INFLUENCE OF CUSPS AND INTERSECTIONS ON THE CALCULATION OF THE WILSON LOOP IN ν-DIMENSIONAL SPACE *

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Abstract

A discussion is given about the influence of cusps and intersections on the calculation of the Wilson Loop in ν-dimensional space. In particular, for the two-dimensional case, it is shown that there are no divergences. (Author)

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1. INTRODUCTION

If we want to describe the dynamics of any system it is of primary importance a good choice of a set of coordinates.

In Gauge Theory the choice of good coordinates is equally important. They not only make the treatment simpler but also provide insight into the dynamics. In non-Abelian Gauge Theories a good choice of coordinator is the expectation value of a non-local gauge invariant operator, the well known Wilson Loop (WL) defined by an appropriate Euclidean functional integral [1].

We can write the Wilson Loop as a perturbative series if we want to make non-trivial calculations involving it. In this case, however, we have the appearence of contour integrals which are divergent. For smooth (i.e. differentiable) and simple (i.e. non-intersecting) loops, it was shown [2] that those integrals are renormalizable in four dimensions, and therefore also the WL, in all orders of perturbation theory. It is, however, insufficient to consider only smooth and simple loops, because for simple loops the equations of motion reduce essentially to their Abelian limit and so contain no informations about the crucial non-Abelian dynamics [3]. Thus, it is essential to consider self-intersecting loops and loops with cusps. The renormalization program for such loops in four dimensions was carried out by Brandt, Neri and Sato [4].

In this paper we investigate the influence of angles and other geometrical factors (lengths, for example) in the residue of the singularity that appears in the contour integrals as a function of the number of dimensions ν of the space. For this we make use of the dimensional regularization method [5] for treating the infinities that appear in our computations. A particular discussion is given for the two dimensional case.

For simplicity we shall limit the discussion to planar curves in ν-dimensional spaces and we consider only the second order term of the perturbative series of the WL which is given by

$$W^{(2)}(C) = -\frac{1}{2}\oint_C dx^{\alpha}\oint_C dy^{\beta} D_{\alpha\beta}(x-y) \qquad (1)$$

where $D_{\alpha\beta}(x-y)$ is the free gluon propagator and in ν dimensions is given by

$$D_{\alpha\beta}(x-y) = \int \frac{d^{\nu}k}{(2\pi)^{\nu}} \left(\frac{g_{\alpha\beta}}{k^2} - A\,\frac{k_{\alpha}k_{\beta}}{k^4} \right) e^{ik.x} \qquad (2)$$

A being a gauge fixing constant.

In configuration space (2) leads to [6]

$$D_{\alpha\beta}(x-y) = \frac{\Gamma\left(\frac{\nu-2}{2}\right)}{4\pi^{\nu/2}} \left\{ g_{\alpha\beta}|x-y|^{2-\nu} - \frac{A}{2(\nu-4)}\,\partial^{x}_{\alpha}\partial^{y}_{\beta}|x-y|^{4-\nu} \right\} \qquad (3)$$

Since in our calculations we shall consider open curves, we shall use the notation V(C) instead of $W^{(2)}(C)$. Then, putting (3) into (1) with $W^{(2)}(C)$ exchanged by V(c), we obtain,

$$V(C) = -\frac{\Gamma\left(\frac{\nu-2}{2}\right)}{8\pi^{\nu/2}} \left\{ \int_C \int_C dx^{\alpha}dy^{\alpha}|x-y|^{2-\nu} + \frac{A}{\nu-4}\,|a-b|^{4-\nu} \right\} \qquad (4)$$

a,b being the end points of C.

Sometimes it is convenient to compute V(C) in momentum space. To do this we proceed in analogy with Abud, Bollini and Giambiagi [7] by defining a linear functional over the curve C,

$$f^{\alpha}_{C}(k) = \int_{C} e^{ik.x} dx^{\alpha} \tag{5}$$

With (5) we can write V(C) as

$$V(C) = -\frac{1}{2(2\pi)^{\nu}} \int d^{\nu}k \left[\frac{g_{\alpha\beta}}{k^2} - \frac{2k_{\alpha}k_{\beta}}{(3-\nu)k^4} \right] f^{\alpha}_{C}(k) f^{\beta *}_{C}(k) \tag{6}$$

where we have taken $A = \frac{2}{3-\nu}$ (dimensional gauge) as in reference (7).

