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QUASICLASSICAL APPROXIMATION FOR ULTRALOCAL SCALAR FIELDS

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Abstract

~~We~~ show how to obtain the quasiclassical evolution of a class of field theories called ultralocal fields. Coherent states that follow the "classical" orbit as defined by Klauder's weak correspondence principle and restricted action principle is explicitly shown to approximate the quantum evolution as $\hbar \to 0$.

Introduction

Ultralocal field theory studies a class of models which differ from relativistic theories by the absence of spatial gradients in the Hamiltonian, e.g. the term $(\nabla\phi)^2$ for scalar fields.

Exact operator solutions have already been obtained for these models[1,2] and they may well provide an alternative route to the perturbative study of quantum field theory where the spatial gradients are to be included as perturbations about the exact, ultralocal solutions. However, up to this date, no one has yet succeeded in finding a representation for the products of spatial derivatives in the context of ultralocal representations.

Ultralocal field theory seems to be particularly appropriate for the study of strong coupling limits to relativistic field theories. For Yang-Mills fields this limit consists of dropping the magnetic field terms from the Hamiltonian i.e. the terms containing spatial derivatives. Similarly, for General Relativity dropping the spatial scalar Ricci curvature term from the Hamiltonian generator can be considered as a Strong Coupling Limit[3]. Although strong coupling Yang-Mills has been studied in the context of lattice gauge theory, this is not mandatory and application of ultralocal methods seems to be promissing. The coordinate invariance of general relativity makes going to a lattice very unnatural and ultralocality ideas may prove to be a sound alternative to understand the quantized theory.

In this paper we address the problem of determining a classical limit of an ultralocal scalar quantum field.The extension of these ideas to general relativity has already been worked out and can be found in Ref.4. Not only canonical but also affine[2] commutation relations will be discussed. The motivation for this kind of field lies on the fact that in some physical systems the dynamical variables are constrained one way or another. This is the case for example when one wants to restrict the spectrum of the field operator to the positive real line. Then canonical commutation relations are not suitable since the canonical momentum will not be self adjoint under this restriction. General Relativity exhibits this peculiarity since the metric tensor on a Cauchy hypersurface must have definite signature. Thus for completeness we treat in this paper both affine and canonical fields.

Roughly the paper is organized as follows. §1 contains an illustration of how one obtains classical limits in the context of quantum mechanics with the help of coherent states and the role of the Weak Correspondence Principle (WCP) and the Restricted Action Principle (RAP) when one tries to generalize these results to field theory. In §2 we present a general discussion about ultralocal scalar fields. The representation we choose is specified by a kind of generalized gaussian vacuum state since this is the most convenient in order to construct coherent states. The evolution problem is tackled in §3 and the coherent states as given by the WCP and RAP will be shown to approximate the evolution of the wave function as $\hbar \to 0$, in analogy with the quantum mechanics case. Discussions and conclusions is the object of §4.

Throughout the paper we denote by the number (n) an equation in the paragraph where the number appears. Numbers ($m.n$) mean equations in other paragraphs, e.g. (3.2) is the second equation in §3.

1. General Remarks on Classical Limits

(1.1) Quantum Mechanics

We start by reviewing briefly in the context of quantum mechanics the methods on which we will base most of our discussion of taking the classical limit of ultralocal field theory. We work in finite dimensions for simplicity but the extension to two dimensional boson scalar fields is found in Hepp[5]. Our presentation will be mostly based on this reference (see also Klauder[6] and the book by Thirring[7]).

Consider a canonical system with the real Hamiltonian function

$$\mathcal{H}(p,q) = \frac{p^2}{2m} + V(q) \tag{1}$$

in the $2n$-dimensional space $\mathbb{R}^{2n}$ with $(p,q) \in \mathbb{R}^{2n}$. If $\operatorname{grad} V \equiv \nabla V$ is Lipschitz around q, then the canonical equations of motion

$$m\dot{q}(t) = p(t) \qquad \dot{p}(t) = -\nabla V(q(t)) \tag{2a}$$

have always a unique local solution for $t \in (-a,a)$, $a > 0$, with initial data

$$q \equiv q(0) \qquad p \equiv p(0). \tag{2b}$$

The corresponding quantum mechanical problem

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = \hat{\mathcal{H}} \psi(x,t) \equiv \left(-\frac{\hbar^2}{2m} \nabla^2 + V(x)\right) \psi(x,t) \tag{3}$$

in the Hilbert space $H = L^2(\mathbb{R}^n)$ with inner product $(\psi,\phi) = \int dx\, \psi^* \phi$ has always global solutions if the self adjoint extensions of ∇^2 and V have a common dense domain D with $\Psi(\cdot,o)\in D$. This solution

$$\Psi(t) = \exp\left(-\frac{i}{\hbar}\hat{\mathcal{H}}_{s.a.}t\right)\Psi(o) \equiv U(t)\,\Psi(o) \tag{4}$$

is expressed in terms of any self adjoint extension $\hat{\mathcal{H}}_{s.a.}$ of the operator $-\hbar^2\nabla^2/2m + V$. In what follows we will not make any distinction between $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}_{s.a.}$.

The discussion of the connection between (2) and (3) is as old as quantum mechanics itself[8]. Several methods have been devised to study this problem. The WKB method relates an asymptotic expansion of solutions of (3) for $\hbar \to 0$ to solutions of the Hamilton-Jacobi equation [9,10] for (2). (An application of the WKB approximation to the gravitational field can be found in the work of Gerlach[11]). The Feynman integral[12] appears to be a very flexible tool but its use is beyond the scope of this paper. The simplest connection between quantum and classical mechanics, however, goes back to the <u>Ehrenfest theorem</u>[8]: for every $\psi\in D$ and V sufficiently regular

$$\begin{aligned}
\frac{d}{dt}\left(\Psi(t),\hat{Q}\,\Psi(t)\right) &= \frac{1}{m}\left(\Psi(t),\hat{P}\,\Psi(t)\right) \\
\frac{d}{dt}\left(\Psi(t),\hat{P}\,\Psi(t)\right) &= -\left(\Psi(t),\nabla V\,\Psi(t)\right)
\end{aligned} \tag{5}$$

where $\hat{P} = -i\hbar\nabla$ and $\hat{Q} = x$. However (5) does not define a solution of (2) since $(\psi(t),\nabla V\psi(t)) \neq \nabla V(\psi(t),\,Q\psi(t))$ unless V is linear. In general only for $\hbar \to 0$ these expectation values define a solution to (2). Another way to relate (5) and (2) when $\hbar \to 0$ is to use minimal uncertainty states for $\hat{P}$ and $\hat{Q}$, i.e. coherent states[14,15]. Since coherent states provide the main tool for obtaining the quasiclassical approximation for ultralocal fields in § 3 we give below an idea of how the method works.

