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A HIERARCHY OF SYSTEMS OF NONLINEAR EQUATIONS+

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Abstract

Imposing isospectral invariance for the one-dimensional Dirac operator yields an infinite hierarchy of systems of chiral invariant nonlinear partial differential equations. The same system is obtained through a Lax pair construction and finally a formulation in terms of Kac-Moody generators is given.

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The first successful solution of the KdV equation led to the development of the inverse scattering transform method. With the help of the construction of Lax pairs or following the AKNS ideas it was possible to generate a large number of solvable nonlinear partial differential equations [1,2]. The next big step by Zhakarov and Shabat [3] showed the connection of the inverse method to the Riemann transform method [4].

Already at the beginning of the development for the KdV equation it was realized that the eigenvalues of the associated Schrödinger operator are left invariant if the potential evolves according to the nonlinear equation. Vice versa one may generate an infinite number of higher order KdV equations by imposing isospectral invariance to the Schrödinger operator.

In this letter we start from the one-dimensional most general self-adjoint Dirac operator on the full line and ask for systems of nonlinear evolution equations for the potentials such that the spectrum of the Dirac operator is left invariant. A simple method, using Feynman-Hellman theorem and an expansion in the spectral parameter allows to characterize these systems iteratively (Theorem 1). Generalizations of the Nonlinear Schrödinger equation and of the Modified KdV equation belong to them. Next we observe that the same systems can be generated following the AKNS scheme. We show that the action of gauge and chiral transformations for the Dirac operator extend naturally to the nonlinear systems. Finally we remark that a third formulation in terms of generators of a Kac-Moody algebra exists. The use of such infinite algebras may shed new light onto infinite dimensional integrable Hamiltonian systems.

All these systems can be solved in a certain gauge by following standard inverse scattering transform methods. Soliton solutions for the Modified KdV equation and the corresponding N = 3 system (see Eq. 19) have been studied recently in connection with a solid state physics model [5,6,7].

Isospectral Flow for the Dirac Equation

Our first derivation of the infinite hierarchy of nonlinear systems follows from imposing isospectral invariance for the one-dimensional Dirac operator with potentials a_1, c_1 and $\underline{v} = (v_1, v_2)$:

$$- iH(\partial_x - ia_1 + \underline{E}\cdot\underline{v})\psi = (\lambda - c_1)\psi \tag{1}$$

where $\underline{E}\cdot\underline{v} = E^1 v_1 + E^2 v_2$ and H, E^1 and E^2 denote generators of $sl(2,C)$ satisfying

$$[H, E^{\substack{1\\2}}] = \pm\, 2iE^{\substack{2\\1}} \,, \qquad [E^{\substack{1\\2}}, E^{\substack{1\\2}}] = 0 \,, \qquad [E^1, E^2] = 2iH \,, \tag{2}$$

which may be expressed by Pauli matrices. We assume that all four potentials and the spinor ψ depend on an additional parameter which we may call time t.

For the discrete spectrum of (1) we impose the condition that it remains time independent. Let ε be an eigenvalue with eigenfunction ψ. From the Feynman-Hellman theorem we deduce that

$$\partial_t \varepsilon = \int_{-\infty}^{\infty} dx \, (I^a \cdot \partial_t a_1 + I^c \cdot \partial_t c_1 + \underline{I}^v \cdot \partial_t \underline{v}) \tag{3}$$

where we introduced densitites

$$I^a = - \psi^\dagger H \psi \,, \qquad I^c = \psi^\dagger \psi \quad \text{and} \quad \underline{I}^v = - i \psi^\dagger H \underline{E} \psi \,. \tag{4}$$