Suppose now that the curve C is contained in some subspace with n dimensions. As in reference (7), we can decompose the vector k as $k_{\mu} = k^{\perp}_{\mu} + \hat{k}_{\mu}$, where $\hat{k}_{\mu}$ is the projection of k_{μ} over the subspace containing C, and $k^{\perp}$ is the orthogonal component of k. Now, performing the integration over $k^{\perp}$ in equation (6) and using the fact that $f_c(k)$ only depends on $\hat{k}$, we get [7]

$$V(C) = -\frac{\Gamma\left(1 - \frac{\nu-n}{2}\right)}{2^{\nu+1}\pi^{(\nu+n)/2}} \int d^n\hat{k} (\hat{k}^2)^{\frac{\nu-n}{2}-1} \left[g_{\alpha\beta} - \frac{2-\nu+n}{3-\nu} \frac{\hat{k}_{\alpha}\hat{k}_{\beta}}{\hat{k}^2} \right] f^{\alpha}_{C}(\hat{k}) f^{\beta *}_{c}(\hat{k}) \tag{7}$$

We shall only consider curves with n = 2, so we can drop the hat over k, and call the components of k by k_1 and k_2. In terms of the integration over k_1 and k_2, V(C) is given by

$$V(C) = \frac{-2\Gamma\left(\frac{4-\nu}{2}\right)}{(4\pi)^{(\nu+2)/2}} \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 (k_1^2 + k_2^2)^{(\nu-4)/2} |f|^2 + \frac{\Gamma(\nu-4)|a-b|^{4-\nu}}{(4\pi)^{(\nu-1)/2}\Gamma[(\nu-1)/2]} \tag{8}$$

2. SINGULAR POINTS

We are interested in the effect of cusps and intersections on the calculation of the Wilson Loop. Initially, we consider the interaction between two segments with lengths L_1 and L_2 (see fig. 1), that touches one another forming an angle .

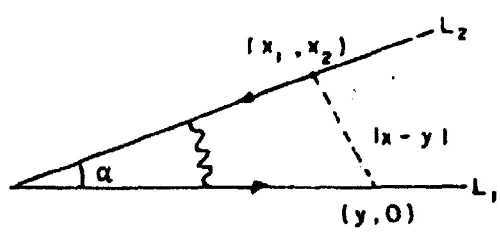

Fig. 1 - One segment just touching another at the end points.

In accordance with the general formalism, the interaction between the segments, in the dimensional gauge, is

$$V = \frac{-2\Gamma[(\nu-2)/2]}{8\pi^{\nu/2}} \int_{seg1} dx^{\beta} \int_{seg2} dy^{\beta} \left[(y-x_1)^2 + x_2^2\right]^{(2-\nu)/2} - \frac{\Gamma[(\nu-2)/2](L_1^{4-\nu} + L_2^{4-\nu} - |\vec{L}_1-\vec{L}_2|^{4-\nu})}{4\pi^{\nu/2}(3-\nu)(4-\nu)} \qquad (9)$$

As in reference (7), we can performe the following change of variables

$$x_1 = \zeta\cos\alpha$$
$$x_2 = \zeta\sin\alpha \qquad (10)$$
$$\eta = (y-x_1) = \lambda x_1 \; .$$

Then, Eq. (9) becomes,

$$V(C) = \frac{-2\Gamma[(\nu-2)/2]}{8\pi^{\nu/2}} \int_{L_2}^{0} d\zeta \int_{-1}^{\infty \operatorname{sgn}(\cos\alpha)} d\lambda\, \zeta^{3-\nu} \cos^2\alpha (\sin\alpha)^{2-\nu} (1 + \lambda^2 \operatorname{cotg}^2\alpha)^{(2-\nu)/2}$$

$$- \frac{\Gamma[(\nu-2)/2](L_1^{4-\nu} + L_2^{4-\nu} - |\vec{L}_1 - \vec{L}_2|^{4-\nu})}{4\pi^{\nu/2}(3-\nu)(4-\nu)} \qquad (11)$$

Integrating Eq. (1_1) over ζ, we obtain

$$V = \frac{2\Gamma[(\nu-2)/2]}{8\pi^{\nu/2}} \frac{L_2^{4-\nu}}{(4-\nu)} \cos^2\alpha(\sin\alpha)^{2-\nu} \int_{-1}^{\infty \mathrm{sgn}(\cos\alpha)} d\lambda (1 + \lambda^2 \mathrm{cotg}^2\alpha)^{(\nu-2)/2} -$$

$$- \frac{\Gamma[(\nu-2)/2](L_1^{4-\nu} + L_2^{4-\nu} - |\vec{L}_1 - \vec{L}_2|^{4-\nu})}{4\pi^{\nu/2}(3-\nu)(4-\nu)} \qquad (1_2)$$

From Eq. (12) we see that there is a pole at $\nu = 3$ associated with the gauge choice which can be removed; at $\nu = 4$, there is a pole whole residue is given by[7]

$$R = \frac{(\pi-\alpha)\mathrm{cotg}\alpha + 1}{4\pi^2} . \qquad (1_3)$$

For $\nu = 5,6,\ldots$ there is no explicit pole. Note that the result (15) is valid when one of the arrows enters and the other leaves the vertex. In the other cases the residue takes a minus sign.