In order to have the powers of $\hbar$ on the right place it is convenient to use a symmetric representation[5] of the CCR (canonical commutation relations)

$$\hat{p}_\hbar = \sqrt{\hbar}\,\hat{p} \qquad\qquad \hat{q}_\hbar = \sqrt{\hbar}\,\hat{q} \tag{6}$$

where $\hat{p} = -i\,\partial/\partial x$ and $\hat{q} = x$. Let $\alpha = (q + ip)/\sqrt{2} \in \mathbb{C}$ with p,q real parameters and define

$$U(\alpha) = \exp(\alpha\hat{a}^{*} - \alpha^{*}\hat{a}) = \exp i(p\hat{q} - q\hat{p}) \tag{7}$$

where $\hat{a} = (\hat{q} + i\hat{p})/\sqrt{2}$. Using the Campbell-Hausdorff formula $e^{Y}X\,e^{-Y} = X + [Y,X] + [Y,[Y,X]] + \dots$

$$U(\alpha)^{*}\,\hat{a}\,U(\alpha) = \hat{a} + \alpha \tag{8}$$

Eq. (8) implies that in the coherent state

$$|\alpha\rangle \equiv U(\alpha)\,|0\rangle \tag{9}$$

with $\hat{a}|0\rangle = 0$, for an arbitrary monomial in the $p_\hbar$'s and $q_\hbar$'s, one has

$$\lim_{\hbar\to 0}\langle \tfrac{1}{\sqrt{\hbar}}\alpha|\,\hat{q}_\hbar \cdots \hat{p}_\hbar\,|\tfrac{1}{\sqrt{\hbar}}\alpha\rangle = q \cdots p \ . \tag{10}$$

As an aside notice that in some cases we have to assume that the vacuum state satisfy the factorization property

$$\lim_{\hbar\to 0}\langle 0|\hat{q}\cdots\hat{p}\,|0\rangle = \langle 0|\hat{q}\,|0\rangle\cdots\langle 0|\hat{p}\,|0\rangle \tag{11}$$

and $\langle 0|\hat{q}|0\rangle = 0 = \langle 0|\hat{p}|0\rangle$ in order for (10) to hold (see Yaffe[16,17]). Hepp[5] shows that (10) is preserved under time evolution as given by U(t) in (4),

$$\lim_{\hbar\to 0}\langle \tfrac{1}{\sqrt{\hbar}}\alpha|\hat{q}_{\hbar}(t)\cdots\hat{p}_{\hbar}(t)|\tfrac{1}{\sqrt{\hbar}}\alpha\rangle = q(t)\cdots p(t) \quad , \qquad (12)$$

as long as the classical orbit q(t), p(t) specified by (2) exists and V is of class C^3 and decreases sufficiently fast when $|x| \to \infty$. Eq. (12) is the coherent state version of the Ehrenfest theorem in the classical limit.

The fact that along coherent states the quantum mechanical evolution $\langle \hbar^{-1/2}\alpha|\hat{a}_{\hbar}(t)|\hbar^{-1/2}\alpha\rangle$ and the classical evolution $\alpha(t) = \langle \hbar^{-1/2}\alpha(t)|\hat{a}_{\hbar}|\hbar^{-1/2}\alpha(t)\rangle$ are in "weak correspondence", which becomes exact for $\hbar \to 0$, has been analysed by Klauder[6] (here $\hat{a}_{\hbar}(t) = U(t)^{*}\hat{a}_{\hbar}U(t) = \frac{1}{\sqrt{2}}(\hat{q}_{\hbar}(t)+i\hat{p}_{\hbar}(t))$ and $\alpha(t) = \frac{1}{\sqrt{2}}(q(t)+ip(t))$. However the only rigorous proof of this fact was given by Hepp. Under the same hypotheses underlying (12) he also proves the following important formula

$$\lim_{\hbar\to 0}\| U(t)\,U(\tfrac{1}{\sqrt{\hbar}}\alpha)|0\rangle - U(\tfrac{1}{\sqrt{\hbar}}\alpha(t))|0\rangle\| = 0 \quad . \qquad (13)$$

This equation is constantly mentioned in Klauder's work [16] but never proven. He writes it loosely (as we will do later) as

$$e^{-\frac{i}{\hbar}\hat{\mathcal{H}}t}\,|p_0,q_0\rangle \overset{\hbar\to 0}{\approx} |p_{cl}(t), q_{cl}(t)\rangle \qquad (14)$$

where $|p_{cl}(t),q_{cl}(t)\rangle = \exp(-\frac{i}{\hbar}p_{cl}(t)\hat{Q})\exp(\frac{i}{\hbar}q_{cl}(t)\hat{P})|0\rangle$, $p_0 = p_{cl}(0)$, $q_0 = q_{cl}(0)$, the subscript "cl" refers to the classical orbit.

There are some systems for which (14) is simply

$$e^{-\frac{i}{\hbar}\hat{\mathcal{H}}t}\,|p_0,q_0\rangle = |p_{cl}(t),q_{cl}(t)\rangle \,. \qquad (15)$$

In these cases the initial coherent state is not deformed during the evolution but remains a coherent state along the classical orbit. Such systems are called <u>exact systems</u>.

(1.2) Generalizations to Field Theory.

