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ABSTRACT

By means of an example for which the effective potential is explicitly calculable (up to the one loop approximation), ~~we~~ discuss how a phase transition takes place as ~~we increase~~ the temperature and pass from spontaneously broken symmetry to a phase in which the symmetry is restored.

Key-words: Finite temperature; Field theory; Statistical mechanics.

§ 1 INTRODUCTION

Quantum field theory (Q.F.T.) has accumulated a significant number of successes in describing the fundamental forces of Nature. The partial unification achieved by the electroweak theory[1] has been put through several experimental tests, culminating with the recent discovery of the W and Z mesons[2]. Quantum chromodynamics (Q.C.D.) gains more and more credibility as the theory of the strong interactions. Its perturbative (high energy) predictions seems to fit well to existing data on deep inelastic scattering and e^+e^-(annihilation)[3] whereas non-perturbative computer calculations[4] provide the first indication that the low energy spectrum of baryon and mesons might also be reproduced.

Nevertheless a complete unified theory of all forces in Nature has not yet been achieved as gravitation has not been successfully incorporated in a complete unified scheme.

The weak and electromagnetic forces are unified, but in a subtle way. The reason being that the Electro-weak interaction is described by a *Spontaneously broken*[5] gauge theory; ie one in which the gauge invariance is not manifest (just as rotational symmetry is not manifest in a magnetized ferromagnet). In fact, the SU(2) x U(1) group has its symmetry reduced to that of the electromagnetic U(1) group. In order to actually have the full invariance, first of all one has to be able to somehow *restore* the gauge symmetry (if that is possible) through the variation of some adjustable parameter of the theory (phase transition).

Statistical mechanics provides us with numerous examples of systems that undergo phase transitions. Such systems may have different degrees of symmetries and pass from one phase to another as we change one or more of the physical parameters. For example, raising the temperature of the ferromagnet beyond the Curie point we transform it into a paramagnet. Analogously we could find that QFT which in the euclidean region is an example of statistical mechanics can also undergo phase transitions as we change the temperature. Thus one might expect that for a high enough temperature, the SU(2) x U(1) symmetry would be valid. Then, weak and electromagnetic forces will have the symmetry restored.

Precisely the standard cosmological models[6] provide us with a theory in which the universe evolves from a big-bang (very hot) state to the rather cold world we live in today. Thus, somewhere along the line we should find the appropriate critical temperature for the transitions to take place. Investigation of the history of the Universe has thus become of great interest to particle-physicist.

Not only in cosmology but also in heavy ion and high energy particle collisions ("little bangs") we can reach temperatures high enough to detect phase transitions. Furthermore, even higher temperatures might be probed by using the experimental richness of comsic-ray physics[7].

The above mentioned arguments justify the study of QFT at finite temperature. Unfortunately the perturbative development of theory presents divergences which have to be regularized and renormalized. For that purpose, dimensional regularization is often used which allows in some cases to obtain results as analytic functions of the number of space-time dimensions $(n = \nu + 1)$.

This also opens a simple way to establish relations between properties at different dimensions[8]. In the above mentioned reference an integral expression was given for the effective potential (in the one loop approximation)as a function of the temperature for any number of dimensions. The effective potential allows the calculation of critical temperature as the point where the type of symmetry changes marking thus a phase transition. We have used the expression obtained in ref. [8] of the effective potential for the $\lambda\psi^4$ theory, which for $\nu = 0$ can be easily computed providing thus a simple model where several properties of finite temperature field theories with broken symmetries can be explicitly studied.

§ 2 THE EFFECTIVE POTENTIAL AT FINITE TEMPERATURE

Let us illustrate some of the features of QFT a finite temperature by looking at a simple example: we shall treat the "$\lambda\psi^4$" theory in a heat bath.

The Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\psi)^2 + \frac{1}{2}\mu^2\psi^2 + \frac{\lambda}{4!}\psi^4 \qquad (2.1)$$

We shall work in ν spatial dimensions, ν being arbitrary. The partition function Z, defined as

$$Z = \mathrm{Tr}\{e^{-\beta H}\} \quad , \quad H = \int d^{\nu}x \, \mathcal{H} \tag{2.2}$$

may be expressed as a functional integral over the field ψ, i.e.

$$Z = N^{-1}\int \mathcal{D}\psi \, \exp\{-\frac{1}{2}\int_0^{\beta} d\tau \int d^{\nu}x [(\nabla\psi)^2 + \mu^2\psi^2 + \frac{\lambda}{12}\psi^4]\} \tag{2.3}$$

τ = it is the euclidean time. The integral over the fields only includes those which obey periodic boundary conditions in τ.

$$\psi(x,\beta) = \psi(x,0) \qquad (\beta = \frac{1}{T}) \tag{2.4}$$

We may add to the curly brackett in 2.3, a source term of the form.

$$\delta S = \int_0^{\beta} d\tau \int d^{\nu}x J(x)\psi(x,\tau) \tag{2.5}$$

where $J(x)$ represents a static external field which couples to ψ. The partition function is then both a function of β and a functional of $J(x)$. The situation is entirely analogous to that of models in statistical mechanics. We may now proceed to obtain the relevant thermodynamic quantities. The Helmholtz free energy is simply

$$F(\beta,J) = -\frac{1}{\beta}\ln Z(\beta,J) \tag{2.6}$$

Its functional derivative with respect to $J(x)$ yields the expectation value of the field at finite temperature:

$$m(\beta,x) = -\frac{\delta F}{\delta J(x)} = \frac{1}{\beta}\int_0^\beta d\tau \; \langle\psi(x,\tau)\rangle \tag{2.7}$$

In the Ising model for example, $J(x)$ will correspond to an external magnetic field that orients the spin and $m(\beta,x)$ to the average value of the spin at a particular point. The magnetization is the average of the last quantity over the volume of the lattice.

It is quite useful to consider yet another thermodynamic function, the mean free energy G(Landau potential) obtained from F via a Legendre transform:

$$G(\beta,m(\beta,x)) \equiv F(\beta,J(x)) + \int d^\nu y J(y) m(\beta,y) \tag{2.8}$$

From which we get,

$$\frac{\delta G}{\delta m(\beta,x)} = J(x) \tag{2.9}$$

This equation provides us with a way of obtaining the vacuum ($J \equiv 0$) of the theory; i.e. the lowest lying state at a given temperature as we turn off the external field. Thus:

$$\frac{\delta G}{\delta m} = 0 \quad ; \quad \frac{\delta^2 G}{\delta m_1 \delta m_2} > 0 \tag{2.10}$$

For theories that are translational invariant (such as $\lambda\psi^4$), the solution $m_v(\beta)$ of 2.10 is independent of x.

The effective potential is obtained by

$$U_{(\nu)} = \Omega^{-1}_{(\nu)} G \qquad (2.11)$$

where $\Omega_{(\nu)}$ is the volume of space.

A systematic (perturbative) way of computing 2.11 is provided by the semiclassical expansion (loop expansion). It amounts to computing Z by expanding the fields around a uniform background.

$$\psi(x,\tau) = \bar{m} + \eta(x,\tau) \qquad (2.12)$$

and then integrating over the "fluctuations" η. The zeroth order (in η) is the zero loop approximation. It gives for $U^0_{(\nu)}(\beta,m)$ the potential appearing in the original lagrangian. When the symmetry is broken, $\mu^2 < 0$ and

$$U^0_{(\nu)}(\bar{m}) = \frac{\lambda}{4!}(\bar{m}^2 - m_\nu^2)^2 \quad ; \quad m_\nu^2 = \frac{6|\mu^2|}{\lambda} \qquad (2.13)$$

For the next approximation we take the quadratic terms in η which are responsible for the one loop result. Performing the corresponding gaussian integration we get (See [9]):

$$U^1_{(\nu)} = \frac{1}{2\beta\Omega_{(\nu)}} \ln \left[\frac{\mathrm{Det}(-\Box + \mu^2 + \frac{\lambda}{2}\bar{m}^2)}{\mathrm{Det}(-\Box + \mu^2 + \frac{\lambda}{2}m_\nu^2)}\right] \qquad (2.14)$$

Where the determinants are restricted to the sub-space of periodic functions (as in 2.4).

