6. International conference on fracture
New Delhi (India) 4-10 Dec 1984
CEA-CONF--7695

CRACK GROWTH IN THE CREEP REGION
CRITERIA BASED ON MATERIAL FORCES

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Meaning of C^* concept is discussed. It is shown that this concept is only a global approach of material force rate concept. The field of C_i^* (material force rate densities) is more representative of creep crack growth than C^* integral. As application corrected expressions of C^* are proposed for non isothermal cases, strain hardening creep and effect of material elasticity.

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1 - INTRODUCTION

The assumption of a crack free structure is not admissible to-day. Below the creep region, this statement has led to the use of linear elastic fracture mechanics at first, then to techniques related to post yield fractures mechanics. The former is only considering the onset of crack propagation, but the last one can take some stable crack propagation into account.

There are similar needs in the creep region. At first the criterion employed for elevated temperature design has been that the time for crack initiation should exceed the design life. As crack free structures do not exist in practice, more attention is now paid to predict the crack growth behaviour under creep conditions. Unfortunately, such a prediction is not easy for the designer, because several parameters have been proposed for characterizing the creep crack growth rate behaviour of metals[1]. As parameters have been proposed the elastic stress intensity factor K_I, the net section stress σ_{net} (or the reference stress) and more recently the parameter C^* analogous to the parameter J used below the creep region[2].

To-day this parameter C^* seems one of the most popular for correlations of experimental data[3], but its field of validity is not well defined. Important studies has been made about that point[4,5,6]. Most of them are related to the strain rate and stress fields near the crack tip. Therefore they are nearer to the studies on the H.R.R. singularity than to the path integral concept. Moreover, they consider very simple formations for creep behaviour (like Norton's law). Nevertheless it seems helpfull to give a more general interpretation of the C^* concept, in the same way of this of J[7] using the material force concept. Such a study must begin with the principles of mechanics used in that field (virtual work for quasi static problems).

2 - VIRTUAL WORK AND CREEP CRACK GROWTH

21 *Principle of virtual work*

One of the most powerful principle in Mechanics is the principle of virtual work. Equilibrium is obtained if

$$\int_V \delta W \, dv = \int_S \overline{X_i} \, \delta u_i \, ds$$

(in the absence of external volume forces)

$\delta W = \sigma_{ij} \, \delta\varepsilon_{ij}$	strain working density
σ_{ij}	stress tensor
ε_{ij}	strain tensor
$\overline{X_i}$	external surface force density
u_i	displacement.

This is only true if continuity of the material is satisfied, it is to say that displacement and strain variations fulfill equations of compatibility. This is not the case of fracture mechanics and a generalized form of the principle of virtual work must be used. A simple writting of this form can be obtained in using "generalized forces and displacements". Generally, the displacement field can be defined by a limited number of parameters u_α called generalized displacements (α from 1 to N)

$$\int \overline{X_i}\ \delta n_i\ ds = X_\alpha\ \delta u_\alpha$$

and the generalized form of virtual work can be written

$$\int_V \delta W\ dv = \delta V = X_\alpha\ \delta u_\alpha - J_\beta\ \delta a_\beta$$

a_β being parameters describing material properties perturbation related to crack advance (a_β can be called generalized material displacements, and J_β generalized material forces)[7].

It is the current practice to made the following assumption : the perturbation created by crack advance is (fairly) characterized by the value of crack extension (only one parameter), this lead to the more current equation

$$\delta V = X\ \delta u - J\ B\ \delta a$$

It must be noted that this lead to

$$\dot{a} = \frac{X\dot{u} - \dot{V}}{JB} \qquad (1)$$

for the value of crack groth rate.

22 *Quasi-static problems*

Strictly speaking, principle of virtual work can only be applied to static cases. If displacements are depending on time, an other presentation of this principle must be chosen : the quasi static formulation[8]. When the time rate of change of external forces and displacement is so gradual that inertial forces can be neglected, it is obvious that the principle of virtual work can be formulated in the conventional manner, except that the time now appears as a parameter. The problem shall be expressed in terms of rate.