The action principle leading to the Schroedinger equation (3) is based on the quantum action funtional

$$I(\psi)=\int dt\,(\psi(t),\,i\hbar\,\dot{\psi}(t)-\hat{\mathcal{H}}\psi(t)) \qquad (16)$$

where the dot means t-derivative. If we replace ψ by a coherent state then the <u>restricted action</u>

$$I(p,q)=\int dt\,\langle p(t),q(t)|\left(i\hbar\frac{\partial}{\partial t}-\hat{\mathcal{H}}\right)|p(t),q(t)\rangle \qquad (17)$$

leads, upon arbitrary variation of p and q, to the classical equations of motion (2a): this is the content of the <u>Restricted Action Principle</u>[18], RAP. To arrive at this result just notice that (17) can be reexpressed as (assume $\langle 0|\hat{P}|0\rangle=0=\langle 0|\hat{Q}|0\rangle$)

$$I(p,q)=\int dt\,(\,p(t)\dot{q}(t)-\mathcal{H}(p(t),q(t))\,) \qquad (18)$$

where(*)

(*) Whenever convenient we do not write explicitly the time dependence.

$$\mathcal{H}(p,q) = \langle p,q|\hat{\mathcal{H}}|p,q\rangle \quad . \tag{19}$$

Then the extremal solutions to (18) are given by

$$\dot{q} = \frac{\partial \mathcal{H}(p,q)}{\partial p} \qquad\qquad \dot{p} = -\frac{\partial \mathcal{H}(p,q)}{\partial q} \tag{20}$$

where $\mathcal{H}(p,q)$) in this case is analogous to the classical Hamiltonian (1) . This way of obtaining a relationship between quantum and classical systems will be explored in our discussion of ultralocal scalar fields in the following sense.

To any quantum system described by a Hamiltonian $\hat{\mathcal{H}}$, a pair of conjugate variables and some vacuum state, we can associate a classical system such as (20). The only problem that might arise is to check whether (19) really defines a classical Hamiltonian. This is not obvious when dealing with quantum field theory as we will do later. In fact a conjecture of Klauder called Weak Correspondence Principle[19], WCP, says that given a quantum generator $\hat{\mathcal{G}}$ then the diagonal coherent state matrix elements of $\hat{\mathcal{G}}$ has the form of the classical generator (as $\hbar \to 0$). Consider for instance a neutral scalar field $\hat{\phi}(\underline{x})$ with canonical momentum $\hat{\Pi}(\underline{x})$ in three dimensional Euclidean space. Coherent states analogous to (9) are

$$|f,g\rangle = U[f,g]\,|0\rangle \tag{21}$$

$$U[f,g] = e^{\frac{i}{\hbar}\int d^3x\,\left(f(\underline{x})\hat{\phi}(\underline{x}) - g(\underline{x})\hat{\pi}(\underline{x})\right)} \tag{22}$$

where $f(\underline{x})$, $g(\underline{x})$ are infinitely differentiable smearing functions with compact support. Then one can show that the momentum, angular momentum and Hamiltonian operators have their classical counterparts as given by the WCP (we assume that these operators annihilate the vacuum state). For example the momentum operator $\hat{P}_k = \int d^3x\, \hat{\pi}(\underline{x}) \nabla_k \hat{\phi}(\underline{x})$ gives

$$P_k(f,g) \equiv \langle f,g | \hat{P}_k | f,g \rangle = \int d^3x\, f(\underline{x}) \nabla_k g(\underline{x}) \tag{19}$$

which is the classical generator. Note that the smearing functions are playing the role of classical fields.

These considerations provide us with a framework for finding a formula analogous to (14) for ultralocal scalar fields in §3. We will make an assumption concerning the form of the vacuum (c.f. (2.20), (2.27)) and then show explicitly that when the smearing functions evolve according to the [illegible] with the Hamiltonian as given by the WCP, the evolution of the wave function can be approximated by coherent states as $\hbar \to 0$.

2. Canonical and Affine Ultralocal Fields

In this section we review the essential aspects of ultralocal quantum field theories. Classically the ultralocal scalar field is obtained by taking the Hamiltonian

$$\tilde{\mathcal{H}} = \int d^3x \left(\tfrac{1}{2}\pi(\underline{x})^2 + (\nabla\phi(\underline{x}))^2 + V(\phi(\underline{x})) \right)$$

and dropping the $(\nabla\phi)^2$ term to obtain

$$\mathcal{H} = \int d^3x \left(\tfrac{1}{2}\pi(\underline{x})^2 + V(\phi(\underline{x})) \right). \tag{1}$$

By dropping the spatial derivatives the evolution of the field at distinct spatial points is independent at all times. The light "cone" at each $\underline{x} \in \mathbb{R}^3$ has collapsed to a [illegible] line passing through $\underline{x}$ [illegible] that the quantization of the Hamiltonian, can be accomplished exactly, without being forced to take $V(\phi)$ as a perturbation.

The quantum theory starts with the introduction of creation and annihilation operators $A^{+}(\underline{x},\lambda)$, $A(\underline{x},\lambda)$, $\lambda\epsilon IR$, $\underline{x}\epsilon IR^{3}$ acting on some Hilbert space H and a unit norm state $|0\rangle\epsilon H$, called vacuum state, such that

$$\left[A(\underline{x},\lambda), A^{\dagger}(\underline{x}',\lambda')\right] = \delta(\underline{x}-\underline{x}')\,\delta(\lambda-\lambda') \tag{2}$$

$$\left[A,A\right] = 0 = \left[A^{\dagger},A^{\dagger}\right] \tag{3}$$

$$A|0\rangle = 0\ . \tag{4}$$

In addition we define the operators

$$B(\underline{x},\lambda) = A(\underline{x},\lambda) + C(\lambda) \tag{5}$$

where $C(\lambda)$ is a real valued function satisfying $C(\lambda) = C(-\lambda)$ called model function. We assume that the vacuum state $|0\rangle$ is cyclic.[10]

An overcomplete set of states in H is formed by

$$|\Psi\rangle = e^{\iint d^3x\, d\lambda\, \Psi(\underline{x},\lambda)\, A^{\dagger}(\underline{x},\lambda)}\,|0\rangle \tag{6}$$

where $\psi\epsilon h$, the Hilbert space of square integrable functions of $\underline{x}$ and λ, henceforth called small Hilbert space. Given the Hilbert space structure on h one infers a Hilbert space structure on H through the normalized inner product

$$\langle\Psi'|\Psi\rangle = \exp\left\{-\frac{1}{2}\|\Psi'\|^{2} - \frac{1}{2}\|\Psi\|^{2} + (\Psi',\Psi)\right\} \tag{7}$$

where $(\psi,\phi)=\int d^3x\, d\lambda\, \psi^*\phi$ is the inner product in h and $\|\ \|$ the associated norm.

