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Quantum theory of a one-dimensional laser with output coupling: II - Nonlinear theory

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ABSTRACT

A previous paper describing the quantum theory of a laser in linear approximation is here extended to the nonlinear case. Instead of the approach of conventional theory - which deals with discrete "cavity-modes" and includes artificial mechanisms to simulate radiation field losses due to beam extraction - ~~we use~~ a more realistic model of optical cavity having output coupling is used, ~~that~~ works entirely within the continuous spectrum, allowing one to obtain the equations for the field both *inside* and *outside* the laser cavity. Besides the quantum noise due to spontaneous emission, a noise term of classical nature due to transmission losses automatically emerges from the present treatment. For single-collective-mode operation the equations for laser field are solved exactly, yielding the transient and steady-state solutions. Inside the laser cavity, the results of nonlinear analysis agree with those found in conventional theory, once ~~we reinterprete~~ the conventional 'mode-amplitude' is reinterpreted as a collective variable. Outside the cavity - unaccessible region in the conventional treatment - the solution for the laser field is also exhibited. Further considerations as concerning the role played by the noise terms in the field buildup are discussed.

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I - INTRODUCTION

This paper is a natural extension of a previous one[1], where a quantum theory of a laser was developed in the linear approximation. We use here the same model of a one-dimensional optical cavity having output coupling in which the field quantization is carried out in terms of the modes of a continuous spectrum defined throughout the space of an entire optical cavity[2], defined by the laser cavity ($z \in [0,\ell]$) and the external region ($z \in (-\infty,0]$). Following the lines of ref. 1 we consider the laser as composed of homogeneously broadened two levels and noninteracting atoms as well as the use of the rotating wave and dipole approximations in the interaction Hamiltonian. The damping term for active atoms was introduced in a phenomenological way, whereas the damping term for the field, due to beam extraction, is automatically built in the model. The approximation of slowly varying amplitude of the field, both in time and space domains, is also employed.

However, instead of assuming here that the atomic population inversion is kept constant - which led to the linear approximation in ref. 1 - we will develop the nonlinear theory for the model, which is able to avoid the breakdown for the linear approximation above threshold and attains to the steady-state solution for the laser field.

In section II, we present a short summary of the model and cavity modes used in ref. 1. In section III, we derive the basic nonlinear equation for the laser field defined as a collective operator. This equation is derived for the laser operation in a single-collective-mode, under the usual adiabatic hypothesis and it is valid in the whole cavity. In section IV, we find the transient and steady-state solutions for the radiation field inside and also outside the laser cavity. Section V contains the discussions of the results obtained in this paper.

II - SUMMARY OF THE MODEL AND CAVITY MODES.

The one-dimensional optical cavity in this model[2] consists of two parallel plates, one totally reflecting placed at $z = \ell$ and the other is semitransparent, placed at $z = 0$. So, the cavity domain is the semi-infinite region $z \in (-\infty, \ell]$. The semitransparent plate simulates an ideal case of a dielectric medium, having very small thickness with a large dielectric constant η. The electrical permissivity of the medium in this model is analitically given as $\varepsilon(z) = \varepsilon_o[1 + \eta\delta(z)]$, where ε_o is the permissivity of vacuum and $\delta(z)$ is the Dirac δ-function.

The normal modes are stationary solution of Maxwell equations, satisfying appropriated boundary conditions[2,3]

$$U_k(z) = \begin{cases} U_k(z_{in}) = L_k \sin[k(z-\ell)], & 0 \leq z \leq \ell \\ \\ U_k(z_{out}) = (2/\pi)^{1/2} \sin(kz - \phi_k), & -\infty < z \leq 0 \end{cases} \qquad (1)$$

where the δ-function normalization has been used[2]. ϕ_k is a phase shift given by: $\phi_k = \sin^{-1}\left[(\pi/2)\, L_k \sin k\ell\right]$ and L_k as given bellow in eq. (2). On assuming that the transparency in the plate is very small, the function L_k^2 becomes strongly peaked around the Fox-Li quasimode frequencies $\omega_n = n(c\pi/\ell)$. In this case the line-width Γ_n, associated with a given Fox-Li frequency ω_{on} is such that $\Gamma_n \ll \Delta\omega = c\pi/\ell$, where $\Delta\omega$ is the spacing between two adjacent resonances. In single-mode operation and for $\Gamma_n \ll \Delta\omega$, the line-shape function L_k can be approximated by the Lorentzian function $M_k(n)$ as[2,3]

$$L_k \doteq M_k(n) = (2/\pi)^{1/2}\Lambda\Gamma\left[(\omega_k - \omega_o)^2 + \Gamma^2\right]^{-1/2} \qquad (2)$$

where, for brevity, we dropped the index n, i.e., $\Gamma_n \rightarrow \Gamma$, $\Lambda_{on} \rightarrow \Lambda_o = \Lambda$, $\omega_{on} \rightarrow \omega_o$, and Γ is determined by the window transparency

$$\Gamma = c/\Lambda^2\ell;\ \Lambda = \eta k_o = n(\eta\pi/\ell) \gg 1 \quad , \qquad (3)$$

$n \approx 10^6 \gg 1$ in the optical domain. The inequallity $\Lambda \gg 1$ leds to $\Gamma \ll \Delta\omega = c\pi/\ell$ and express the requirement that the transparency in the window is very small. Further details the reader will find in ref. 1.