Therefore, will be considered displacement rate $\dot{u}_i$ and strain rates $\dot{\varepsilon}_{ij}$. The main question is "What about straining work W such $\delta W = \sigma_{ij}\ \delta\varepsilon_{ij}$?".

The substitution of the strain rate to the strain leads to consider the following function

$$W^{*} \text{ such } \delta W^{*} = \sigma_{ij}\ \delta\ \dot{\varepsilon}_{ij} \qquad (2)$$

With this definition, the principle of virtual work can be written

$$\int_V \delta W^{*}\ dv = \int_S \overline{X_i}\ \delta \dot{u}_i\ ds$$

(if external volume forces are present $\int_V X_i\ \delta\ \dot{u}_i\ dv$ must be added to second hand).

Like in the conventional formulation, this is only true if the variations of strain rate and displacement rate satisfy conditions of compatibility. Such conditions being not satisfied in case of crack growth, the quasi static formulation must be generalized. As it can be seen below, the generalization introduce the concept of material displacements and material forces. The C^{*} concept is only a particular case of material force.

23 *Meaning of W^{*} and $\int \overline{X}\ \dot{u}\ ds$*

In many papers W is called strain energy density and W^{*} strain energy rate density. Such a practice can leads to some misunderstanting. *W^{*} is not the rate of W*

$$\frac{dW}{dt} = \dot{W} = \sigma_{ij}\ \dot{\varepsilon}_{ij} \neq W^{*}$$

$$W^{*} = W^{\cdot} - \varepsilon_{ij}\ \dot{\sigma}_{ij}$$

- The same remark can be made about $\int X_i\ \delta\ \dot{u}_i$ ds often written $\delta V^{*} = X_i\ \delta\ \dot{u}_i$, to designate V^{*}, it can be found expressions like "power","energy rate", "creep energy dissipation rate". It must be pointed out that *V^{*} is not the power of external forces.* This power is

$$\dot{V} = X\ \dot{u}_i \neq V^{*}$$

- When a Norton's law is considered for creep $\dot{\varepsilon} = B\ \sigma^{n}$, it is easy to see that W^{*} is a state function related to the rate of strain working density $\dot{W}$

$$W^{*} = \frac{n}{n+1}\ \dot{W}$$

and

$$V^{*} = \frac{n}{n+1}\ \dot{V} = \frac{n}{n+1}\ X\ \dot{u}$$

leading to write equation (1) as follow

$$\dot{a} = \frac{n+1}{n} \; \frac{V^{*} - \int W^{*} \, dv}{JB}$$

. It is the author thinking that these differences are important, they are very like the one between C^{*} and the rate of J.

3 - MATERIAL DISPLACEMENT CONCEPT. MATERIAL FORCE RATE

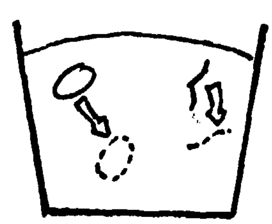
31 *Spatial and material displacements*

A point of the continuum is identified by its cartesian coordinates x_i in the initial state. This is a Lagrangian formulation x_i coordinates are only identifying a particle of material. Due to the action of external forces X_i (volume forces) and $\overline{X_i}$ (surface forces), this point is displaced, reaching $x_i + u_i$ in an external referential, and exhibits a state of strain rate ε_{ij} and a state of stress σ_{ij}. It must be pointed out that u_i is a geometrical or spatial displacement (a conventional one), and x_i and $x_i + u_i$ are initial and present coordonates related to an external referential linked for instance to the walls of the laboratory (spatial coordinates). On the contrary, the coordinates of this point are yet x_i if related to the continuum itself (material coordinates).

It is introduced

$$\delta W^{*} = \sigma_{ij} \, \delta \, \dot{\varepsilon}_{ij} \qquad (2)$$

Now, perturbations are considered as possible in the continuum itself. It is to say that the initial organisation can be changed and that the material coordinates of one point can change. A trivial exemple of such a situation can be given by pastry making. If dried fruits are included in cake dough, these dried fruits can present some displacement before baking. Their displacement is a spatial (or conventional displacement) if it is related to the cake mold, it is a material displacement, if it is related to the initial state of the dough.