We wish to have operators acting on H satisfying canonical commutations relations. For the ultralocal representations that we are studying this is not always possible. The operator representations are

$$\hat{\phi}(\underline{x})=\int_{-\infty}^{+\infty} d\lambda\, B^{\dagger}(\underline{x},\lambda)\ \lambda\ B(\underline{x},\lambda) \qquad (8)$$

$$\hat{\pi}(\underline{x})=\int_{-\infty}^{+\infty} d\lambda\, B^{\dagger}(\underline{x},\lambda)\frac{\hbar}{i}\frac{\partial}{\partial\lambda} B(\underline{x},\lambda) \qquad (9)$$

satisfying

$$\left[\hat{\phi}(\underline{x}),\hat{\pi}(\underline{x}')\right]=i\hbar\,\delta(\underline{x}-\underline{x}')\int_{-\infty}^{+\infty} d\lambda\, B^{\dagger}(\underline{x},\lambda)B(\underline{x}',\lambda)\ . \qquad (10)$$

Up to the factor $\int d\lambda\, B^{\dagger}B$ (which formally commutes with $\hat{\phi}$ and $\hat{\pi}$), (8) and (9) are canonically conjugates. For the most interesting representations (the irreducibles) one has $\int d\lambda\, C(\lambda)^2=\infty$ and consequently $\int d\lambda\, B^{\dagger}B$ and $\hat{\pi}(\underline{x})$ are not well defined[1]. The fact that the conjugate momentum is undefined makes using the Weak Correspondence Principle to obtain a classical limit for these theories more difficult. In fact we were unable to use the WCP in those cases where the model function is not square integrable and the discussion about canonical fields is concentrated on those models where the model function satisfy $\int d\lambda\, C(\lambda)^2=M<\infty$. The affine momentum however (see below) is well defined even for non square integrable C.

A consequence of the representation (8) of $\hat{\phi}$ is that the expectation functional

$$E(f) = \langle 0 | e^{i\int d^3x\, f(\underline{x})\hat{\phi}(\underline{x})} | 0 \rangle \quad , \tag{11}$$

where $f(\underline{x})$ is a differentiable function with compact support, satisfies the condition

$$E(f_1 + f_2) = E(f_1)\, E(f_2) \tag{12}$$

for all f_1 and f_2 with disjoint supports. This statistical independence of disjoint spatial volumes is the essence of ultralocality. In addition, the truncated Green's functions are all proportional to products of δ-functions[1] .

The representation of the Hamiltonian operator is given by

$$\hat{\mathcal{H}}(\underline{x}) = \int d\lambda\, B^{\dagger}(\underline{x},\lambda)\, h\, B(\underline{x},\lambda) \tag{13a}$$

where

$$h = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial\lambda^2} + v(\lambda) \quad . \tag{13b}$$

The potential $v(\lambda)$ is determined by the condition

$$\int d\underline{x}\, \hat{\mathcal{H}}(\underline{x})|0\rangle = 0 \tag{14}$$

which implies

$$v(\lambda) = \frac{\hbar^2 C''(\lambda)}{2C(\lambda)} \quad . \tag{15}$$

Observe that the operator $\mathcal{h}$ may also be written in the form

$$\mathcal{h} = a^{\dagger} a \tag{16}$$

with

$$a = \frac{\hbar}{\sqrt{2}} C(\lambda) \frac{\partial}{\partial \lambda} C(\lambda)^{-1} .$$

The matrix elements of the Hamiltonian in the state $|\psi\rangle$ are particularly simple

$$\langle \Psi | \hat{\mathcal{H}}(\underline{x}) | \Psi' \rangle = \langle \Psi | \Psi' \rangle \int d\lambda \, \psi^{*}(\underline{x},\lambda) \mathcal{h} \, \psi'(\underline{x},\lambda) . \tag{17}$$

The operator $\mathcal{h}$ acts only on the λ-dependence of ψ and not on its $\underline{x}$-dependence. The time evolution operator also has a simple action on $|\psi\rangle$

$$e^{-\frac{i}{\hbar}\hat{\mathcal{H}}t} \, |\Psi\rangle = |e^{-\frac{i}{\hbar}\mathcal{h}t} \, \psi\rangle . \tag{18}$$

($\hat{\mathcal{H}} = \int d\underline{x} \, \hat{\mathcal{H}}(\underline{x})$). From (18) it is clear that the dynamics in the field Hilbert space H reduces to that in the small Hilbert space h. In other words instead of doing quantum field theory in H we can equivalently do quantum mechanics in h. The simplicity of ultralocal field theory is to a large extent a consequence of this fact.

The above discussion points out the important role played by the function C. It determines the form of the potential and whether or not $\hat{\pi}$

is a well defined operator. If C is square integrable then $\hat{\pi}$ is well defined and the representation is reducible. In addition the ground state is not unique. If C is not square integrable and $\int \lambda^2(\lambda^2+1)^{-1}C^2(\lambda)d\lambda = \infty$ then neither $\hat{\phi}$ nor $\hat{\pi}$ are well defined[1]. Thus we always assume

$$\int_{-\infty}^{+\infty} d\lambda\, C^2 \frac{\lambda^2}{\lambda^2+1} = N < \infty \quad . \tag{19}$$

This condition also guarantees that (11) is well defined. An example of classes of square integrable and non square integrable model functions satisfying (19) are given respectively by

$$C(\lambda) = \sqrt{|\lambda|}\; e^{-\frac{1}{\hbar} y(\lambda)} \tag{20a}$$

$$C(\lambda) = \frac{1}{\sqrt{|\lambda|}}\, e^{-\frac{1}{\hbar} y(\lambda)} \tag{20b}$$

where $y(\lambda)$ is an even polynomial. For concreteness one could take

$$y(\lambda) = \frac{1}{2} m \lambda^2 \quad . \tag{21}$$

With (21) the potential (15) corresponding to (20) becomes, respectively,

$$v(\lambda) = -\frac{\hbar^2}{8\lambda^2} + \frac{1}{2} m^2\lambda^2 - m\hbar \tag{22a}$$

$$v(\lambda) = \frac{3\hbar^2}{8\lambda^2} + \frac{1}{2} m^2 \lambda^2 \quad . \tag{22b}$$

An alternative ultralocal theory that we will consider is based on the affine commutation relations[2]. The basic structure is as above except that we will replace λ with k which is restricted to $k > 0$. We will represent operators $\hat{\phi}(\underline{x})$ and $\hat{K}(\underline{x})$ satisfying the affine commutation relations

$$[\hat{K}(\underline{x}), \hat{\phi}(\underline{x}')] = i\hbar\, \delta(\underline{x}-\underline{x}')\, \hat{\phi}(\underline{x}) . \tag{23}$$

This is essentially the field version of the commutation relations of the Lie algebra of the affine group.