III - DERIVATION OF THE BASIC EQUATION FOR THE LASER FIELD.

Assuming, as usual, the dipole and rotating wave approximations, the total Hamiltonian of the system reads[1]

$$H = H_F + H_A + H_I \tag{4}$$

where H_F is the Hamiltonian for the free radiation field

$$H_F = \int_0^\infty \hbar \omega_k a_k^\dagger a_k dk \ , \tag{5}$$

H_A is the Hamiltonian for the atoms

$$H_A = \sum_m \hbar \ \Omega_m \ \sigma_{m_2}^\dagger \sigma_{m_2} \tag{6}$$

and H_I is the interaction Hamiltonian

$$H_I = \sum_m \int_0^\infty \hbar \ (g_{km} a_k^\dagger \sigma_{m_1}^\dagger \sigma_{m_2} + \ H.c.)dk, \tag{7}$$

where H.c. stands for the Hermitian conjugate and

$$g_{km} = i\Omega_m (1/2 \hbar \omega_k)^{1/2} U_k (z_m) p_m \tag{8}$$

is the coupling constant between atom and field, p_m is the z component of the dipole matrix element for the mth atom. The creation and annihilation operators for the field $a_k^\dagger$, a_k, satisfy the standard commutation relations

$$\left[a_k, a_{k'}^\dagger\right] = \delta(k-k') \; ; \; \left[a_k, a_{k'}\right] = \left[a_k^\dagger, a_{k'}^\dagger\right] = 0 \qquad (9)$$

while the creation and annihilation operators for the upper and lower energy states for the mth atom, $\sigma^\dagger_{m2(1)}$ $\sigma_{m2(1)}$, satisfy the anticommutation relations

$$\left[\sigma_{mi}, \sigma^\dagger_{mj}\right]_+ = \delta_{ij} \; ; \left[\sigma_{mi}, \sigma_{mj}\right]_+ = \left[\sigma^\dagger_{mi}, \sigma^\dagger_{mj}\right]_+ = 0 \; . \qquad (10)$$

The electric field in terms of the creation and annihilation operators $a_k^\dagger$, a_k is the collective operator[4]

$$E(z,t) = i\int_0^\infty (\hbar\omega_k/2)^{1/2}\left[a_k(t) - a_k^\dagger(t)\right] U_k(z)\,dk . \qquad (11)$$

Taking the negative frequency part of the field operator we have

$$E^-(z,t) = -i\int_0^\infty (\hbar\omega_k/2)^{1/2} a_k^\dagger(t) U_k(z)\,dk . \qquad (12)$$

The application of eqs. (4)-(10) in the Heisenberg equations for the field and atomic operators yields

$$(d/dt)a_k^\dagger = i\omega_k a_k^\dagger + i \sum_m g_{km}^* \sigma_{m_2}^\dagger \sigma_{m_1}(t) , \qquad (13)$$

$$(d/dt)(\sigma_{m_2}^\dagger \sigma_{m_1}) = i\Omega_m(\sigma_{m_2}^\dagger \sigma_{m_1}) - \gamma_m(\sigma_{m_2}^\dagger \sigma_{m_1})$$

$$- i \int_0^\infty g_{km} a_k^\dagger(t)\, P_m(t) dk , \qquad (14)$$

$$(d/dt)(\sigma_{m_2}^\dagger \sigma_{m_2}) = \lambda_{m_2} - \lambda_m(\sigma_{m_2}^\dagger \sigma_{m_2})$$

$$+ i \int_0^\infty (g_{km} a_k^\dagger(t)\, \sigma_{m_1}^\dagger \sigma_{m_2}(t) - g_{km}^* a_k(t)\, \sigma_{m_2}^\dagger \sigma_{m_1}(t)) dk , \qquad (15)$$

where $P_m(t) = (\sigma_{m_2}^\dagger \sigma_{m_2}(t) - \sigma_{m_1}^\dagger \sigma_{m_1}(t))$ is the atomic population inversion and γ_m is the phenomenological damping term for the atomic polarization; $\lambda_{m_2}(\lambda_{m_1})$ is the phenomenological pumping rate for the upper (lower) level of the mth atom. Similar equations for the operators a_k, $\sigma_{m_1}^\dagger \sigma_{m_2}$, $\sigma_{m_1}^\dagger \sigma_{m_1}$ can also be found.

