IFT. P-01/85

Dirac Particle on S^2

P. LEAL FERREIRA

&

B. E. PALLADINO

ift INSTITUTO DE FÍSICA TEÓRICA

INSTITUTO DE FÍSICA TEÓRICA - rua pamplona, 145 - c.e.p. 01405 - são paulo - brasil

DIRAC PARTICLE ON S^2

P.LEAL FERREIRA* and B.E.PALLADINO

INSTITUTO DE FÍSICA TEÓRICA, Rua Pamplona 145,

01405 - São Paulo, SP.

SUMMARY

The problem of a Dirac particle in stationary motion on S^2 - a two dimensional sphere embedded in Euclidean space E^3 - is discussed. It provides a particularly simple case of an exactly solvable constrained Dirac particle whose properties are here studied, with emphasis on its magnetic moment.

* With financial support of FINEP and CNPq, Brazil.

1. INTRODUCTION

The quantum mechanical problem of a nonrelativistic particle constrained to move on a N-sphere S^N is well-known to admit exact solutions[1]. For the corresponding problem of a constrained Dirac particle on S^N, solutions, to our knowledge are not known for a general value of N, although the particular case of N=2 (the two-dimensional sphere S^2, embedded in Euclidean space E^3), as we shall see, admits exact solutions. This is a direct consequence of the embedding property of S^2 in E^3, which allow us to derive stationary solutions on S^2 as a restriction (r=R) imposed on the free motion parametrized by spherical coordinates (r, θ, φ) in E^3, R being the radius of the 2-sphere.

After discussing the energy spectrum of a Dirac particle on S^2 in section 2, we carry out, in section 3, perturbative calculations of the magnetic moment of the system. Section 4 is devoted to some final remarks on the problem.

2. THE ENERGY SPECTRUM OF THE FREE DIRAC PARTICLE ON S^2.

Our starting point are the stationary solutions of a free Dirac particle in E^3, in spherical coordinates. Since this is a rather well-known subject we go directly to the main results to be used in the sequel.

The Dirac equation for a free particle of mass M ($\hbar=c=1$) is

$$\left(\underset{\sim}{\alpha} \cdot \underset{\sim}{p} + \beta M - E \right) \psi = 0 . \qquad (2.1)$$

Spherical solutions of (2.1) may be written in the form[2]

$$\psi(r,\theta,\varphi) = \begin{pmatrix} g(r)\, \mathcal{Y}^{j_3}_{j l_A}(\theta,\varphi) \\ i f(r)\, \mathcal{Y}^{j_3}_{j l_B}(\theta,\varphi) \end{pmatrix} \tag{2.2}$$

where g(r) and f(r) are real functions of the Euclidean distance r in E^3 and the $\mathcal{Y}^{j_3}_{jl}$'s are spinor harmonics, responsible for the angular dependence of ψ .

The two radial functions obey the equations

$$\begin{cases} \dfrac{d}{dr} f(r) + (1-k)\dfrac{f(r)}{r} + (E-M)\, g(r) = 0 \\ \dfrac{d}{dr} g(r) + (1-k)\dfrac{g(r)}{r} - (E+M)\, f(r) = 0 \end{cases} \tag{2.3}$$

in which - k is the eigenvalue of the Dirac K operator, $K = \beta(\underset{\sim}{\Sigma}\cdot\underset{\sim}{L}+1)$.

Constraining the particle to move on S^2, of radius R, we put

$$\begin{aligned} g(r) &= g(R) = a \quad , \quad \frac{dg}{dr} = 0 \\ f(r) &= f(R) = b \quad , \quad \frac{df}{dr} = 0 \,. \end{aligned} \tag{2.4}$$

Consequently, the two real constants a and b will obey the linear system

$$\begin{cases} (E-M)Ra - (k-1)b = 0 \\ (k+1)\, a - (E+M)Rb = 0 \,. \end{cases} \tag{2.5}$$

The normalization of the wave-function on S^2 is defined by

$$\int \psi^+ \psi \; dS = 1 , \tag{2.6}$$

with $dS = R^2 d\Omega$.