The affine ultralocal operators are

$$\hat{\phi}(\underline{x}) = \int_0^\infty dk\, B^\dagger(\underline{x},k)\, k\, B(\underline{x},k) \tag{24}$$

$$\hat{K}(\underline{x}) = \int_0^\infty dk\, B^\dagger(\underline{x},k) \frac{\hbar}{2i}\left(\frac{\overleftarrow{\partial}}{\partial k} k - k \frac{\overrightarrow{\partial}}{\partial k}\right) B(\underline{x},k) . \tag{25}$$

Observe that the spectrum of $\hat{\phi}(\underline{x})$ is positive since $k > 0$.

The Hamiltonian for the affine case is taken to be

$$\hat{\mathcal{H}}(\underline{x}) = \int dk\, B^\dagger(\underline{x},k)\, h\, B(\underline{x},k) \tag{26a}$$

with

$$h = -\hbar^2 \frac{\partial}{\partial k} k \frac{\partial}{\partial k} + \frac{\hbar^2 \left(\frac{\partial}{\partial k} k \frac{\partial}{\partial k} C\right)}{C} \tag{26b}$$

so that $\hat{\mathcal{H}}|0\rangle = 0$. Following (20b) we take the model function to be

$$C(k) = \frac{1}{\sqrt{k}}\, e^{-\frac{1}{2} y(k)} \tag{27}$$

but in this case y(k) does not need to be even. As in the canonical case there is a singularity in the potential at k = 0 but this will cause no problem in applying the Weak Correspondence Principle and the Restricted Action Principle to obtain the quasiclassical approximation.

3 Quasiclassical Approximation

A formula similar to (1.14) will be obtained for the affine field. Canonical commutation relations are treated later in this section. These results were obtained by the author in collaboration with M. Pilati. It is implicit in Klauder's work that he was aware of the possibility of obtaining a quasiclassical approximation of this kind using the W.C.P.. However, he has never done it explicitly.

Coherent states analogous to (1.9) are given by the following overcomplete set

$$|p,q\rangle = U[p,q]|0\rangle \tag{1}$$

$$U[p,q] = e^{-\frac{i}{\hbar}\int d^3x\, q(\underline{x})\hat{\phi}(\underline{x})}\, e^{\frac{i}{\hbar}\int d^3x\, \ln p(\underline{x})\hat{K}(\underline{x})} \tag{2}$$

with $p(\underline{x}) > 0$. The real smearing functions p, q defined on $\mathbb{R}^3$ will be taken to be infinitely differentiable and of compact support. The unitary operators (2) expressed in terms of the affine fields $\hat{\phi}(\underline{x})$ and $\hat{K}(\underline{x})$ constitute a representation in the field Hilbert space H of the affine group since $U[p,q]U[p',q'] = U[pp', q+p^{-1}q']$ (the affine group on $\mathbb{R}$ is a two parameter, non abelian group, given by $x \to a^{-1}x + b$, $\forall x \in \mathbb{R}$). Analogously when we work with canonical variables the unitary operators of the theory (c.f. (3.30)) constitute a representation of the Heisenberg group in H.

The restricted action for the states (1) reads

$$I = \int dt \langle p,q| \left(i\hbar \frac{\partial}{\partial t} - \hat{\mathcal{H}} \right) |p,q\rangle$$

$$= \int dt \langle 0| e^{-\frac{i}{\hbar}\int d^3x \ln p \hat{K}} \int d^3y \left(\dot{q}\hat{\phi} - \frac{\dot{p}}{p}\hat{K} \right) e^{\frac{i}{\hbar}\int d^3x \ln p \hat{K}} |0\rangle - \int dt \langle p,q|\hat{\mathcal{H}}|p,q\rangle$$

$$= \int dt d^3x \left(p\dot{q} - \langle p,q|\hat{\mathcal{H}}|p,q\rangle \right) \tag{3}$$

where $\hat{\mathcal{H}} = \int d^3\underline{x}\, \hat{\mathcal{H}}(\underline{x})$ and $\hat{\mathcal{H}}(\underline{x})$ is given by (2.26). To obtain such a result we have imposed

$$\langle 0|\hat{\phi}(\underline{x})|0\rangle = 1 \tag{4}$$

and also we have used

$$U^{\dagger}[p,q](\alpha\hat{\phi}(\underline{x}) + \beta\hat{K}(\underline{x}))U[p,q] = \alpha p(\underline{x})\hat{\phi}(\underline{x}) + \beta(\hat{K}(\underline{x}) + p(\underline{x})q(\underline{x})\hat{\phi}(\underline{x})) \tag{5}$$

Arbitrary variations with respect to p and q in (3) give the following "classical" equations of motion for the smearing functions

$$\dot{q}_{cl}(\underline{x},t) = \frac{\delta\mathcal{H}(p,q)}{\delta p(\underline{x},t)} \tag{6a}$$

$$\dot{p}_{cl}(\underline{x},t) = -\frac{\delta\mathcal{H}(p,q)}{\delta q(\underline{x},t)} \tag{6b}$$

where we have defined

$$\mathcal{H}(p,q) = \langle p,q|\hat{\mathcal{H}}|p,q\rangle . \tag{7}$$

From the Weak Correspondence Principle we interpret (7) as a "classical" Hamiltonian. Thus we have all the necessary ingredients to approximate the evolution of a wave vector $|\psi\rangle$ (2.18). Before doing this it is convenient to compute in more detail the form of the Hamiltonian (7).