Solving eq. (15) and its similar for $\sigma_{m_1}^\dagger \sigma_{m_1}$ we find

$$P_m(t) = P_m^{(0)} + 2i \int_0^t e^{-\gamma_m(t-t')} \int_0^{\infty} \Big[g_{km} a_k^{\dagger}(t')\, \sigma_{m_1}^{\dagger} \sigma_{m_2}(t')$$

$$- g_{km}^{*} a_k(t')\, \sigma_{m_2}^{\dagger} \sigma_{m_1}(t') \Big]\, dk\, dt' \,. \qquad (16)$$

In the zeroth order approximation, i.e., in the absence of coupling between field and atoms, $P_m(t)$ has an equilibrium value $P_m^{(0)}$ established via pumping and damping mechanisms. Also $P_m^{(0)}$ is equal to $N\ell$, where N is the number of atoms by unit of length in the z-direction.

The substitution of eq. (14) and its Hermitian conjugate, plus the use of eq. (16) and the elimination of the rapidly oscillating component of $a_k^{\dagger}(t)$, by replacing $a_k^{\dagger}(t) \to a_k^{\dagger}(t) \exp(-i\omega_0 t)$ lead the eq. (13) to

$$\dot{a}_k^{\dagger}(t) - i\Delta\omega_k a_k^{\dagger}(t) = f_A(z_{in},t) - \sum_m \frac{g_{km}^{*} P_m^{(0)}}{\left[i(\Omega_m - \omega_0) - \gamma_m\right]} \int_0^{\infty} g_{\mu m} a_{\mu}^{\dagger}(t)\, d\mu$$

$$+ 2 \sum_m \frac{g_{km}^{*} P_m^{(0)}}{\gamma_m \left[i(\Omega_m - \omega_0) + \gamma_m\right]\left[i(\Omega_m - \omega_0) - \gamma_m\right]} \int_0^{\infty} g_{\mu m} a_{\mu}^{\dagger}(t)\, g_{\rho m} a_{\rho}^{\dagger}(t)\, g_{\sigma m}^{*} a_{\sigma}(t)\, d\mu\, d\rho\, d\sigma$$

$$- 2 \sum_m \frac{g_{km}^{*} P_m^{(0)}}{\gamma_m \left[i(\Omega_m - \omega_0) - \gamma_m\right]^2} \int_0^{\infty} g_{\mu m} a_{\mu}^{\dagger}(t)\, g_{\rho m}^{*} a_{\rho}(t)\, g_{\sigma m} a_{\sigma}^{\dagger}(t)\, d\mu\, d\rho\, d\sigma \qquad (17)$$

where

$$f_A(z_{in},t) = i \sum_m g^*_{km} \sigma^\dagger_{m_2} \sigma_{m_1}(0) \exp[i(\Omega_m - \omega_0) - \gamma_m] t \quad . \tag{18}$$

In eq. (17) all the terms decaying faster than atomic relaxation have been neglected[1], except for $f_A(z_{in},t)$ which plays important role in the laser buildup from vacuum. Next, using eq. (8) plus the following approximation[1]

$$\int_0^\infty g_{km} a_k^\dagger(t) dk = - \frac{\Omega_m p_m}{\hbar \omega_0} \mathcal{E}^-(z_m,t) \tag{19}$$

and limiting our considerations to identical atoms, which implies $\gamma_m = \gamma$, $p_m = p$ and $\Omega_m = \Omega$, the eq. (17) in the resonance case $(\Omega = \omega_0)$ reads

$$\dot{a}_k^\dagger(t) - i\omega_k a_k^\dagger(t) = iM_k (1/2\hbar\omega_k)^{1/2} H(f_A(z_{in},t), \mathcal{E}^-(z_{in},t), \mathcal{E}^+(z_{in},t)), \tag{20}$$

where

$$H\left(f_A(z_{in},t), \mathcal{E}^-(z_{in},t), \mathcal{E}^+(z_{in},t)\right) = -i\omega_o p^* \sin[k(z_{in}-\ell)]\,\sigma^\dagger_{m_2}\,\sigma_{m_1}(0)\,e^{-\gamma t}$$

$$+\frac{\omega_o p^2 N\ell}{\hbar\gamma\sqrt{2}}\,\mathcal{E}^-(z_{in},t)$$

$$-\frac{2\omega_o p^4 N\ell}{(\hbar\gamma)^3\sqrt{2}}\Big[\mathcal{E}^-(z_{in},t)\,\mathcal{E}^-(z_{in},t)\,\mathcal{E}^+(z_{in},t)$$

$$+\mathcal{E}^-(z_{in},t)\,\mathcal{E}^+(z_{in},t)\,\mathcal{E}^-(z_{in},t)\Big]. \qquad (21)$$