The case $\nu = 2$ will be considered later.

Now, we are interested in the case of intersection, and we take two segments of equal lengths L (see Fig. 2)

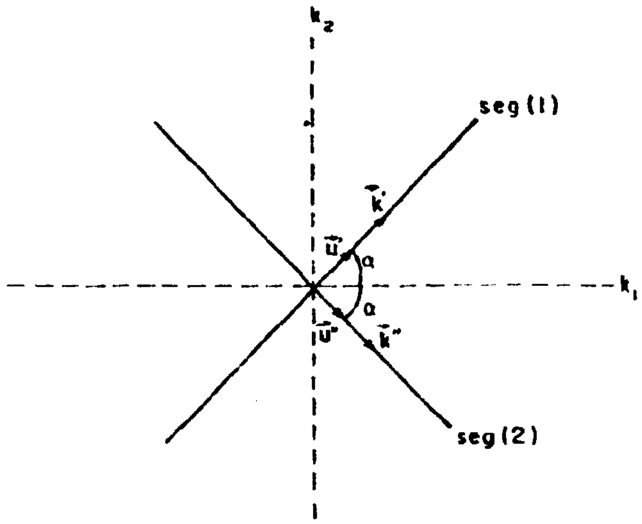

Fig. 2 - Two segment that cross each other forming an angle 2α.

From expression (5) we have for a segment along the k_1 axes the following result for the functional over it

$$f_{seg} = \int_{-L/2}^{L/2} e^{ik_1x_1} dk_1 = \frac{2}{k_1} \sin(k_1 \frac{L}{2}) \qquad (14)$$

or, with the use of the vectorial notation,

$$\vec{f}_{seg(1)} = \frac{2}{k'} \sin(k' \frac{L}{2}) \vec{u}'$$

$$\vec{f}_{seg(2)} = \frac{2}{k''} \sin(k'' \frac{L}{2}) \vec{u}'' \qquad (15)$$

where $\vec{u}'$ and $\vec{u}''$ are unit vectors in the directions of $\vec{k}'$ and $\vec{k}''$, respectively.

It is easy to see that k', k'', k_1 and k_2 are related by

$$k'k'' = k_1^2\cos^2\alpha - k_2^2\sin^2\alpha \quad . \qquad (16)$$

From the general formalism, the interaction between the segments, in the Fourier space, is given by

$$V = \frac{-16\Gamma[(4-\nu)/2]}{(4\pi)^{(\nu+2)2}}\int_{-\infty}^{+\infty}dk_1\int_{-\infty}^{+\infty}dk_2\cos(2\alpha)\,\frac{\sin(k_1\frac{L}{2}\cos\alpha + k_2\frac{L}{2}\sin\alpha)\,\sin(k_1\frac{L}{2}\cos\alpha - k_2\frac{L}{2}\sin\alpha)}{(k_1^2\cos^2\alpha - k_2^2\sin^2\alpha)(k_1^2 + k_2^2)^{(4-\nu)/2}} \qquad (17)$$

where we have used that $|f|^2 = 2f_{seg\,(1)}f_{seg\,(2)}$ to take into acount the total contribution of the interaction.

In (17) we take the gauge term equal to zero.

Using

$$\sin(k_1\frac{L}{2}\cos\alpha + k_2\frac{L}{2}\sin\alpha)\sin(k_1\frac{L}{2}\cos\alpha - k_2\frac{L}{2}\sin\alpha) =$$

$$= \sin^2(k_1\frac{L}{2}\cos\alpha) - \sin^2(k_2\frac{L}{2}\sin\alpha) \tag{18}$$

in (17) and the following integral formulas [8]

$$\int_0^\infty x^{\lambda-1}(1+\alpha x^p)^{-\mu}(1+\beta x^p)^{-\nu}dx = \frac{1}{p}\,\alpha^{-\lambda/p}B(\frac{\lambda}{p}, \mu+\nu-\frac{\lambda}{p})\;{}_2F_1(\nu, \frac{\lambda}{p}; \mu+\nu; 1-\frac{\beta}{\alpha}) \tag{19}$$