Now we ask for a time evolution of the potentials such that $\partial_t \varepsilon = 0$. The ansatz for

$$\partial_t \varepsilon = \int_{-\infty}^{\infty} dx \, \partial_x (a_2 I^a + c_2 I^c + \underline{b}\cdot\underline{I}^v) \tag{5}$$

as a total differential leads after using (1) and comparing (3) with (5) to the relations

$$\partial_t a_1 = \partial_x a_2$$

$$\partial_t c_1 = \partial_x c_2 - 2i\underline{b}\cdot\sigma_2 \underline{v} \qquad (6)$$

$$\partial_t \underline{v} = -2ic_2 \sigma_2 \underline{v} + \partial_x \underline{b} + 2i(c_1 - \epsilon)\sigma_2 \underline{b}$$

where the Pauli matrix σ_2 enters via the structure constants of sl(2,C). Next we expand a_2, $\underline{b}$ and c_2 in powers of ϵ

$$a_2 = \sum_{n=0}^{N} \epsilon^n a_2^{(n)} , \quad \underline{b} = \sum_{n=0}^{N} \epsilon^n \underline{b}^{(n)} , \quad c_2 = \sum_{n=0}^{N} \epsilon^n c_2^{(n)} , \qquad (7)$$

where the expansion coefficients $a_2^{(n)}$, $\underline{b}^{(n)}$ and $c_2^{(n)}$ depend on all potentials and their derivatives. Introducing (7) into (6) leads to recurrence relations

$$\partial_x a_2^{(n)} = 0 , \quad 1 \leq n \leq N , \qquad (8)$$

$$\underline{b}^{(N)} = 0 , \quad 2\underline{b}^{(n)} = -2c_2^{(n+1)} \underline{v} - i\sigma_2 D_x \underline{b}^{(n+1)} , \quad 0 \leq n < N$$

$$\partial_x c_2^{(N)} = 0 , \quad \partial_x c_2^{(n)} = 2i \underline{b}^{(n)}\cdot\sigma_2 \underline{v} , \quad 1 \leq n < N ,$$

together with time evolution equations for the potentials

$$\partial_t a_1 = \partial_x a_2^{(o)}$$

$$\partial_t c_1 = \partial_x c_2^{(o)} - 2i \underline{b}^{(o)}\cdot\sigma_2 \underline{v} \qquad (9)$$

$$D_t \underline{v} = D_x \underline{b}^{(o)}$$

where D_t and D_x denote covariant derivatives

$$D_t = \partial_t + 2i\sigma_2 c_2^{(o)} , \qquad D_x = \partial_x + 2i\sigma_2 c_1 . \qquad (10)$$

Remark

One of the fields c_1 or $c_2^{(o)}$ may be chosen arbitrarily as a function of t and x. The same holds for a_1 and $a_2^{(o)}$; these two even obey evolution equations which decouple completely from the other equations in (9).

Remark on gauge invariance

It has been realized some time ago that certain nonlinear equations are gauge equivalent to each other [8].

Here we note that the invariance of the Dirac equation (1) under gauge and chiral transformations

$$\psi \to \exp(i\Lambda - iH\chi)\psi , \qquad \underline{v} \to \exp(-2i\sigma_2\chi)\underline{v}$$
$$a_1 \to a_1 + \partial_x\Lambda , \qquad c_1 \to c_1 + \partial_x\chi , \tag{11}$$

implies gauge invariance of (9): Under the transformation (11) together with

$$a_2^{(o)} \to a_2^{(o)} + \partial_t\Lambda \quad \text{and} \quad c_2^{(o)} \to c_2^{(o)} + \partial_t\chi \tag{12}$$

the systems (8) and (9) remain invariant. Clearly, only the chiral part of the transformation of ψ has nontrivial consequences for the system.

In an attempt to generate systems of the hierarchy from Eq. (8) one fixes N and is faced with the problem of integrating the equation for $\partial_x c_2^{(n)}$ for $n = N-1, N-2, \ldots, 1$. It turns out that the r.h.s. for this quantity is always a total derivative which allows to formulate

Theorem 1

The iteration procedure Eqs. (8) and (9) determines systems of nonlinear equations for the potentials of the Dirac equation leaving its spectrum invariant. All expansion coefficients in Eqs. (7) are polynomials in the potentials and their derivatives.