According to the definition of the electric field operator (cf. eq. (11)), we multiply eq. (20) by $\left[-i(\hbar\omega_k/2)^{1/2}U_k(z)\right]$ and integrate in the k-domain, $k\in[0,\infty)$, in order to put eq. (20) in terms of the collective operator as defined by eq. (12). In this way we have

$$\dot{\mathcal{E}}^-(z_{in},t) = \int_0^\infty (\hbar\omega_k/2)^{1/2}(\Delta\omega_k)U_k(z)a_k^\dagger(t)dk + \frac{1}{2}\int_0^\infty M_k U_k(z)dk$$

$$\times\ H\left(f_A(z_{in},t),\ \mathcal{E}^-(z_{in},t), \mathcal{E}^+(z_{in},t)\right). \qquad (22)$$

Substituting the formal solution of eq. (20), namely

$$a_k^\dagger(t) = a_k^\dagger(0)e^{i\Delta\omega_k t} + iM_k(1/2\hbar\omega_k)^{1/2}$$

$$\times\ \int_0^t e^{i\Delta\omega_k(t-t')}H\left(f_A(z_{in},t'), \mathcal{E}^-(z_{in},t')\,\mathcal{E}^+(z_{in},t')\right)dt' \quad (2.$$

in eq. (22), we obtain

$$\dot{\mathcal{E}}^{-}(z,t) - I^{-}(z,t) - \frac{i}{2}\int_0^t\int_0^\infty (\Delta\omega_K) M_k U_k(z) e^{i\Delta\omega_k(t-t')} H\left(f_A(z_{in},t'), \mathcal{E}^{-}(z_{in},t'), \mathcal{E}^{+}(z_{in},t')\right) dk\,dt'$$

$$= \frac{1}{2}\int_0^\infty M_K U_k(z) dk \left[H\left(f_A(z_{in},t), \mathcal{E}^{-}(z_{in},t), \mathcal{E}^{+}(z_{in},t)\right)\right] \qquad (24)$$

where $I^{-}(z,t)$, given by

$$I^{-}(z,t) = \int_0^\infty (\hbar\omega_k/2)^{1/2} (\Delta\omega_k) U_k(z) a_k^{\dagger}(0) e^{i\Delta\omega_k t} dk\ , \qquad (25)$$

is a noise term due to the initial field. Later on, we will return to this point for a more detailed discussion.

The equation (24) is the work equation for the field, valid both *inside* and *outside* the laser cavity. In the next section we will devote ourselves to finding the solutions of this equation.

IV - SOLUTIONS OF THE LASER EQUATION OF MOTION.

A. The field inside the laser cavity.

In order to find the field inside the cavity we use $U_k(z)$ for $z \in [0,\ell]$, as given in eq. (1). Thus, we have

$$\dot{\mathcal{E}}^-(z_{in},t) - I^-(z_{in},t) = \frac{i}{2}\int_0^t\int_0^\infty (\Delta\omega_k) M_k \{M_k \sin[k(z_{in}-\ell)]\} e^{i\Delta\omega_k(t-t')}$$

$$\times \left[H(f_A(z_{in},t'),\mathcal{E}^-(z_{in},t'),\mathcal{E}^+(z_{in},t'))\right] dk\,dt' = \frac{1}{2}\int_0^\infty M_k\{M_k \sin[k(z_{in}-\ell)]\}\,dk$$

$$\times H(f_A(z_{in},t),\mathcal{E}^-(z_{in},t),\mathcal{E}^+(z_{in},t)). \qquad (26)$$

The integration in k-domain, after the application of the residue technique, leads the foregoingequation to

$$\dot{\mathcal{E}}^-(z_{in},t) - I^-(z_{in},t) + \frac{\Gamma M^2}{2\sqrt{2}}\int_0^t e^{-\Gamma(t-t')} H(f_A(z_{in},t'),\mathcal{E}^-(z_{in},t'),\mathcal{E}^+(z_{in},t'))\,dt'$$

$$= \frac{M^2}{2\sqrt{2}} H(f_A(z_{in},t),\mathcal{E}^-(z_{in},t),\mathcal{E}^+(z_{in},t)) , \qquad (27)$$

where

$$M^2 = \int_0^\infty M_k^2\,dK = 2/\ell \qquad (28)$$

and, for the sake of simplification, the usual approximation $\sin[k(z_{in} - \ell)] \to 1/\sqrt{2}$ has been used. Let us write $\mathcal{E}^-(z_{in},t)$ as

$$\mathcal{E}^-(z_{in},t) = \mathcal{C}^-(z_{in},t) + (M^2/2\sqrt{2})\int_0^t H\left(f_A(z_{in},t'),\mathcal{E}^-(z_{in},t'),\mathcal{E}^+(z_{in},t')\right)dt'. \qquad (29)$$