Consider the derivative

$$\frac{\delta \mathcal{H}(p,q)}{\delta q(\underline{x})} = \frac{i}{\hbar}\langle p,q|[\dot{\phi}(\underline{x}),\hat{\mathcal{H}}]|p,q\rangle = 2\langle p,q|\hat{K}(\underline{x})|p,q\rangle = 2p(\underline{x})q(\underline{x}).$$

So

$$\mathcal{H}(p,q) = \int d^3x \, p q^2 + W(p) \qquad (8)$$

where the "potential" W(p) is

$$W(p) = \mathcal{H}(p,0) = \langle 0| e^{-\frac{i}{\hbar}\int d^3x \ln p \hat{K}} \, \hat{\mathcal{H}} \, e^{\frac{i}{\hbar}\int d^3x \ln p \hat{K}} |0\rangle$$

$$= \langle 0|\left\{\hat{\mathcal{H}} + \left(-\frac{i}{\hbar}\right)\int d^3x \ln p [\hat{K},\hat{\mathcal{H}}] + \frac{1}{2!}\left(-\frac{i}{\hbar}\right)^2 \iint d^3x \, d^3x' \ln p \ln p' [\hat{K},[\hat{K}',\hat{\mathcal{H}}]] + \cdots\right\}|0\rangle$$

$$= \langle 0|\left\{\hat{\mathcal{H}} - \hbar^2 \sum_{n=1}^{\infty} \iint d^3x \, dk \frac{(-\ln p)^n}{n!} B^{\dagger} \vec{\partial} k \vec{\partial} B + \hbar^2 \sum_{n=1}^{\infty} \int d^3x \, dk \, B^{\dagger} \frac{(\ln p \, k \vec{\partial})^n}{n!} \left(\frac{\vec{\partial} k \vec{\partial} c}{c}\right) B\right\}|0\rangle$$

$$= \iint d^3x \, dk \langle 0| B^{\dagger}(\underline{x},k) \, h(kp) \, B(\underline{x},k)|0\rangle$$

$$= \iint d^3x \, dk \, C(k) \, h(kp) \, C(k) \qquad (9)$$

and use has been made of $e^{gk\partial} f(k) = f(ke^g)$ for differentiable f, g with

$\vec{\partial} \equiv \frac{\partial}{\partial k}$. It can be shown that, writing the operator $\hbar$ as

$$\hbar = a^{+} a \tag{10}$$

for $-i\hbar\sqrt{k}\; C(k) \frac{\partial}{\partial k} C(k)^{-1}$, (9) becomes

$$W(\rho) = \iint d^3x\, dk\, \rho\, |\sqrt{k}\, a\, C(k\rho^{-1})|^2$$

$$= \hbar^2 \iint d^3x\, dk\, \rho^{-1} k\, C^2(k) \left| \frac{\partial}{\partial k} \frac{C(k\rho^{-1})}{C(k)} \right|^2 . \tag{11}$$

The evolution (2.18) can be expressed in the small Hilbert space h as

$$e^{-\frac{i}{\hbar} \hbar t}\, \Psi(x, k) = \Psi(x, k, t) . \tag{12}$$

Coherent states in the small Hilbert space that approximate the r.h.s. of (12) are obtained as follows. Since the states (2.6) are eigenstates of A,

$$A(x, k)\, |\Psi\rangle = \Psi(x, k)\, |\Psi\rangle , \tag{13}$$

the vector $|\phi_{p,q}\rangle$ of the Hilbert space H corresponding to a coherent state $|p, q\rangle \in H$ is determined by the following element of h:

$$\Psi_{p,q}(x, k) = e^{-\frac{i}{\hbar} q(x) k} (p(x))^{-1/2}\; C(k\, p(x)^{-1}) - C(k) \tag{14}$$

(just compute $A(x, k)|p, q\rangle$ to obtain $\Psi_{p,q}(x, k)|p, q\rangle$).

The Schroedinger equation in h reads

$$h\Psi(k) = i\hbar\dot{\Psi}(k) \tag{15}$$

where it is not necessary to write explicitly the $\underline{x}$-dependence since, due to ultralocality, the dynamics at the spatial point x_1 is a copy of that at $\underline{x}_2$, for any $\underline{x}_1 \neq \underline{x}_2$. Thus our problem has been reduced to "quantum mechanics in k-space". When some initial condition is chosen, (15) is equivalent to (12). We will always assume that the initial state is a coherent state

$$\Psi_0 \equiv \Psi_{p_0,q_0} \tag{16}$$

with $p_0 \equiv p(0)$, $q_0 \equiv q(0)$. Observe that if we were dealing with an exact system then (16) would evolve to another coherent state and we could have written (15) as

$$h\Psi_{p,q} = i\hbar\dot{\Psi}_{p,q} \tag{17}$$

where the t-dependence is entirely contained in p and q. For instance this kind of evolution occurs when the model function is constant. In general we will show that when C(k) has the form (2.27) then system (6) with Hamiltonian (8)

$$\dot{q}_{cl} = q^2 + \frac{\delta W}{\delta p} \tag{18a}$$

$$\dot{p}_{cl} = -2pq \tag{18b}$$

give, as $\hbar \to 0$, an approximate solution to the Schroedinger equation that is

$$\mathfrak{h}\,\Psi_{p_{cl},q_{cl}} \simeq i\hbar\,\dot{\Psi}_{p_{cl},q_{cl}} \;. \tag{19}$$

In other words the right hand side of (12) can be approximated by properly selected coherent states:

$$e^{-\frac{i}{\hbar}\mathfrak{h}t}\,\Psi_{p_{cl}(0),q_{cl}(0)} \simeq \Psi_{p_{cl}(t),q_{cl}(t)} \;. \tag{20}$$

To obtain (19) just compute its right and left hand sides $\left(\Psi^c_{p,q} \equiv \Psi_{p,q}\cdot C\right)$:

$$\mathfrak{h}\,\Psi_{p,q} = \left(-\hbar^2\vec{\partial}k\vec{\partial} + \hbar^2 C^{-1}(\vec{\partial}k\vec{\partial}C)\right)\Psi^c_{p,q}$$

$$= -\hbar^2\left(\vec{\partial} + k\vec{\partial}^2\right)\Psi^c_{p,q} + \hbar^2\left(\frac{1}{4k} + \frac{1}{\hbar^2}ky'^2 - \frac{k}{\hbar}y''\right)\Psi^c_{p,q}$$

$$= -\hbar^2\left\{\left(-\frac{i}{\hbar}q + C'_p\rho^{-1}C_p^{-1}\right) + k\left[\left(C''_p\rho^{-2}C_p^{-1} - C'^2_p\rho^{-2}C_p^{-2}\right) + \left(-\frac{i}{\hbar}q + C'_p\rho^{-1}C_p^{-1}\right)^2\right] - \left(\frac{1}{4k} + \frac{1}{\hbar^2}ky'^2 - \frac{k}{\hbar}y''\right)\right\}\Psi^c_{p,q}$$

$$= \left\{\hbar k\left[y''_p\rho^{-2} - y''\right] + \left[kq^2 - 2iqk\rho^{-1}y'_p + ky'^2 - ky'^2_p\rho^{-2}\right]\right\}\Psi^c_{p,q} \tag{21}$$