We may rewrite 2.14 as

$$U^1_{(\nu)} = \frac{1}{2\beta\Omega_{(\nu)}} \mathrm{Tr}\ \ln\left[1 + D_{(\beta)}\left(\frac{\lambda}{2}(\bar{m}^2 - m_\nu^2)\right)\right] \qquad (2.15)$$

where $D_{(\beta)}$ is given by:

$$D_{(\beta)} = \frac{1}{(\frac{2\pi n}{\beta})^2 + p^2} \tag{2.16}$$

Taking the trace is equivalent to an integration over p and a summation over n.

The result is given in ref. [8]:

$$U^1_{(\nu)} = - \frac{1}{\beta^{1+\nu}\pi^{\frac{\nu}{2}}\Gamma(1+\frac{\nu}{2})} \int_{\mu}^{\infty} dt\,(t^2 - u^2)^{\frac{\nu}{2}} \coth t \tag{2.17}$$

with $u = \frac{1}{2}\beta\psi \quad ; \quad \psi^2 = \frac{\lambda}{2}\bar{m}^2 - |\mu^2|$

It is easy to see that

$$\frac{dU^1_{(\nu)}}{d\psi^2} = - \frac{1}{4\pi} U^1_{(\nu-2)} \tag{2.18}$$

Having computed the effective potential for arbitrary dimension ν, we can use it to obtain the vacuum of the theory at different temperatures ($T = \beta^{-1}$). Once this is accomplished, one can compute the effects of temperature on the couplings by taking derivatives of the potential at the vacuum state:

$$\mu^2(\beta) = \left.\frac{d^2 U_{(\nu)}}{d\bar{m}^2}\right|_{\bar{m} = m_\nu} \tag{2.19}$$

$$\lambda(\beta) = \left.\frac{d^4 U_{(\nu)}}{d\bar{m}^4}\right|_{\bar{m} = m_\nu} \tag{2.20}$$

In the next section we shall consider a simple example - the case $\nu = 0$ - where we have a closed form for U_ν and by studying the vacua at different temperatures we will obtain information about a phase transition in T.

§ 3 THE PHASE TRANSITION

Our aim now is to illustrate, from an analysis of the effective potential at finite temperature, how a phase transition can emerge. Up to one loop order, we have to sum the zero loop contribution given by 2.13 with the one-loop potential of 2.17. However, the latter presents in general a singularity (or cut) for $\psi \leq 0$ which is due to the presence of cotgh $t \sim \frac{1}{t}$ near $t = 0$. So, this approximation breaks down near $\bar{m}^2 = \frac{2}{\lambda}|\mu|^2 (\psi = 0)$. We are thus inclined to concentrate our analysis in the vecinity of the minima of the zero-loop potential $\bar{m} \simeq m_\nu$. Then we can use 2.18 to write a Taylor development of U_ν near that value, keeping only up to quadratic terms

$$U_\nu(\beta\psi^2) = \bar{U}_\nu - \frac{\bar{U}_{\nu-2}}{4\pi}(\psi^2 - \bar{\psi}^2) + \frac{1}{2!}\,\frac{\bar{U}_{\nu-4}}{(4\pi)^2}(\psi^2 - \bar{\psi}^2)^2 + \ldots \quad (3.1)$$

where $\bar{U}_\alpha = U_\alpha(\beta\bar{\psi}^2)$ $\quad (\bar{\psi}^2 = \frac{\lambda}{2}m_\nu^2 - |\mu|^2)$

The effective potential being given by (adding the zero loop 2.13).

$$U_\nu = \bar{U}_\nu - (\psi^2 - \bar{\psi}^2)\frac{\bar{U}_{\nu-2}}{4\pi} + (\psi^2 - \bar{\psi}^2)^2\left(\frac{1}{6\lambda} + \frac{\bar{U}_{\nu-4}}{32\pi^2}\right) + \ldots \quad (3.2)$$

It is then seen, from 3.2 that the constants $\bar{U}_{\nu-2}$, $\bar{U}_{\nu-4}$ modify the position of the minima of U_ν and the curvature of U_ν; i.e., the effective mass of the elementary excitation. Of course, this modification depends on the temperature. When the temperature is raised, the effective mass is decreased until a value is reached ($\beta = \beta_c$) at which the curvature is zero, the minimum desappears, showing then a phase transition. We pass from a broken symmetry phase to a restored symmetry one. This analysis is qualitative for any ν but it can be made quantitative for the case $\nu = 0$ when we have a compact expression for the effective potential.