Putting eq. (29) into eq. (27), we have

$$\dot{\mathcal{C}}^-(z_{in},t) = I^-(z_{in},t) \qquad (30)$$

which is easily integrated to give

$$\mathcal{C}^-(z_{in},t) = \mathcal{C}^-(z_{in},0) + \int_0^t I^-(z_{in},t')dt'. \qquad (31)$$

Combining eqs. (27), (29)-(31) and using $\mathcal{E}^-(z_{in},0) = \mathcal{C}^-(z_{in},0)$, cf. eq. (29), we find

$$\dot{\mathcal{E}}^-(z_{in},t) + \Gamma\mathcal{E}^-(z_{in},t) = (M^2/2\sqrt{2})H\left(f_A(z_{in},t),\mathcal{E}^-(z_{in},t),\mathcal{E}^+(z_{in},t)\right)$$

$$+ \Gamma\int_0^t I^-(z_{in},t')dt' + I^-(z_{in},t). \qquad (32)$$

Next, we use the explicit form of $I^-(z_{in},t)$ given by the substitution of $U_k(z)$ by $U_k(z_{in})$, $z_{in} \in [0,\ell]$, in eq. (25) and exchanging the order integration in the second term of the right-hand side of eq. (32), we obtain

$$\dot{\mathcal{E}}^-(z_{in},t) + \Gamma\mathcal{E}^-(z_{in},t) = F^-(z_{in},t) + G^-(z_{in},t) + \alpha M^2 \mathcal{E}^-(z_{in},t)$$

$$- \beta M^2 \mathcal{E}^+(z_{in},t)\mathcal{E}^-(z_{in},t)\mathcal{E}^-(z_{in},t) \qquad (33)$$

where

$$F^-(z_{in},t) = \int_0^\infty (\Delta\omega_k - i\Gamma)(\hbar\omega_k/2)^{1/2} u_k(z_{in}) a_k^+(0) e^{i\hbar\omega_k t} , \qquad (34)$$

$$G^-(z_{in},t) = -\frac{i\omega_0 p M^2}{2\sqrt{2}} \sin[k(z_{in}-\ell)]\, \sigma_{m_2}^+ \sigma_{m_1}(0) e^{-\gamma t} \qquad (35)$$

and

$$\alpha = \frac{\omega_0 p^2 N\ell}{4\hbar\gamma} \; ; \quad \beta = \frac{\omega_0 p^4 N\ell}{(\hbar\gamma)^3} . \qquad (36)$$

The expression (33) is the basic equation for the field *inside* the cavity. The terms $F^-(z_{in},t)$ and $G^-(z_{in},t)$ in eq. (33)

are noise operators which depend on the initial value of the field and atomic operators, respectively. It should be stressed that the noise term $F^-(z_{in},t)$ emerges from the model of optical cavity and has a classical nature. However, unlike $F^-(z_{in},t)$ the noise term $G^-(z_{in},t)$ has essentially a quantum origin. Also, as can be seen in eq. (35), the noise atomic operator $G^-(z_{in},t)$ may be neglected after a long time in comparison to the lifetime of the field inside the laser cavity. In the absence of $G^-(z_{in},t)$, eq. (33) becomes

$$\dot{\mathcal{E}}^-(z_{in},t) + \Gamma\mathcal{E}^-(z_{in},t) = F^-(z_{in},t) + \alpha M^2\mathcal{E}^-(z_{in},t) - \beta M^2\mathcal{E}^-(z_{in},t)\mathcal{E}^-(z_{in},t)\mathcal{E}^-(z_{in},t) \tag{37}$$

Equation (37) is antinormally-ordered[5], which is a suitable form for the application of coherent representation[6] allowing one to transform quantum operators into ordinary functions. Accordingly, the Hermitian conjugate of eq. (37) is transformed into

$$\dot{V}(z_{in},t) + \Gamma V(z_{in},t) = \alpha M^2 V(z_{in},t) - \beta M^2 |V(z_{in},t)|^2 V(z_{in},t) + \mathcal{F}(z_{in},t), \tag{38}$$

where $\mathcal{F}(z_{in},t)$ is the representation of the noise operator $F^-(z_{in},t)$ in the coherent basis

$$\mathcal{J}(z_{in},t) = \int_0^\infty (\Delta\omega_k - i\,\Gamma)(\hbar\omega_k/2)^{1/2} u_k(z_{in}) v_k(0) e^{-i\Delta\omega_k t} dk. \quad (39)$$

