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# SESSION 34: STATISTICAL SAMPLING PLANS

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In auditing for such anomalies in an accountability system as clerical errors and procedural violations, attributes inspection sampling plans are used. Such plans are also used as part of the verification of facility MUF, where, as an initial step, assurance must be provided that there is not an intolerable frequency of large discrepancies between book and actual values before closer inspection using the $\hat{D}$ statistic is implemented.

In attributes inspection, each sampling unit or item is classified as being either acceptable or a defect based on some defect criteria. Specifically, the statistical problem in inspection planning for attributes inspection may be stated as follows:

Let N = number of items in population
n = number of items in sample (sample size)
D = number of defects in population of size N
d = number of defects in sample of size n.

The number of defects, d, is observed. If d equals or exceeds some critical value, $d_o$, then the audit is declared to be unacceptable. From an inspection design viewpoint, the problem is to select values for n, the sample size, and $d_o$, the critical value.

To choose n and $d_o$, two criteria are set up:

1. If $D = D_o$, conclude that the audit is unacceptable with small probability, $\alpha$. $D_o$ corresponds to "acceptable" quality.

2. If $D = D_1$, conclude that the audit is unacceptable with large probability, $(1 - \beta)$. $D_1$ corresponds to "unacceptable" quality.

The problem is not a simple one to solve. For a population of finite size, N, the random variable, number of defects in the sample, is distributed according to the hypergeometric probability density function. Although some tables exist that give solutions to the problem, the tables are necessarily very limited in scope because of the large number of parameters involved.

Two solutions to the problem are discussed in the lecture. One solution is given for N large relative to n, in which case the random variable is approximately distributed according to the binomial density function. The solution is based on an approximation to this latter function. In the other formulation of the problem, the special case in which $d_o = 1$ is considered. This solution is often used by the IAEA in their attributes inspection plans, and is very simple to remember and apply. The sample size is given by

$$n = N(1 - \beta^{1/D}) \quad (1)$$

In applying this formula to the problem in which a quantitative verification of the facility MUF is to be made, it is applied in each stratum in two ways:

1. In attributes inspection for "gross" defects, the largest defect is assumed to be $\bar{x}$, the average amount of element per item. The goal quantity of M units (same units as x) is assumed to be achieved in each stratum. The number of defects, $D_1$, is then $M/\bar{x}$.

2. In attributes inspection using variables measurements, a "medium" defect is assumed to be one that would escape detection if inspected by the attributes tester. The size of a medium defect is assumed to be $\gamma x$ so that $D_1$ is $M/\gamma\bar{x}$.

As a final part of quantitative verification, the $\hat{D}$ difference statistic discussed previously is used to investigate the significance of small biases. Specifically, their cumulative effect on the facility MUF is measured by the $\hat{D}$ statistic. From an inspection planning viewpoint, the problem is to choose the number of measurements to perform in each stratum. This is done to meet the following type criterion:

<u>Criterion</u>: If the true value for the difference statistic, $\hat{D}$, is M units, detect this fact with a statistical test using $\hat{D}$ with probability $(1 - \beta)$. The significance level of the statistical test is $\alpha$. This is a common type statistical problem in selecting a sample size and critical value, but is somewhat complicated by a number of considerations:

1. One must not only determine the entire sample size, but must also allocate the total sample size among the various strata. This is done by allocating such that the variance of $\hat{D}$ is minimized for fixed total sample size.

2. Because of limitations imposed by systematic errors, it may not be possible to meet the criterion. In this case, the relationship between sample size and is examined and some compromising value is chosen for the sample size.

3. The variance of $\hat{D}$ under the alternative when its mean is not zero may be larger than that under the null hypothesis when its mean is zero. This will affect the sample size, and, in planning, an inflation factor on this variance should be applied.

In a full scale general solution to the problem there are a number of parameters that may be identified. In addition to assigning values to M, $\alpha$, $\beta$, and $C^2$ (the variance inflation factor), one can also perform the planning for the (MUF-$\hat{D}$) statistic rather than the $\hat{D}$ statistic. The general formula is solved for the specific case in which $\alpha = \beta = 0.05$, $C^2 = 4$, and the $\hat{D}$ statistic is used. In this event, the sample size is inversely proportional to

$$0.2053\ m^2 - 0.1642\ m\ \sqrt{6.0886\ m^2} \qquad (2)$$

where m is the ratio of M to the systematic error standard deviation for the $\hat{D}$ statistic.