In other words, material displacements are corresponding to a flow of material properties through the body (included holes and cracks).

32 *Material force rate*

Due to such a virtual material displacement δx_k, there is some change in the distribution of ε_{ij} and W^* in the body. The change in strain rate is obviously equal to $\dot{\varepsilon}_{ij,k}\,\delta x_k$ leading to a change of $\dot{W}^*$ equal to $\sigma_{ij}\,\dot{\varepsilon}_{ij,k}\,\delta x_k$. Part of this change is due to the δx_k as a spatial displacement, it is equal to $\dot{W}^*_{,k}\,\delta x_k$. Finally the variation of $\dot{W}^*$ caused by material variation only is given by

$$\delta \dot{W}^* = \sigma_{ij}\,\dot{\varepsilon}_{ij,k}\,\delta x_k - \dot{W}^*_{,k}\,\delta x_k$$

which can be written

$$\delta \dot{W}^* = -\,\dot{c}_k\,\delta x_k$$

$$\dot{c}_k = \dot{W}^*_{,k} - \sigma_{ij}\,\dot{\varepsilon}_{ij,k} \qquad (3)$$

- $\dot{c}_k$ is dual of material displacement δx_k, having the same dimension as a force rate density. It can be called *material force rate* density.

- Perhaps it is useful to pay attention to the good definition of stress and strain. If u_i, $\dot{u}_i$, ε_{ij} and $\dot{\varepsilon}_{ij}$ are small there is no problem and conventional definitions can be used. If they are not small[9], displacement gradient must chosen as strain $\varepsilon_{ij} = u_{i,j}$, and stress is the BOUSSINESQ nominal stress tensor (so that stress working density can be written $\delta W = \sigma_{ij}\,\delta u_{i,j}$ and $\delta \dot{W}^* = \sigma_{ij}\,\delta \dot{u}_{i,j}$).

- $\dot{c}_k$ is volume density of material force rate. If surface discontinuties exist in the body, surface density $\bar{\dot{c}}_k$ can appear . When the material displacement δx_k lead to cross a surface discontinuity, variation of displacement is finite Δu_i, it is to say $\Delta u_{i,k}\,\delta x_k$ in this direction and $\delta\Delta \dot{W}^* = T_i\,\Delta\dot{u}_{i,k}\,\delta x_k$.

- Part of this variation is only caused by spatial displacement $\Delta \dot{W}^*\,n_k\,\delta x_k$. Therefore the material variation is

$$\delta \dot{W}^* = -\,\bar{\dot{c}}_k\,\delta x_k$$

$$\bar{\dot{c}}_k = \Delta \dot{W}^*\,n_k - T_i\,\Delta\dot{u}_{i,k} \qquad (4)$$

- Only translations δx_k have been considered. From a rigorous point of view, local rotations of material properties must be considered, leading to the introduction of material couple force rates. For simplicity sake they will not be considered here.

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* *A comma followed by suffixes will denote differentiation with respect to x, for instance* $\dot{\varepsilon}_{ij,k} = \dfrac{\delta\dot{\varepsilon}_{ij}}{\delta x_k}$.

4 - GENERALIZED FORMATION OF VIRTUAL WORK PRINCIPLE FOR QUASI STATIC PROBLEMS - INTRODUCTION OF C^*

In the previous section, it has been shown that material disorders like holes and cracks evolutions, etc. can be identified by the material displacement concept. The resulting variation of W^* has been computed, leading to the definition of material force rate c_i^* (volume density) and $\bar{c}_i^*$ (surface density).