In order to solve the eq. (38), we begin by assuming the initial condition is such that $\mathcal{J}(z_{in},t) = 0$. This is accomplished by taking, e.g., $v_k(0) = C_0 M_k$ in eq. (39). Hence, in the absence of the noise term $\mathcal{J}(z_{in},t)$ and setting

$$V(z_{in},t) = a(z_{in},t)\, \exp\left[-\, i\phi(z_{in},t)\right] \quad (40)$$

in eq. (38), we have

$$\dot{a}(z_{in},t) = s_0 a(z_{in},t) - \beta M^2 a^3 (z_{in},t) \quad (41)$$

$$\dot{\phi}(z_{in},t) = 0 \quad (42)$$

where $s_0 = (\alpha M^2 - \Gamma)$ is the effective gain[1] and gives the threshold condition $s_0 = 0$. Eq. (42) is readily solved yielding: $\phi(z_{in},t) = \phi(z_{in},0) = \phi_0$ which has a constant value in time domain. As regards eq. (41), the transient solution for the intensity is readily found

$$a^2(z_{in},t) = \frac{1}{a^{-2}(z_{in},0)e^{-2s_0t} + a^{-2}(z_{in},\infty)} \quad (43)$$

where $a^2(z_{in},0)$ is the initial value for the intensity of the radiation field, whereas

$$a^2(z_{in},\infty) = s_0/\beta M^2 = (\alpha M^2 - \Gamma)/\beta M^2 \quad (44)$$

is its asymptotic value above threshold as $t \to \infty$.

The expression (44), which also follows by setting $\dot{a}(z_{in},t) = 0$, is a well know result in the conventional laser theory. However, it should be emphazised that this result refers here to a *collective* variable $V(z,t)$ rather than to an *individual* amplitude.

For small times such that $s_0 t \ll 1$ we have

$$a^2(z_{in},t) \simeq a^2(z_{in},0)e^{2s_0t} \quad (45)$$

which coincides with the linear approximation[1] - valid for small times.

From the above discussion it follows that the general solution of eq. (38), when $\mathcal{J}(z_{in},t) = 0$, is

$$V(z_{in},t) = a(z_{in},t)e^{-i\phi_0} \quad (46)$$

with $a(z_{in},t)$ given by eq. (43). Hence, above threshold and in the absence of the noise terms $F^{-}(z_{in},t)$, $G^{-}(z_{in},t)$, the complex amplitude $V(z_{in},t)$ evolves towards a value that loses memory of its initial modulus (cf. eq. (44)) but retains the initial phase.

Let us see what happens when $\mathcal{J}(z_{in},t) \neq 0$. Setting

$$\mathcal{J}(z_{in},t) = f(z_{in},t)e^{-i\theta(z_{in},t)} \tag{47}$$

we get, instead of eqs. (41) and (42),

$$\dot{a}(z_{in},t) = s_0 a(z_{in},t) - \beta M^2 a^3(z_{in},t) + f^{-}(z_{in},t)\cos\psi(z_{in},t) \tag{48}$$

$$\dot{\phi}(z_{in},t) = \left[f(z_{in},t)/a(z_{in},t)\right]\sin\psi(z_{in},t) \ , \tag{49}$$

where $\psi(z_{in},t) = \theta(z_{in},t) - \phi(z_{in},t)$. Hence, the effect of the operator $F^{-}(z_{in},t)$ is to produce fluctuations in both amplitude and phase. However, as $a(z_{in},t)/f(z_{in},t)$ becomes very large for $t \to \infty$, due to the effect of laser amplification on $a(z_{in},t)$, we may then neglect fluctuations in the amplitude. But this is not so for the fluctuation in the phase.

Although being a quantum treatment, the present approach becomes essentially semiclassical when the noise atomic operator $G^{-}(z_{in},t)$ is neglected. In this case, as seen in the present section, the results coincide with those of a previous semiclassical treatment for the field inside the cavity[2].

B. The field outside the laser cavity.

Going back to eq. (24), let us investigate it for the field outside the cavity. First, we rewrite this equation in the form

$$\dot{\mathcal{E}}^-(z,t)-I^-(z,t)-(1/2)\int_0^t H\left(f_A(z_{in},t'),\mathcal{E}^-(z_{in},t'),\mathcal{E}^+(z_{in},t)\right)\frac{1}{c}\frac{\partial}{\partial\tau}\int_{-\infty}^{\infty} M_k U_k(z)e^{i\Delta\omega_k\tau}d(\Delta\omega_k)dt$$

$$=(1/2c)\int_{-\infty}^{\infty} M_k U_k(z)d(\Delta\omega_k)H\left(f_A(z_{in},t),\mathcal{E}^-(z_{in},t),\mathcal{E}^+(z_{in},t)\right). \qquad (50)$$

and take $U_k(z)=U_k(z_{out})$, $z_{out}\in(-\infty,0]$, as given by[2]

$$U_k(z_{out})=\frac{1}{\Lambda\Gamma}\left[(\Delta\omega_k)M_k\sin kz-\Gamma M_k\cos kz\right], \qquad (51$$