Condition (2.6) gives immediately

$$R^2 (a^2 + b^2) = 1 . \tag{2.7}$$

The solutions of the system (2.5), with condition (2.7) are:

(i) if $k = +1$:

$$a = \frac{M}{(1+M^2R^2)^{1/2}}, \quad b = \frac{1}{R(1+M^2R^2)^{1/2}} \quad \text{for } E = M, \tag{2.8$_1$}$$

$$a = 0 \qquad , \quad b = \frac{1}{R} \qquad \text{for } E = -M;$$

(ii) if $k = -1$: (2.8$_2$)

$$a = \frac{1}{R} \quad , \qquad b = 0 \qquad \text{for } E = M,$$

$$a = \frac{1}{R(1+M^2R^2)^{1/2}}, \quad b = \frac{M}{(1+M^2R^2)^{1/2}} \quad \text{for } E = -M ;$$

(iii) if $k \neq \pm 1$,

$$a = \frac{1}{R}\left[1 + \frac{(E-M)^2R^2}{(k-1)^2}\right]^{-\frac{1}{2}}, \quad b = \frac{1}{R}\left[1 + \frac{(E+M)^2R^2}{(k+1)^2}\right]^{-\frac{1}{2}}$$

with

$$E = \pm\frac{1}{R}\left[M^2R^2 + (k^2-1)\right]^{\frac{1}{2}} \tag{2.8$_3$}$$

$$(k = \pm 1, \pm 2, \ldots) .$$

The energy levels are seen to have a double degeneracy with respect to k. Since there is still a $(2j+1)$ - fold degeneracy due to j_3, for the total degeneracy d we, of course have

$$d = 2(2j+1) = 4|k| . \tag{2.9}$$

The parity π of a state of given k is

$$\pi = \begin{cases} (-1)^{|k|} & \text{if } k>0 \\ (-1)^{|k|-1} & \text{if } k<0 . \end{cases}$$

Therefore, the parities of the states k and $-k$ are opposite. It follows that the double degeneracy corresponds to a parity doubling of the spectrum.

3. THE MAGNETIC MOMENT

We will now discuss the magnetic moment of a Dirac particle, of charge Q, constrained to move on S^2.

If we assume that a weak, constant and uniform magnetic field, taken in the $0z$ direction, is acting on our system, the magnetic moment μ will be given by[3]

$$\begin{aligned}\mu &= \frac{Q}{2}\int \psi^{+}(\underset{\sim}{r}\times\underset{\sim}{\alpha})_3\,\psi\,dS \\ &= \frac{QR^3}{2}\int \psi^{+}(\hat{\underset{\sim}{r}}\times\underset{\sim}{\alpha})_3\,\psi\,d\Omega\ ,\end{aligned} \tag{3.1}$$

where $\hat{r} = \frac{r}{R}$ and ψ is taken for $j_3 = j$:

$$\psi = \begin{pmatrix} a\,\mathcal{Y}^{j}_{jl_A} \\ ib\,\mathcal{Y}^{j}_{jl_B}\end{pmatrix} = \begin{pmatrix} a\,\mathcal{Y}^{j}_{jl_A} \\ -ib\,\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}}\,\mathcal{Y}^{j}_{jl_A}\end{pmatrix}. \tag{3.2}$$

The last step in eq. (3.2) is a consequence of the equality

$$\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{r}}\ \mathcal{Y}^{j_3}_{jl_A} = -\,\mathcal{Y}^{j_3}_{jl_B}\ , \tag{3.3}$$

where $\underset{\sim}{\sigma}$ are 2x2 Pauli spin matrices.