It is now possible to write the quasi static principle of virtual work when compatibility conditions are not satisfied. It is enough to add the part due to material displacement

$$\int_V \delta W^* \, dv = \int_S \bar{X}_i \, \delta\dot{u}_i \, ds - \int_V c_i^* \, \delta x_i \, dv - \int_\Sigma \bar{c}_i^* \, \delta x_i \, ds \qquad (5)$$

$\delta W^* = \sigma_{ij} \, \delta\dot{\varepsilon}_{ij}$

σ_{ij} stress tensor (norminal)

$\dot{\varepsilon}_{ij}$ strain rate tensor (displacement gradient)

$\dot{u}_i$ displacement rate

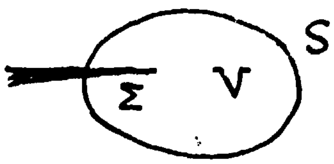
δx_i virtual material displacement

$\bar{X}_i$ external surface forces

Σ discontinuity surface (like cracks)

(external volume forces X_i have not been considered).

. For practical use of this principle, it is needed to define the virtual material displacement corresponding to the type of perturbation considered. The most simple is a translation of material properties, indicated by δx_i having the same value in each point. If a crack is present, there is just a virtual advance of the crack tip. Applying this principle to a part around this crack tip



$$\int_V \delta W^* \, dv = \int_S \bar{X}_i \, \delta\dot{u}_i \, ds - \delta x_i \left[\int_V c_i^* \, dv + \int_\Sigma \bar{c}_i^* \, ds \right]$$

a simple calculation show that

$$\int_V c_i^* \, dv + \int_\Sigma \bar{c}_i^* \, ds = \int_S (W^* n_i - T_j \dot{u}_{j,i}) \, ds \qquad (6)$$

the right hand term is the well known integral C^*.

The integral C^* relating to a surface S is the resultant of all the material force rate (volume density and density on discontinuity surfaces Σ) included in the volume V surrounded by S.

. As a matter of fact this relation means that material force rate applied to a part of a body are satisfying equilibrium equations. Hence an attempt can be made to introduce what correspond to a stress tensor. Below the creep region it is possible to introduce the energy momentum tensor[10]. When creep can occurs it is better to choice

$$\theta^*_{kj} = W^* \delta_{kj} - \sigma_{ij} \dot{u}_{i,k}$$

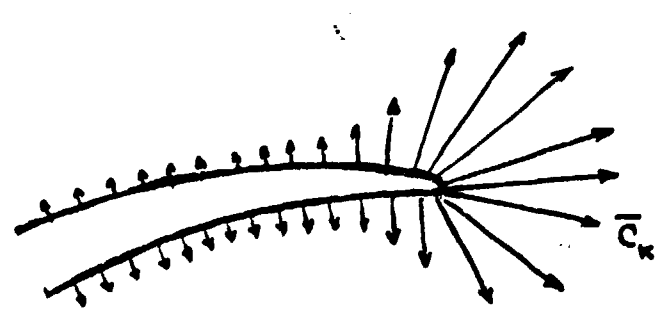
where δ_{kj} is the Kronecher's tensor. And expressions of material force rate can be written as follow

$$\begin{cases} c^*_k = \theta^*_{kj,j} \\ \bar{c}^*_k = - \theta^*_{kj} n_j \end{cases} \qquad (7)$$

θ^*_{ij} could be called energy momentum rate.

5 - ON PATH-INDEPENDANCE OF C^* INTEGRAL

. The preceding results show that C^* concept is only a consequence of the distribution of material force density c_i in the material. Obviously the field of c^*_i in the vicinity of the crack tip is a better indication of the process than C^*. It must be pointed out that on the free surface of crack, there is a field of surface density $\bar{c}^*_i$ which is normal to the surface, the intensity being higher near the crack tip. Therefore path independence of C^* is only possible in case of plane cracks (with limitation to C^* components in the crack plane).



As for J-integral it is useful to precise conditions of path independence of C^*. Such a property seems lead to a characterization of stress and strain rate fields near the crack tip.

In application of the preceding relations the condition of path independence is $\int_V \dot{C}_i \, dv \equiv 0$ for all volumes but those including the crack edges.