to find, after some algebraic manipulations including the use of residue technique, the corresponding differential equation for the field *outside* the laser cavity

$$\dot{\mathcal{E}}^-(z_{out},t) - I^-(z_{in},t) - (\Gamma M^2/\Lambda 2\sqrt{2})\int_0^t H\left(f_A(z_{in},t'),\mathcal{E}^-(z_{in},t'),\mathcal{E}^+(z_{in},t')\right)e^{-\Gamma(t-t')}dt'$$

$$= (-M^2/\Lambda 2\sqrt{2})\, H\left(f_A(z_{in},t),\mathcal{E}^-(z_{in},t),\mathcal{E}^+(z_{in},t)\right). \quad (52)$$

At this point we set

$$\mathcal{E}^-(z_{out},t) = \mathcal{E}_0^-(z_{out},t) - (\Gamma M^2/\Lambda 2\sqrt{2})\int_0^t H\left(f_A(z_{in},t'),\mathcal{E}^-(z_{in},t'),\mathcal{E}^+(z_{in},t')\right)e^{-\Gamma(t-t')}dt' \quad (53)$$

and proceed as previously, from eq. (29) to eq. (32), to get

$$\dot{\mathcal{E}}^-(z_{out},t) + \Gamma\mathcal{E}^-(z_{out},t) = F^-(z_{out},t) + G^-(z_{out},t) - \frac{1}{\Lambda}\Big[\alpha M^2\mathcal{E}^-(z_{in},t) -$$

$$- \beta M^2\mathcal{E}^+(z_{in},t)\,\mathcal{E}^-(z_{in},t)\,\mathcal{E}^-(z_{in},t) \quad (54)$$

where

$$F^-(z_{out},t) = \int_0^\infty (\Delta\omega_K - i\Gamma)(\hbar\omega_k/2)^{1/2}\, u_k(z_{out})\, a_k^+(0)\, e^{i\Delta\omega_K t}\, dk \quad (55)$$

and the assumption $\mathcal{G}^{-}(z_{out},0) = 0$ has been used.

Neglecting again the noise term $G^{-}(z_{in},t)$ and applying the coherent representation in eq. (54), as in the previous section, we have

$$\dot{V}(z_{out},t) + \Gamma V(z_{out},t) = -\frac{1}{\Lambda}\{V(z_{in},t)\left[\alpha M^2 - \beta M^2|V(z_{in},t)|^2\right]\} + \mathcal{F}(z_{out},t) \quad (56)$$

where

$$\mathcal{F}(z_{out},t) = \int_0^{\infty}(\Delta\omega_k + i\Gamma)(\hbar\,\omega_k/2)^{1/2}U_k(z_{out})v_k(0)e^{-\Delta\omega_k t}dk\ . \quad (57)$$

We assume the same initial condition $v_k(0) = C_0 M_k$, which also leads to $\mathcal{F}(z_{in},t) = 0$. So, putting

$$V(z_{out},t) = a(z_{out},t)\ \exp\left[-i\phi(z_{out},t)\right] \quad (58)$$

in eq. (56) and using eq. (40) we obtain

$$\dot{a}(z_{out},t) + \Gamma a(z_{out},t) = -\frac{1}{\Lambda}a(z_{in},t)\left[\alpha M^2 - \beta M^2 a^2(z_{in},t)\right]\cos\psi(z_{out},t) \quad (59)$$

$$\dot{\phi}(z_{out},t) = \left[\Gamma a(z_{in},t)/\Lambda a(z_{out},t)\right]\sin\psi(z_{out},t) \quad (60)$$

where

$$\psi(z_{out},t) = \phi(z_{out},t) - \phi(z_{in},t) \ . \tag{61}$$

In order to have $\dot{\phi}(z_{out},t) = 0$, in agreement with eq. (42) which led to the phase stabilization for the field *inside* the cavity, we must set $\sin\psi(z_{out},t) = 0$, or $\phi(z_{out},t) = \phi(z_{in},t)$ and the stable solution for $\phi(z_{out},t)$ equals that for $\phi(z_{in},t) = \phi_0$. Substituting this result and eq. (44) in eq. (59), we find the intensity *outside* the laser cavity

$$a^2(z_{out},\infty) = a^2(z_{in},\infty)/\Lambda^2 \tag{62}$$

where $a^2(z_{in},\infty)$ is given by eq. (44).

The transient solution for the amplitude $a(z_{out},t)$ outside the cavity can also be found by integration of eq. (59), yielding the formal result

$$a(z_{out},t) = \int_0^t e^{-\Gamma(t-t')}\xi(z_{in},t')dt' \tag{63}$$

where $\xi(z_{in},t)$ is all the right-hand side (the inhomogeneous term) of the differential equation (59), with $\cos\psi(z_{out},t) = 1$ and $a(z_{in},t)$ given by eq. (43).