and

$$\int_0^\infty x^{\mu-1}\sin^2(ax)dx = -\frac{\Gamma(\mu)\cos(\frac{\mu\pi}{2})}{2^{\mu+1}a^\mu} \tag{20}$$

we obtain the following result

$$V = \frac{16\Gamma[(4-\nu)/2]}{(4\pi)^{(\nu+2)/2}}\;B(\frac{1}{2}, \frac{5-\nu}{2})\cos(2\alpha)\;\frac{\Gamma(\nu-4)\cos(\frac{\nu-4}{2}\pi)}{L^{\nu-4}}$$

$$\times\left\{(\sin^2\alpha)^{(2-\nu)/2}{}_2F_1(1,\frac{1}{2};\frac{6-\nu}{2};\frac{1}{\sin^2\alpha}) + (\cos^2\alpha)^{(2-\nu)/2}{}_2F_1(1,\frac{1}{2};\frac{6-\nu}{2};\frac{1}{\cos^2\alpha})\right\} \tag{21}$$

Using the relations

$$B(x,y) = \frac{\Gamma(x)\ \Gamma(y)}{\Gamma(x+y)} \quad \text{and} \quad \Gamma(n+1) = \Gamma(n) \qquad (22)$$

we can write (21) as

$$V = \frac{16\cos(2\alpha)\ \cos[(\nu\pi)/2]\,\Gamma[(5-\nu)/2]\,\Gamma(\nu-4)}{(4\pi)^{(\nu+1)/2}\ L^{\nu-4}\ (4-\nu)} \left\{ \frac{{}_2F_1(1, \frac{1}{2}; \frac{6-\nu}{2}; \frac{1}{\sin^2\alpha})}{(\sin^2\alpha)^{\nu-2/2}} \right.$$

$$\left. + \frac{{}_2F_1(1, \frac{1}{2}; \frac{6-\nu}{2}; \frac{1}{\cos^2\alpha})}{(\cos^2\alpha)^{\nu-2/2}} \right\} \qquad (23)$$

For $\nu = 4$, Eq. (23) has one pole with residue

$$R = -\frac{\mathrm{cotg}(2\alpha)}{2\pi} \qquad (24)$$

The above result corresponds to the case in which the arrows leave the vertex (the present case) or enter the vertex. On the other cases the sign must be changed. Note that the result (24) can be obtained by considering the residue associated with each pair of segments that compose the actual case.

To analyze the cases $\nu = 3,5,6,\dots$, we make use of the formulae 9.131-2 and 9.132-1 of ref. (8) and of the relation $\Gamma(1-x)\ \Gamma(x) = \pi/\mathrm{sen}(\pi x)$. With these we have the final result

$$V = \frac{16\cos(2\alpha)\ \cos[(\nu\pi)/2]\,\Gamma[(5-\nu)/2]\,\Gamma[(6-\nu)/2]\,\Gamma(\nu-4)}{(4\pi)^{(\nu+1)/2}\ L^{\nu-4}\,(4-\nu)}$$

$$\times \left\{ \frac{-\,tg^2\alpha}{(\sin^2\alpha)^{(\nu-2)/2}}\ \frac{\Gamma(-1/2)}{\Gamma(1/2)\ \Gamma[(4-\nu)/2]}\ {}_2F_1(1,\ \tfrac{5-\nu}{2};\ \tfrac{3}{2};\ -\,tg^2\alpha)\ + \right.$$

$$+ \frac{(-\,tg^2\alpha)^{1/2}}{(\sin^2\alpha)^{(\nu-2)/2}}\ \frac{\Gamma[(\nu-3)/2]\ \sin[\pi/2(\nu-3)]}{\Gamma(1/2)}\ {}_2F_1(\tfrac{1}{2},\ \tfrac{4-\nu}{2};\ \tfrac{1}{2};\ -tg^2\alpha)\ +$$

$$+ \frac{\Gamma[(3-\nu)/2]}{(\cos^2\alpha)^{(\nu-2)/2}\,\Gamma[(4-\nu)/2]\,\Gamma[(5-\nu)/2]}\ {}_2F_1(\ \tfrac{1}{2};\ \tfrac{\nu-1}{2};\ -\,tg^2\alpha)\ +$$

$$\left. + \frac{(-\,tg^2\alpha)^{(3-\nu)/2}}{(\cos^2\alpha)^{(\nu-2)/2}}\ \frac{\Gamma[(\nu-3)/2]}{\Gamma(1/2)}\ {}_2F_1(\tfrac{4-\nu}{2};\ \tfrac{5-\nu}{2};\ \tfrac{5-\nu}{2};\ -\ tg^2\alpha) \right\} . \qquad (25)$$