51 *Conventional creep*

. If only creep occurs , following NORTON's law $\dot{\varepsilon} = B\,\sigma^n$ - for instance, it is possible to write

$$\sigma_{ij} = \frac{\partial W^*}{\partial \dot{\varepsilon}_{ij}} \qquad \text{with} \qquad W^* = \frac{\dot{\varepsilon}_e^{\frac{1}{n}+1}}{(1+\frac{1}{n})\,B^n}$$

($\dot{\varepsilon}_e$ equivalent strain rate).

. This relation must be considered as general

$$\sigma_{ij} = \frac{\partial W^*}{\partial \dot{\varepsilon}_{ij}} \tag{8}$$

the main question being of what variables W^* is dependent.

The simpler case is W^* depending only of $\dot{\varepsilon}_{ij}$. As

$$W^*_{,k} = \frac{\partial W^*}{\partial \dot{\varepsilon}_{ij}} \dot{\varepsilon}_{ij,k} = \sigma_{ij}\, \dot{\varepsilon}_{ij,k}$$

it is clear that $\dot{c}_i \equiv 0$ in every point of the material, therefore $\int \dot{c}_i \, dv \equiv 0$ and C^* is not path dependent.

. *If the material is such that* $\sigma_{ij} = \frac{\partial W^*}{\partial \dot{\varepsilon}_{ij}}$ *(viscosity) with* W^* *depending only of* $\dot{\varepsilon}_{ij}$, C^* *is not path dependent.*

52 *Non isothermal problems and strain hardening creep*

. It is well know that B (in NORTON's law) is temperature dependent, such must be W^*. Therefore if temperature is not uniform in the body, the preceding conclusion is not applicable

$$W^*_{,k} = \sigma_{ij}\ \varepsilon_{ij,k} + \frac{\partial W^*}{\partial \theta}\ \theta_{,k}$$

and

$$c^*_i = \frac{\partial W^*}{\partial \theta}\ \theta_{,k}$$

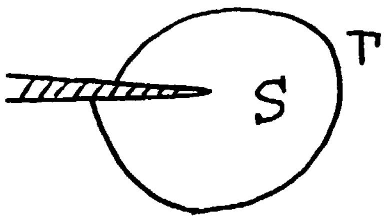
. This effect is spurious and do not concern conditions of crack growth. It must be eliminated and a corrected value of C^* must be used

$$C^*_i \text{ corr} = \int_\Sigma \bar{c}^*_i \, ds$$

$$C^*_i \text{ corr} = \int_S (W^*\ n_i - T_j\ u_{j,i})\ ds - \int_V \frac{\partial W^*}{\partial \theta}\ \theta_{,i}\ dv$$

. or in plane case

$$C^* \text{ corr} = \int_\Gamma W^*\ dy - T_i \left(\frac{\partial \dot{u}_i}{\partial x}\right) ds - \int_S \frac{\partial W^*}{\partial \theta}\ \frac{\partial \theta}{\partial x}\ dv$$



In the preceding case, if temperature is not uniform, C^* must be corrected in soustracting $\int_S \frac{\partial W^*}{\partial \Theta}\ \frac{\partial \theta}{\partial x}\ dv$.

. More important are hardening effects (B depending of time of strain). If strain hardening is noticeable W^* is depending of ε_{ij}, and the preceding correction must be extended in soustracting also

$$\int_S \frac{\partial W^*}{\partial \varepsilon_{ij}}\ \frac{\partial \varepsilon_{ij}}{\partial x}\ dv$$

53 *Creep and linear elasticity*

Unfortunately, elastic behaviour of material cannot always be neglected, and in many cases NORTON's law must be written $\dot{\varepsilon} = B\,\sigma^n + \dot{\sigma}/E$. Therefore the general formula giving σ_{ij} shall be written

$$\sigma_{ij} = \frac{\partial W^*}{\partial \dot{\varepsilon}_{ij}}$$

with W^* function of $\dot{\varepsilon}_{ij} - D_{ijkl}\,\dot{\sigma}_{kl}$, D_{ijkl} being the material elasticity matrix.