A numerical application is made to the model plant discussed in previous lectures.

In inspection planning, it is assumed that all M units (the goal quantity) is diverted by the particular route to be responded to by the given inspection. For example, in determining the sample size for attributes inspection in stratum k, it is assumed that all M units are diverted through large defects (data falsifications) in that particular stratum. Clearly, if any amount smaller than M units is so diverted, the probability of detection will be less than the design value of $(1 - \beta)$ for that particular part of the inspection.

There are, of course, a virtually limitless number of strategies that might be used by the diverter to accumulate his goal quantity of M units in a material balance period. For any given strategy, one can calculate the probability of non-detection (or its complement, the probability of detection) for the statistical tests employed. "Detection" occurs if at least one of the following conditions occurs:

1. A gross defect is found in at least one of the strata using the attributes tester

2. A defect is found in at least one of the strata using the variables tester in the attributes mode.

3. The absolute value ot the $\hat{D}$ statistic exceeds its critical value, i.e., there is statistical evidence that the mean of $\hat{D}$ is not zero.

4. The operator's calculated MUF exceeds its critical value, i.e., there is statistical evidence that the mean value of MUF is not zero.

As an alternate to steps (3) and (4), one may not perform separate tests of significance for $\hat{D}$ and MUF but may choose to detect the combined effects of two diverter strategies (diversion by small data falsifications and into MUF). Thus, (3) and (4) may be replaced by:

5. The (MUF - $\hat{D}$) statistic exceeds its critical value, i.e., there is statistical evidence that the mean value of (MUF - $\hat{D}$) is not zero.

There are distinct operational advantages using (MUF - $\hat{D}$) as the test statistic rather than $\hat{D}$ and MUF separately. Most importantly, both $\hat{D}$ and MUF require information about the operator's systematic errors. This information is often difficult to develop or, if available, may be poorly based and somewhat unreliable. On the other hand, the (MUF - $\hat{D}$) statistic is independent of the operator's systematic errors. It does, of course, require information about the inspector's systematic errors but such information is easier to derive and, from the inspector's viewpoint at least, should be more reliable.

As another advantage of the (MUF - $\hat{D}$) statistic, when calculating the probability of non-detection by the $\hat{D}$ and MUF tests separately administered, one must take into account the covariance between $\hat{D}$ and MUF. This can be done, but the computations can be complicated involving table look-up in a table of bivariate normal distribution. Computer programs do exist that perform the calculation of non-detection probabilities for $\hat{D}$ and MUF, but unless such a program is available to the user, or unless a table of the bivariate normal distribution is available, the non-detection probabilities for the $\hat{D}$ and MUF test in combination cannot even be calculated. This is not the case with the (MUF - $\hat{D}$) statistic. In passing, it is noted that one cannot simply ignore the covariance between $\hat{D}$ and MUF and assume that the test statistics are independent; this is far from true and gives incorrect and misleading results.

*The interesting relationship among MUF, $\hat{D}$, and (MUF - $\hat{D}$) vari-*ances is, for the case in which both parties do not commit the same systematic errors:

$$V(MUF - \hat{D}) = V(\hat{D}) - V(MUF) \quad (3)$$

This may also be written as:

$$\text{Covariance } (\hat{D}, MUF) = V(MUF) \quad (4)$$

Equation (3) is basic in the evaluation of the inspection plans. Since $V(\hat{D})$ and V(MUF) will already have been calculated, V(MUF - $\hat{D}$) follows immediately.

Restricting further attention to points (1), (2), and (5) detailed above, the probability of non-detection for a given diverter strategy reduces to

$$Q = \beta^{a_2} Q1 \quad (5)$$

where $a_2$ is the fraction of M diverted into some combination of large and medium data falsifications, where β is the design parameter for all strata in the attributes inspection (or the largest such value if is not the same for all strata), and $Q_1$ is the probabilily of non-detection of an amount $(1 - a_2)M$ with the (MUF - $\hat{D}$) test. The probability $Q_1$ is a function of how the diverter splits the amount $(1 - a_2)M$ into MUF and into D. Thus, the strategy space open to the diverter involves his choice of $a_2$ and his further choice on how much of the remaining amount of M, the goal quantity, is diverted into MUF.

The quantity $Q_{Max}$ is that value of Q corresponding to optimal diverter strategy, i.e., that strategy which yields the largest value for Q.

An example application dealing with the inspection of the model plant is given. The example illustrates how strongly dependent on diverter strategy is the probability of non-detection.