$$U_{(o)}(\beta,\bar{m}) = \frac{\lambda}{4!}(\bar{m}^2 - m_v^2)^2 + \frac{1}{\beta}\,\ell n\left[\frac{sh\,(\frac{1}{2}\beta\psi)}{sh\,(\frac{1}{2}\beta\psi_v^2)}\right] \tag{3.3}$$

where $m_v^2 = \frac{6|\mu^2|}{\lambda}$, $\psi^2 = \frac{\lambda}{2}\bar{m}^2 - |\mu^2|$, and $\psi_v^2 = 2|\mu^2|$.

We have chosen to compute the potential in the broken phase.

In order to look for the values of $\bar{m}$ that are extrema of $U_{(o)}$ we shall introduce:

$$\phi \equiv \frac{\lambda}{2}(\bar{m}^2 - m_v^2) \tag{3.4}$$

The derivative of 3.1 with respect to ϕ yields:

$$U'_{(o)} = \frac{\phi}{3\lambda} + \frac{1}{2\psi}\,ctgh\,(\frac{1}{2}\beta\psi) \tag{3.5}$$

The solutions of $U'_{(o)} = 0$ are the given by

$$tgh\,(\frac{1}{2}\beta\psi) = -\frac{3\lambda}{2\psi(\psi^2 - 2|\mu^2|)} \tag{3.6}$$

The graph for the two functions, members of the equality 3.6, is shown in figure 1.

From that graph we see that solutions (other than $\bar{m} = 0$) will only exist in the range:

$$0 \le \psi \le \sqrt{2|\mu^2|}$$

This corresponds to:

$$\frac{2|\mu^2|}{\lambda} \le \bar{m}^2 \le \frac{6|\mu^2|}{\lambda} \quad ; \quad \frac{m_v^2}{3} \le \bar{m}^2 < m_v^2 \qquad (3.7)$$

In fact the minimum of the full curve occurs at a value of ψ given by:

$$\psi_M = \sqrt{\frac{2}{3}|\mu^2|} \quad ; \quad \bar{m}_M^2 = \frac{5}{9} m_v^2 \qquad (3.8)$$

A further condition for the existence of non-zero extrema is then that the value of the right-hand side of 3.6 at ψ_M, be smaller or equal to one.

$$-\frac{3\lambda}{2\psi_M(\psi_M^2 - 2|\mu^2|)} \cdot \sqrt{\frac{3}{2}} \cdot \frac{9}{8} \cdot \frac{\lambda}{|\mu^2|^{3/2}} \le 1 \qquad (3.9)$$

Let us see how a phase transition might be inferred from our analysis in the case where 3.7 is satisfied. At zero temperature ($\beta \to \infty$) the curve (c) in fig. 1 tends to a step function at the origin. Thus there will be the solutions to 3.6 at A and B. It is easy to check that B corresponds to symmetric minimum whereas A corresponds to symmetric maximum of $U_{(o)}$. This is illustrated in figure 2, where the zero loop potential, represented by the dotted curve, is also exhibited.

The one-loop approximation breaks down near $\bar{m} = \pm\sqrt{\frac{2|\mu^2|}{\lambda}}(\psi = 0)$, where the potential diverges. Thus we are inclined to interpret the maxima at A not too seriously and concentrate our analysis in the vicinity of the minima. As the temperature is raised, the intersection points occur at C and D of figure 1. D will correspond to the minima and what we see is that the value of $U_{(o)}$ is lower there that at the minima of T = 0. Furthermore, the concavity is "softer" indicating that the mass of the fluctuations (near the minima) is decreasing with increasing temperature. We will eventually reach a situation where the curve (c) just touches curve (a). At this point the mass vanishes:

$$\mu^2(T_c) = \left.\frac{d^2U_{(0)}(\beta_{c1}\bar{m})}{d\bar{m}^2}\right|_{\bar{m}=m_V} = 0; \;\; \beta_c = T_c^{-1} \qquad (3.10)$$

At T_c the non zero extrema disappear. The critical temperature may be roughly estimated from:

$$\text{tgh}\,\frac{\beta_c}{2}\sqrt{\frac{2}{3}|\mu^2|} \simeq \frac{1}{2}\sqrt{\frac{3}{2}}\,\frac{9}{8}\cdot\frac{\lambda}{|\mu^2|^{3/2}} \qquad (3.11)$$

The system thus exhibits two phases. One, the low temperature phase, where the expectation value of the field m(T) is different from zero. And another, above T_c, where m_v vanishes.