For $\nu = 5$, (25) has no pole. (The pole of $\Gamma(\frac{5-\nu}{2})$ is compensated by the zero of $\cos(\frac{\nu\pi}{2})$). For ν even ($\nu > 8$) there is a pole associated with $\Gamma(\frac{6-\nu}{2})$. For $\nu = 3$, V given by (25) has one pole whose residue is

$$R = -\,\frac{L}{\pi}\,\cos(2\alpha) \qquad (26)$$

Then, for $\nu = 3$ and 4, V given by (25) has poles whose residue depends on geometric factors as the length of the segments and the angle between them.

3. TWO-DIMENSIONAL CASE

In the case of intersection of two-segments, for $\nu = 2$, the analysis is simple. As we can see from (25) there is no pole for such value of ν. In the case of two segments that form an angle α we proceed in different way.

As we know, being $\Gamma(\frac{\nu-2}{2})\, r^{2-\nu}$ the potencial in ν dimensions, we obtain the two dimensional potential lnr from $\frac{d}{d\nu}(\nu-2)\;\Gamma(\frac{\nu-2}{2})\, r^{2-\nu}\Big|_{\nu=2}$. Then, for $\nu = 2$ we take the following expression for the interaction between the segments

$$V(\nu=2) = \frac{d}{d\nu}\left[-\frac{2(\nu-2)\Gamma[(\nu-2)/2]}{8\pi^{\nu/2}}\frac{L_2^{4-\nu}}{(4-\nu)}\cos^2\alpha(\sin\alpha)^{2-\nu}\int_{-1}^{\infty\,\mathrm{sgn}(\cos\alpha)} d\lambda\,(1+\lambda^2\cot g^2\alpha)^{(\nu-2)/2} \right.$$

$$\left. -\frac{(\nu-2)\Gamma[(\nu-2)/2]\,(L_1^{4-\nu}+L_2^{4-\nu}-|\vec{L}_1-\vec{L}_2|^{4-\nu})}{4\pi^{\nu/2}(3-\nu)(4-\nu)}\right]_{\nu=2} \tag{27}$$

The gauge term in Eq. (27) is finite, as well as all the terms in which appear a derivative of a function of ν multiplied by the integral. Consider, now, the term in which appear the derivative of the integral. For this we use the following integral formulae [8]

$$\int_0^1 (1+bx^m)^{\ell}dx = \sum_{k=0}^{\infty}\binom{\ell}{k}\frac{b^k\Gamma(1+km)}{\Gamma(2+km)}, \text{ with } b^2>1 \tag{28}$$

$$\int_0^{\infty} x^{\lambda-1}(k + \alpha x^p)^{-\mu-\nu}\, dx = \frac{1}{p}\, \alpha^{-1/p}\, B\left(\frac{\lambda}{p}, \mu + \nu - \frac{\lambda}{p}\right). \qquad (29)$$

With the use of (28) and (29) we have

$$\int_{-1}^{\infty} d\lambda (1 + \lambda^2 \mathrm{cotg}^2\alpha)^{(\nu-2)/2} = \sum_{k=0}^{\infty} \binom{\frac{\nu-2}{2}}{k} \frac{(\mathrm{cotg}^2\alpha)^k}{2k + 1} + \frac{1}{2}\, \frac{B(\frac{1}{2}, \frac{1-\nu}{2})}{\mathrm{cotg}\alpha} \qquad (30)$$

Taking $\frac{d}{d\nu}$ of Eq. (30) and putting $\nu = 2$, we see that the result is finite, because for this value of ν the term $B(\frac{1}{2}, \frac{1-\nu}{2})$ has no divergences. Therefore, for $\nu = 2$, the interaction between the segments is finite.

Then, by joining smoothly the two segments that form an angle or that have an intersection we get a loop with cusp and intersection, respectively. As we have shown that the divergences are absent in two dimensions, in the case in which the contour has an angle or generally an intersection, we conclude that the two dimensional Wilson Loop evaluated around a closed contour with a cusp or an intersection is finite. This is an interesting result became we know that the Wilson Loop expectation value for pure gauge theories gives a trivial area law in two dimensions.

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