By a simple manipulation, we have

$$\mu = \frac{QR^2}{2}\int \psi^+ \mathcal{O}\, \psi \, d\Omega \tag{3.4}$$

where

$$\psi^+ \mathcal{O}\, \psi = 2Rab\, \mathcal{Y}^{j\,+}_{jl_A} \begin{pmatrix} -\sin^2\theta & \sin\theta\cos\theta e^{-i\varphi} \\ \sin\theta\cos\theta e^{+i\varphi} & \sin^2\theta \end{pmatrix} \mathcal{Y}^{j}_{jl_A} . \tag{3.5}$$

We shall consider separately two cases, according to whether $k = \pm\left(j+\frac{1}{2}\right)$.

(i) $k = -(j+\frac{1}{2})$, hence $l_A = j-\frac{1}{2}$.

In this case, we get

$$\psi^+ \mathcal{O}\, \psi = -2Rab \left| Y^{j-\frac{1}{2}}_{j-\frac{1}{2}}(\theta,\varphi)\right|^2 \sin^2\theta . \tag{3.6}$$

With the help of the integral

$$\int \left| Y^{j-\frac{1}{2}}_{j-\frac{1}{2}}\right|^2 \sin^2\theta \, d\Omega = \frac{l_A+1}{l_A+\frac{3}{2}} , \tag{3.7}$$

we obtain

$$\mu = -QR^3ab\,\frac{l_A+1}{l_A+\frac{3}{2}} = -QR^3ab\,\frac{k}{k-\frac{1}{2}}\,. \tag{3.8}$$

(ii) $k = j+\frac{1}{2}$, hence $l_A = j+\frac{1}{2}$.

We get

$$\begin{aligned}\psi^+ \mathbb{O}\,\psi = -\frac{2Rab\sin\theta}{2j+2}\Big[& Y_{j+\frac{1}{2}}^{j-\frac{1}{2}*}\,Y_{j+\frac{1}{2}}^{j-\frac{1}{2}}\sin\theta\, + \\ & +(2j+1)^{1/2}\,Y_{j+\frac{1}{2}}^{j-\frac{1}{2}*}\,Y_{j+\frac{1}{2}}^{j+\frac{1}{2}}\cos\theta\, e^{-i\varphi} + \\ & +(2j+1)^{1/2}\,Y_{j+\frac{1}{2}}^{j+\frac{1}{2}*}\,Y_{j+\frac{1}{2}}^{j-\frac{1}{2}}\cos\theta\, e^{i\varphi} + \\ & -(2j+1)\,Y_{j+\frac{1}{2}}^{j+\frac{1}{2}*}\,Y_{j+\frac{1}{2}}^{j+\frac{1}{2}}\sin\theta\Big]\,. \end{aligned} \tag{3.9}$$

Now, making use of the following expressions for the spherical harmonics involved in (3.9), namely

$$Y_\ell^\ell = \left[\frac{(2\ell+1)!!\,2^\ell}{(2\ell)!!\,4\pi}\right]^{\frac{1}{2}} \hat{x}_+^{\,\ell} ,$$

$$Y_\ell^{\ell-1} = \left[\frac{(2\ell+1)!!\,2^{\ell-1}}{(2\ell-2)!!\,4\pi}\right]^{\frac{1}{2}} \hat{x}_0\, \hat{x}_+^{\,\ell-1}$$

$$\left(\hat{x}_+ = \frac{-1}{\sqrt{2}}(\hat{x} + i\hat{y}) , \quad \hat{x}_0 = \frac{z}{R} \right),$$

we calculate the four terms appearing in that equation and o' ain for μ the result

$$\begin{aligned} \mu &= \frac{1}{2}\,\frac{QR^3ab}{2j+2}\,\frac{(2\ell_A+1)!!}{(2\ell_A-1)!!}\int_0^\pi \sin^{2\ell_A+1}\theta\, d\theta \\ &= +\, QR^3ab\,\frac{\ell_A}{\ell_A+\frac{1}{2}} \\ &= +\, QR^3ab\,\frac{k}{k+\frac{1}{2}} \, . \end{aligned} \qquad (3.10)$$

Therefore, the two cases (i) and (ii) can be written together as

$$\mu_{\pm} = \pm\, QR^{3} ab \,\frac{k}{k \pm \frac{1}{2}}$$
$$= \pm\, QR^{3} ab \,\frac{j+\frac{1}{2}}{j+1} \qquad (3.11)$$

corresponding to $k = \pm\left(j+\frac{1}{2}\right)$ respectively.

