_1 \\ [E_j^+, E_j^-] &= \delta_{ij} H_j , \quad i,j \neq 1 \\ [H_i, E_j^\pm] &= a_{ij} E_j^\pm \end{aligned} \quad . \tag{28}$$

It is then obvious that we can write

$$U_0 = \sum A_j(xt)\, H_j + \sum A_j'(xt)\, E_j^+ + \sum A_j''(xt)\, E_j^-$$
$$V_1 = \sum C_j(xt)\, E_j^- + \sum C_j'(xt)\, H_j + \sum C_j''(xt)\, E_j^+$$
$$U_1 = \sum B_j(xt)\, E_j^+ + \sum B_j'(xt)\, H_j + \sum B_j''(xt)\, E_j^- . \quad (29)$$

Imposing the Z_n reduction on the matrices U and V with the help of the matrix Q defined through

$$Q\,(E_j^+) = q^{-1} E_j^+$$
$$Q\,(H_j) = H_j \quad (30)$$
$$Q\,(E_j^-) = q\, E_j^-$$

we observe that V_0, U_1 and V_1 do possess the following decomposition

$$U_0 = \sum A_j(xt)\, H_j$$
$$U_1 = \sum B_j(xt)\, E_j^+ \quad (31)$$
$$V_1 = \sum C_j(xt)\, E_j^- \,.$$

Now using the compatibility equations (26) and the super-algebra (28) one can reproduce the following nonlinear equation known as supersymmetric Dodd-Bullough equation

$$\theta_{xt} = -e^{\theta}(e^{\theta} + \psi_1\psi_2) + e^{\theta}$$
$$\psi_{1x} = -2e^{\theta}\psi_2 \quad (32)$$
$$\psi_{2t} = -2e^{\theta}\psi_1 .$$

where θ and $\psi_1\psi_2$ are, respectively, the bosonic and fermionic part of the super-field Φ. To explore the structure of the Kac-Moody algebra we again impose the restriction of the reduction condition on $\gamma(\lambda)$ which now belongs to a graded Lie algebra. The analogue of condition (20) reads

$$\chi(\lambda) \in A^{(4)}(0,2)$$
$$Q\ \chi(\lambda)\ Q^{-1} = q\,\gamma(\lambda) \quad (33)$$
$$q = e^{\pi i}$$

Taking recourse to the infinitesimal expansion of $\gamma(\lambda)$ and using (33) we get

$$\gamma(\lambda) = 1 + \sum \Omega_n^j E_j^+ \lambda^n$$
$$+ \sum \Delta_n^j E_j^- \lambda^n \quad (34)$$
$$+ \sum K_n^j H_j \lambda^n$$

along with

$$Q\,\Omega_n^j\,Q^{-1} = \Omega_n^j\, q^{n+1}$$
$$Q\ \Delta_n^j\,Q^{-1} = \Delta_n^j\, q^{n-1} \quad (35)$$
$$Q\ K_n^j\,Q^{-1} = K_n^j .$$

Again the grading is obtained for different odd and even values of n. We can actually solve these sets of equations; for example for n = 1

$$\Omega_1 = \begin{pmatrix} a & o & c & o \\ o & f & o & h \\ i & o & K & o \\ o & n & o & p \end{pmatrix} \quad (36)$$

for $n = 2$

$$\Omega_2 = \begin{pmatrix} o & b & o & d \\ e & o & g & o \\ o & j & o & e \\ n & o & K & o \end{pmatrix} \qquad (37)$$

Similarly for other cases, a,c, f, g, etc. are arbitary constants. It is quite easy to observe that $\Omega_1, \Omega_2, \Delta_1, \Delta_2$ etc. again belong to particular sub-algebra of the $A^{(4)}(0,2)$.

V. CONCLUSION

In this paper we have given a detailed exposition of a concrete method to derive hidden symmetries from linearization systems of various integrable models. Since here we were able to cover a wider range of integrable systems with more completed algebraic structure and with supersymmetry one is led to believe that all two-dimensional integrable models could have an infinite parameter hidden symmetry of the Kac-Moody type. One should think that parts of the method can be carried over to higher dimensional integrable models. Take for example the three-dimensional KdV equation (the Kadomstev-Petviashvilie equation). Here we also find a reduction system [7] and our method here seems also to work with such a reduction system. However, since such a reduction system has no explicit λ-dependence which can only be recovered in the asymptotic limit $\chi \to \infty$, it is necessary to prove the group property on the scattering data rather than on the wave function.

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