. It is now possible to made a computation like the preceding one

$$W^*_{,k} = \frac{\partial W^*}{\partial \dot{\varepsilon}_{ij}}\,\dot{\varepsilon}_{i,,k} + \frac{\partial W^*}{\partial \dot{\sigma}_{mn}}\,\dot{\sigma}_{mn,k}$$

as

$$\frac{\partial W^*}{\partial \dot{\sigma}_{mn}} = -\,D_{ijmn}\,\frac{\partial W^*}{\partial \dot{\varepsilon}_{ij}}$$

$$W^*_{,k} = \sigma_{ij}\,(\dot{\varepsilon}_{ij,k} - D_{ijmn}\,\dot{\sigma}_{mn,k})$$

. This can be written

$$W^*_{,k} = \sigma_{ij}\,(\dot{\varepsilon}_{ij,k} - \dot{e}_{ij,k})$$

where

$\dot{e}_{ij} = D_{ijmn}\,\dot{\sigma}_{mn}$ is the elastic strain rate

(if elasticity matrix is not time dependent).

Hence

$$C^*_k = -\,\sigma_{ij}\,\dot{e}_{ij,k}$$

C^*_k is path dependent, and should be replaced by

$$C^*_k(\text{corr}) = \int_S (W^*\,n_i - T_j\,\dot{u}_{j,i})\,ds + \int_V \sigma_{ij}\,e_{ij,k}\,dv$$

or in plane case

$$C_k^{\bullet} \text{ (corr)} = \int_{\Gamma} W^{\bullet} \, dy - T_i \left(\frac{\partial \dot{u}_i}{\partial x} \right) ds + \int_S \sigma_{ij} \frac{\partial \dot{e}_{ij}}{\partial x} dv$$

. *If elasticity cannot be neglected, C^* must be corrected in adding the surface integral* $\int_S \sigma_{ij} \frac{\partial \dot{e}_{ij}}{\partial x} dv$ *where e_{ij} is the elastic strain.*

. It is worth noting that the present situation is analogous of this of J when plasticity occurs[11]. A volume integral could be considered

$$\int_V (W^{\bullet}_{,k} - \sigma_{ij} \dot{f}_{ij,k}) \, dv$$

where $f_{ij,k}$ is the creep strain rate. Unfortunately, this integral cannot be reduced to a surface integral, because f_{ij} do not meet continuity condition (obviously, such a reduction would possible in using a creep distorsion tensor as shown in[11]).

6 - CONCLUSIONS

- In the creep region, quasi static formulation of the principle of virtual work lead to introduce functions like $W^{\bullet} = \int \sigma \, d\dot{\varepsilon}$ which are not equal to the rate of straining work $W = \int \sigma \, d\varepsilon$. This seems the reason why C^* is not the rate of J.

- As continuity conditions are not met in crack growth a generalized formulation of this principle must be used. This practice leads to introduce material force rate density c_k^* and $\overline{c}_k^*$. The virtual work of this material force must be added to the conventional work.

- C^* integral is the resultant of all the material force rate included in the region surrounded by the integral contour.

- If material behaviour is characterized by $\sigma = \partial W^{\bullet}/\partial \dot{\varepsilon}_{ij}$ where $W^{\bullet}$ is only dependent of $\dot{\varepsilon}_{ij}$, C^* is no path dependent.

- On the contrary if $W^{\bullet}$ is dependent of other parameters, C^* is path dependent. In order to keep its meaning, corrected expressions must be used.

  . For non isothermal cases

$$C^{*} \text{(corr)} = C^{*} - \int_S \frac{\partial W^{*}}{\partial \theta} \frac{\partial \theta}{\partial x} dv$$

. For strain hardening creep

$$C^{*}(\text{corr}) = C^{*} - \int_{S} \frac{\partial W^{*}}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial x} \, dv$$

. When the rate of elastic strain e_{ij} is not negligible related to creep strain rate

$$C^{*}(\text{corr}) = C^{*} + \int_{S} \sigma_{ij} \frac{\partial \dot{e}_{ij}}{\partial x} \, dv$$

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