In eq. (3.11), the factor R^3ab may be more conveniently calculated as follows. By multiplying the first of eqs. (2.5) by b, the second by a and adding, we have get

$$R^{3} a b = \frac{1}{2M}\left(1+k-2kR^{2}b^{2}\right)$$
$$= \frac{1}{2M}\left[1+k-\frac{2k}{1+\frac{(E+M)^{2}R^{2}}{(k+1)^{2}}}\right], \qquad (3.12)$$

where in the last step we made use of eq. (2.8_2). By applying (3.12) to (3.11), we obtain

$$\mu_{\pm} = \pm\,\frac{Q}{2M}\left[1+k-\frac{2k}{1+\frac{(E+M)^{2}R^{2}}{(k+1)^{2}}}\right]\frac{k}{k\pm\frac{1}{2}}\,. \qquad (3.13)$$

Finally, it may be of interest to point out that for the magnetic moment calculated via the iterated Dirac equation, as the expectation value of

$$\mu = \frac{Q}{2E}\left(L_3 + \Sigma_3\right), \qquad (3.14)$$

we obtained

$$\mu_\pm = \pm \frac{Q}{2E} k \begin{cases} 1 - \dfrac{R^2 a^2}{k+\frac{1}{2}} & \text{if } k>0, \\ 1 + \dfrac{R^2 b^2}{k-\frac{1}{2}} & \text{if } k<0. \end{cases} \qquad (3.1\cdot)$$

where the expressions for a and b are given by eq.s (2.8$_3$).

4. FINAL REMARKS

The solutions obtained are remarkably simple and are given in exact form for any value of k (or j). The energy spectrum as given by formula (2.8$_3$) is a discrete one and corresponds to parabolic Regge trajectories ($E^2 \propto j^2$). It also presents a double degeneracy in k, which may be interpreted as a parity doubling.

We remark that this degeneracy can be easily broken up - and exact solutions still obtained - if we introduce in the Dirac Hamiltonian an additional term of the form F (K, J^2), F denoting an arbitrary function of the commuting angular operators K and J^2.

Finally, a few words about the magnetic moment calculations are in order. It should be noted that magnetic moments calculated via (3.12) or via (3.14) give, in general, different results[*], the differences being of the order of 25% .

An exceptional case is that of the ground state (E=M) with k=-1. In this case, eq.(3.12) gives $\mu_{-1}=0$ whereas, from eq. (3.15) the result $\frac{Q}{2M}$ is obtained. This may be easily understood having in mind that in this state b=0 and remarking that in the first case the magnetic moment is given by the product a b whereas in the second, it is given by a sum of two terms, one proportional to a^2, the other to b^2.

Let us add a few words about the future outlook. This work can proceed in at least two directions: in the first, a generalization to N-spheres is envisaged. In the second, instead of spheres, the Dirac particle is constrained to move on a surface of different symmetry as, for instance, a circular cylinder of given length, a case also amenable to an exact treatment. We hope to return to these interesting topics at a later date.

ACKNOWLEDGMENT

One of us (P.L.F.) is grateful to Prof. E.Ley Koo, of UNAM, México for interesting discussions.

REFERENCES AND NOTES

(1) For a recent discussion, see P.Leal Ferreira, Rev.Brasil. Fís., Special volume, p.183, July (1984).

(2) For the Dirac equation and related quantities we follow the notation and conventions of J.J.Sakurai, "Advanced Quantum Mechanics", Addison -Wesley Publ. Co., Reading, MA (1967).

(3) This definitions is consistent with the secular equation resulting from the double degeneracy since the off-diagonal elements of the perturbation vanish in first order.

(4) This fact was recognized long ago by H.Margenau, Phys. Rev. 57, 383 (1940).