$$i\hbar\dot{\Psi}_{p,q} = i\hbar\left\{-\frac{i}{\hbar}\dot{q}k - \frac{1}{2}\rho^{-1}\dot{\rho} - C'_p\rho^{-2}\dot{\rho}k\,C_p^{-1}\right\}\Psi^c_{p,q}$$

$$= \left\{\dot{q}k + iy'_p\rho^{-2}\dot{\rho}k\right\}\Psi^c_{p,q} \tag{22}$$

where $\mathfrak{h}$ is given in (2.26b) and $C(k)$ in (2.27). The symbols y', y'_p mean derivative with respect to k and kp^{-1} respectively and $y \equiv y(k)$, $y_p \equiv y(kp^{-1})$. To know under what conditions (19) holds we equate (21) to (22) as $\hbar \to 0$ and obtain

$$k q^2 + k y'^2 - k y_\rho'^2 \rho^{-2} \approx \dot{q} k \tag{23}$$

$$2 i q k \rho^{-1} y_\rho' \approx i y_\rho' \rho^{-2} \dot{\rho} k \ . \tag{24}$$

Multiplying (23) by $\rho^2 C(k\rho^{-1})^2$ and integrating over k from 0 to ∞ one gets the following relations (assume $\int_0^\infty dk\, k C(k)^2 = 1$ as in (4))

$$\dot{q} \approx q^2 + V(\rho) \tag{25a}$$

$$\dot{\rho} \approx -2\rho q \tag{25b}$$

where

$$V(\rho) = \int dk\, \rho^{-2} k C_\rho^2 \left(y'^2 - y_\rho'^2 \rho^{-2} \right) \ . \tag{26}$$

Observe that we must check that the term $\hbar k[y_\rho'' \rho^{-2} - y'']$ dropped from (21) does not blow up when integrated over $\rho^{-2} C(k\rho^{-1})^2 dk$ because we are dealing with non square integrable C(k). Indeed

$$\int_0^\infty dk\, \rho^{-2} C(k\rho^{-1})^2 \hbar k \left[y_\rho'' \rho^{-2} - y'' \right] < \infty \ . \tag{27}$$

Comparing (25) with (18) it is clear that the coherent state $\psi_{\rho_\alpha(t),\, q_\alpha(t)}$ will approximate the right hand side of (20) if we could show that

$$V \approx \frac{\delta W}{\delta \rho} \tag{28}$$

as $\hbar \to 0$. To see that (28) holds we just take the derivative of the "potential" W as given by (11)

$$\frac{\delta W}{\delta \rho} \approx \int dk\, \rho^{-2} k\, C_\rho^2 \left(y'^2 - y_0'^2 \rho^{-2}\right) - 2\int dk\, \rho^{-3} k^2\, C_\rho^2\, y'' \left(y_0' - y' \rho\right). \qquad (29)$$

Define a change of variables by setting $\hbar^{-1} y(k) \equiv \hbar^{-1} k\, z(k) \equiv u\, z(\hbar u)$. Then the second term in the right hand side of (29) is clearly of higher order in $\hbar$ than the first and thus can be dropped. This gives the desired result.

The conclusion is that the coherent state whose smearing functions evolve according to the "classical" system (18) give the approximation (20).

The same analysis can now be applied to affine or canonical fields whose model function is square integrable. This later topic will be discussed next while the unsettled case of non square integrable canonical models does not seem to fit the present framework (see below).

Suppose we take $C(\lambda)$ as given by (2.20a). Using (2.13b) we compute under what conditions the approximation holds. The coherent states analogous to (2) are in this case constructed from the unitary operators

$$U[p,q] = e^{-\frac{i}{\hbar}\int d^3x\, p(x)\hat{\phi}(x)}\, e^{\frac{i}{\hbar}\int d^3x\, q(x)\hat{\pi}(x)} \qquad (30)$$

in terms of the pair of canonical fields $\hat{\phi}(\underline{x})$ and $\hat{\pi}(\underline{x})$. The coherent state

$$|p,q\rangle = U[p,q]|0\rangle \tag{31}$$

corresponds to the following element of the small Hilbert space

$$\Psi_{p,q}(\xi,\lambda) = e^{-\frac{i}{\hbar}p(x)\lambda}\, C(\lambda+q(x)) - C(\lambda) \quad . \tag{32}$$

Now we can compute both sides of (19) ($\Psi^c_{p,q} \equiv \Psi_{p,q} + C$):

$$h\Psi_{p,q} = \left(-\frac{\hbar^2}{2}\partial^2 + \frac{\hbar^2}{2}C^{-1}C''\right)\Psi^c_{p,q}$$

$$= \left\{-\frac{\hbar^2}{2}\left[\left(C_q'' C_q^{-1} - C_q'^2 C_q^{-2}\right) + \left(-\frac{i}{\hbar}p + C_q' C_q^{-1}\right)^2\right] + \frac{\hbar^2}{2}\left(-\frac{1}{4\lambda^2} - \frac{1}{\hbar}y'' - \frac{1}{\hbar\lambda}y' + \frac{1}{\hbar^2}y'^2\right)\right\}\Psi^c_{p,q}$$

$$= \left\{\frac{\hbar^2}{2}\left(\frac{1}{4(\lambda+q)^2} - \frac{1}{4\lambda^2}\right) + \frac{\hbar}{2}\left(y_q'' - y'' + \frac{1}{q+\lambda}y_q' - \frac{1}{\lambda}y' + \frac{ip}{\lambda+q}\right) + \frac{1}{2}\left(p^2 - 2ipy_q' + y'^2 - y_q'^2\right)\right\}\Psi^c_{p,q} \tag{33}$$

$$i\hbar\dot{\Psi}_{p,q} = i\hbar\left\{-\frac{i}{\hbar}\dot{p}\lambda + C_q'\dot{q}C_q^{-1}\right\}\Psi^c_{p,q}$$

$$= i\hbar\left\{-\frac{i}{\hbar}\dot{p}\lambda + \dot{q}\left(\frac{1}{2(\lambda+q)} - \frac{1}{\hbar}y_q'\right)\right\}\Psi^c_{p,q}$$

$$= \left\{\dot{p}\lambda + \frac{i\hbar\dot{q}}{2(\lambda+q)} - i\dot{q}y_q'\right\}\Psi^c_{p,q} \quad . \tag{34}$$