A brief comment is in order about the singularity in $U_{(o)}(\beta,\bar{m})$ at $\bar{m}^2 = \frac{1}{3}m_v^2(\psi = 0)$. This singularity is an artifact of the loop expansion and may or may not appear in a completely non perturbative treatment.

Two-loop calculations, as well as other methods of estimating T_c are being developed in order to clarify this point. Preliminary results show that the singularity subsists, for the same value of ψ.

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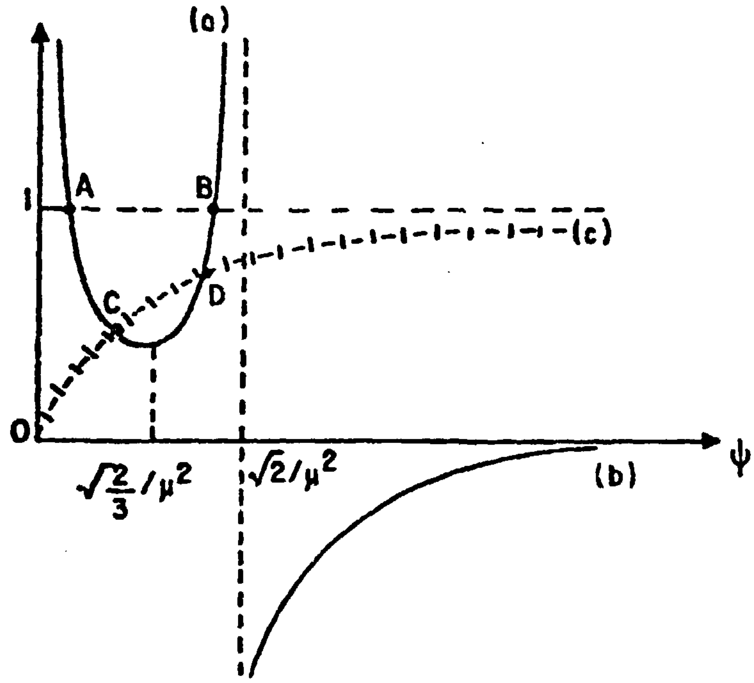

FIG.1

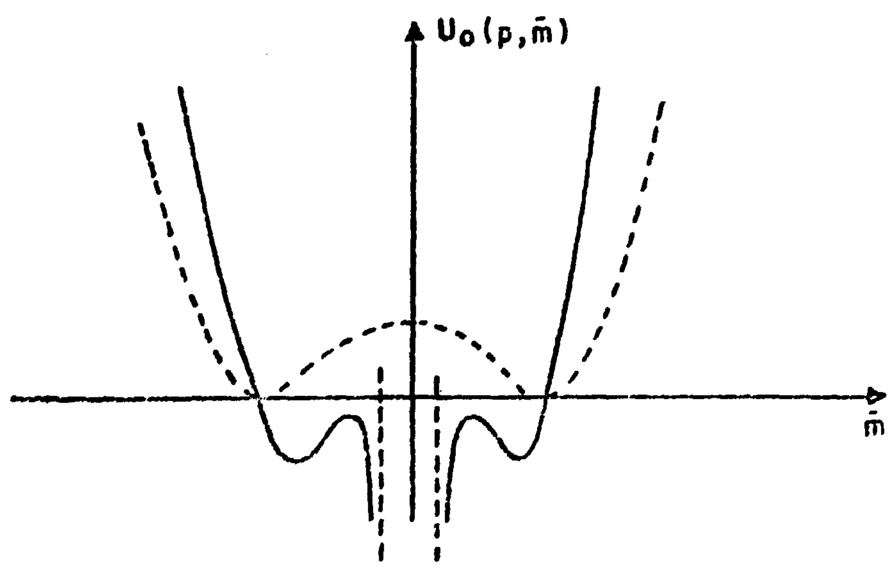

FIG.2