V - CONCLUDING REMARK

In this paper, as in ref. 1, the field quantization is carried out in terms of modes of the continuous spectrum, defined throughout the space (cf. eq.(1)). In contrast to the usual procedure, the transmission loss is already part of the continuum treatment, which avoids the inclusion of unrealistic loss mechanisms simulated by artificial loss-reservoirs.

The theory is developed by expanding the field as a linear superposition of the cavity-modes $U_k(z)$ (cf. eq.(11)). The definition of a collective operator (cf. eq.(11)) which contains into itself informations about the entire optical cavity allows one to obtain the field *inside* and *outside* the cavity. The present approach not only gives the results found in the conventional laser theory but even allows us to find the field in external region of the laser cavity. As in ref.1, we assume the following approximations: (i) the transparency of the window is required to be very small (cf. eq.(3)); (ii) the laser operation in single-collective-mode and (iii) two levels and homogeneously broadened active atoms. However, unlike ref. 1, the population inversion is no longer kept constant, which leads to the nonlinear theory. In this way, the work equation (24) was derived for the radiation field, which is valid both inside and outside the laser cavity. For the field inside the cavity and in absence of noise terms we are led to a Bernoulli-type nonlinear differential equation (see eq.(40) which gives the transient and steady-state solutions for the field amplitude, as obtained in eqs.(43) and (44). Once the field inside the cavity is known, a little modification of eq.(24) leading to eq.(50) gives the field *outside* the cavity (cf.eqs.(61)-(63)). This latter result cannot be derived (or even postulated) in the conventional laser theory found in the current literature.

A qualitative approach on the role played by the noise terms $F^-(z,t)$ and $G^-(z,t)$ on the laser field buildup from vacuum can be done by considering eq.(33) in the linear approximation, valid for small times,

$$\dot{\mathcal{E}}^+(z_{in},t) = s_0\mathcal{E}^+(z_{in},t) + F^+(z_{in},t) + G^+(z_{in},t) \tag{64}$$

whose formal solution, in a coherent representaión, reads

$$V(z_{in},t) = V(z_{in},0)e^{s_0 t} + \int_0^t e^{s_0(t-t')}\mathcal{F}(z_{in},t')dt' + \int_0^t e^{s_0(t-t')}G^+(z_{in},t')dt' \tag{65}$$

where, cf. eq.(12),

$$V(z_{in},0) = \int_0^\infty (\hbar\omega_k/2)^{1/2}\, U_k(z_{in})v_k(0)dk \tag{66}$$

and $\mathcal{F}(z_{in},t)$ is given by eq.(39).

So, assuming the initial field in vacuum state, we have $V(z_{in},0) = 0$ and $v_k(0) = 0$ which implies, according to eq.(39), $\mathcal{J}(z_{in},t) = 0$. This means that $F^+(z_{in},t)$ is not a true inhomogeneous term of the differential equation (56). Hence, the true inhomogeneous term in this equation is the noise atomic term $G^+(z_{in},t)$, which provides the laser buildup from vacuum.

As regards to the term $F^-(z_{in},t)\left(F^+(z_{in},t)\right)$, the initial conditions by which it is determined (cf. eqs.(34),(39)), are usually stochastic, instead of deterministic (see ref.2), so that $F^-(z_{in},t)$ $\left(F^+(z_{in},t)\right)$ actually plays the role of a noise source term and eq. (37) becomes a Langevin-type equation.

In the absence of the noise term $G^+(z_{in},t)\left(G^-(z_{in},t)\right)$ the present treatment becomes essentially semiclassical. In fact, the transient and steady-state solutions for the field *inside* the laser cavity (see eqs.(43),(44)) coincide with the previous results of a semiclassical treatment, as found in ref.2. Also, the results coincide with those found in the conventional laser theory; but the 'laser mode' of the usual approach should be interpreted as a collective concept and the 'mode amplitude' is actually a *collective* variable.

In a future paper we will treat the quantum coherence functions for the field inside and outside the laser cavity. Also, a development for nonlinear theory that removes the adiabatic hypothesis is under investigation. At this point, according to ref.1, eq.(52), we might expect a modification in the solution given by eq. (43), yielding

$$a^2(z_{in},t) = \frac{1}{a^{-2}(z_{in},0)e^{-2s_{01}t} + a^{-2}(z_{in},\infty)} \qquad (67)$$

where $a^2(z_{in},\infty) = s_{01}/\beta M^2$; $s_{01} = -(\gamma + \Gamma)/2 + [(\gamma + \Gamma)/2 + \gamma(\alpha M^2 - \Gamma)]^{1/2}$ is the effective gain in the absence of adiabatic hypothesis as found in ref.1.

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