The symbols y', y_q' mean derivative with respect to λ and $q+\lambda$ respectively and $y \equiv y(\lambda)$, $y_q \equiv y(q+\lambda)$. Equating (33) with (34) and taking $\hbar \to 0$ one obtains

$$\dot{q} \approx p \tag{35}$$

$$\dot{p}\lambda \approx \frac{1}{2}\left(p^2 - \left(y_q'^2 - y'^2\right)\right) . \tag{36}$$

Multiplying (36) by $\frac{1}{\lambda}C(\lambda)^2$ and integrating over λ from $-\infty$ to $+\infty$ one gets (from now on assume that the model function is normalized: $\int_{-\infty}^{\infty} C(\lambda)^2 d\lambda = 1$)

$$\dot{q} \approx p \tag{37a}$$

$$\dot{p} \approx -\frac{1}{2}\int d\lambda \, \frac{1}{\lambda} C(\lambda)^2 \left(y_q'^2 - y'^2\right) \equiv V(q) . \tag{37b}$$

Notice that

$\int_{-\infty}^{\infty} d\lambda \frac{1}{\lambda} C(\lambda)^2 = 0$ since $C(\lambda) = C(-\lambda)$. The next step will be to show that (37a) and (37b) are the "classical" solutions obtained from the restricted action principle thus confirming this approximation scheme.

The Restricted Action

$$I = \int dt \langle p,q | \left(i\hbar \frac{d}{dt} - \hat{\mathcal{H}}\right) | p,q \rangle = \iint d^3x \, dt \left(p\dot{q} - \langle p,q | \hat{\mathcal{H}}(\underline{x}) | p,q \rangle\right) \tag{38}$$

gives the following extremal solutions for the smearing functions

$$\dot{q}_{cl}(\underline{x},t) = \frac{\delta \mathcal{H}(p,q)}{\delta p(\underline{x})} \tag{39a}$$

$$\dot{p}_{cl}(\underline{x},t) = -\frac{\delta \mathcal{H}(p,q)}{\delta q(\underline{x})} . \tag{39b}$$

Again, according to the Weak Correspondence Principle,

$$\mathcal{H}(p,q) = \int d^3x \langle p,q | \hat{\mathcal{H}}(\underline{x}) | p,q \rangle \tag{40}$$

is interpreted as a "classical" Hamiltonian. One can show that

$$\mathcal{H}(p,q)=\frac{1}{2}\int d\lambda\, C(\lambda)^2\int d^3x\; p(\underline{x})^2 + W(q) \tag{41}$$

where

$$W(q)=\mathcal{H}(0,q)=\langle 0|\, e^{-\frac{i}{\hbar}\int d^3x\, q(\underline{x})\hat{\pi}(\underline{x})}\,\hat{\mathcal{H}}\, e^{\frac{i}{\hbar}\int d^3x\, q(\underline{x})\hat{\pi}(\underline{x})}|0\rangle$$

$$=\iint d^3x\, d\lambda\, C(\lambda)\, h(\lambda+q)\, C(\lambda)$$

$$=\iint d^3x\, d\lambda\, C(\lambda-q)\, h(\lambda)\, C(\lambda-q) \equiv \iint d^3x\, d\lambda\, C_q\, h\, C_q\,.$$

As $\hbar \to 0$ this becomes

$$W(q) \approx \frac{1}{2}\iint d^3x\, d\lambda\, C_q^2\,(y'^2 - y_q'^2) \tag{42}$$

(the subscript q means the corresponding quantity depends on λ-q because we have made the change of variable λ+q→λ). Eq. (39) is now rewritten as

$$\dot{q}_{cl} = p \tag{43a}$$

$$\dot{p}_{cl} = -\frac{\delta W}{\delta q} \tag{43b}$$

where, from (42)

$$-\frac{\delta W}{\delta q} \approx \frac{1}{2}\int_{-\infty}^{+\infty} d\lambda\, 2C_q C_q' \,(y'^2 - y_q'^2) - \frac{1}{2}\int_{-\infty}^{+\infty} d\lambda\, C_q^2\,(2y_q' y_q'') \tag{44}$$

$$= \int_{-\infty}^{+\infty} d\lambda\, C_q C_q' (y'^2 - y_q'^2) = \int_{-\infty}^{+\infty} d\lambda\, C_q^2 \left(\frac{1}{2(\lambda-q)} - \frac{1}{\hbar} y_q'\right)(y'^2 - y_q'^2)$$

since the second integrand in (44) is odd and drops out. Changing variables again $\lambda - q \to \lambda$, and dropping another odd integrand, one gets finally

$$-\frac{\delta W}{\delta q} \approx \frac{1}{2}\int_{-\infty}^{+\infty} d\lambda\, C(\lambda)^2 (y_q'^2 - y'^2)$$

which is nothing more than V(q), that is (37) corresponds to (39) as $\hbar \to 0$.

If the model function is not square integrable then we run into trouble when trying to use the Weak Correspondence Principle because the first term in the right hand side of (41) turns out to be infinite since it is multiplied by $\int d\lambda\, C(\lambda)^2$. It is true that equations similar to (35) and (36) can still be obtained in this case but we cannot integrate (36) over $\lambda^{-1} C(\lambda)^2$ to obtain an equation for $\dot{p}$. It is not known as yet how to solve this drawback.

6. Discussion and Conclusions

The simplicity of ultralocal field theory stems from the fact that we can work in the small Hilbert space with finite degrees of freedom instead of the field Hilbert space. We have seen that affine fields are less singular than canonical fields and that the approximation scheme in the small Hilbert space studied so far breaks down when non square integrable model functions are used in the canonical context. This is because we were unable to get rid of an infinity of the form $\int d\lambda\, C(\lambda)^2$ which completely hinders the use of the WCP and the restrited action to define the "classical" orbit.

The ideas presented in this paper have already been applied to some cosmological situations where the gravitational field exhibits a kind of spontaneously decoupled dynamics in the asymptotic region close to the initial singularity (the gravitational field is ultralocal in that region for a large class of solutions to Einsten field equations [21,22,23]). Strong Coupling Yang-Mills fields can also be treated as an ultralocal field and this topic will be tackled in a future publication.

Although we recognize that the concept of ultralocality must be weakened in order that operators like $(\nabla\phi)^2$ be represented the essential ideas proved to be a useful guideline for future developments in those theoretical contexts where ultralocality can be given a physical meaning.

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