Proof

We will show by complete induction that $\partial_x c_2^{(n)}$ is a total derivative for all n; without loss of generality we put

$$c_2^{(N)} = 1 \qquad \text{and} \qquad \underline{b}^{(N-1)} = -\underline{v} .$$

Suppose the assertion is true for $\partial_x c_2^{(j)}$, $n+1 \leq j \leq N$; we will show that $\partial_x c_2^{(n)}$ is a total derivative in two steps. First note that

a) $\underline{b}^{(m)} \cdot \sigma_2 \, \underline{b}^{(r)} = \underline{b}^{(m-1)} \cdot \sigma_2 \, \underline{b}^{(r+1)}$ + total derivative, which follows by using the second and third line of Eq. (8) twice and by "partial integration".

b) From a) we obtain the chain

$$\partial_x c_2^{(r)} = 2i \, \underline{b}^{(N-1)} \cdot \sigma_2 \, \underline{b}^{(n)} = 2i \, \underline{b}^{(N-2)} \cdot \sigma_2 \, \underline{b}^{(n+1)} + \text{total deriv.} = \ldots =$$

$$= 2i \, \underline{b}^{(N-s-1)} \cdot \sigma_2 \, \underline{b}^{(n+s)} + \text{total deriv.} = \ldots$$

Next we distinguish two cases: If $N+n-1 = 2p$ is even we arrive at $\underline{b}^{(p)} \cdot \sigma_2 \, \underline{b}^{(p)}$ which is zero; if $N+n-1 = 2q-1$ is odd, we arrive at $\underline{b}^{(q)} \cdot \sigma_2 \, \underline{b}^{(q-1)}$ which can be seen from a) to be a total differential too.

The AKNS Scheme - Lax Pair Construction

The second way to obtain the above mentioned hierarchy follows the AKNS scheme, which is similar to a construction of a Lax pair. Multiplying (1) by iH from the left yields

$$\partial_x \psi = X\psi , \qquad X = iH(\lambda - c_1) + ia_1 - \underline{E} \cdot \underline{v} . \tag{13}$$

The time evolution should be given by a T-operator according to

$$\partial_t \psi = T\psi . \tag{14}$$

(13) and (14) are integrable iff X and T fulfill

$$[\partial_x - X, \partial_t - T] = 0 \; . \tag{15}$$

Expanding next T in a power series in λ

$$T = \sum_{n=0}^{N} \lambda^n (i a_2^{(n)} - i H c_2^{(n)} - \underline{E} \cdot \underline{b}^{(n)}) \tag{16}$$

yields after inserting (13) and (16) into (15) systems (8) and (9).

Remark

Special examples of the hierarchy have been treated in the literature. We may omit a_1 and $a_2^{(o)}$ since they decouple from the system.

For N = 1 we obtain as evolution equations

$$\partial_t c_1 = \partial_x c_2^{(o)} \quad \text{and} \quad D_t \underline{v} = D_x \underline{v} \; . \tag{17}$$

A suitable gauge transformation allows to put $c_1 = c_2^{(o)} = 0$.

For N = 2 a generalization of the nonlinear Schrödinger equation is obtained

$$\partial_t c_1 = \partial_x c_2^{(o)} - \partial_x (\underline{v} \cdot \underline{v}) \quad \text{and} \quad D_t \underline{v} = i\sigma_2 D_x^2 \underline{v} \; ; \tag{18}$$

taking $c_1 = 0$ and $c_2^{(o)} = \underline{v} \cdot \underline{v}$ yields the standard form [9,10].

For N = 3 we obtain a generalization of a coupled system of MKdV equations

$$\begin{aligned} \partial_t c_1 &= \partial_x c_2^{(o)} - 2i \partial_x (\underline{v} \cdot \sigma_2 D_x \underline{v}) \\ D_t \underline{v} &= 2 D_x (\underline{v} (\underline{v} \cdot \underline{v})) - D_x^3 \underline{v} \end{aligned} \tag{19}$$

which goes over to the standard form if $c_2^{(o)} = 2i \, \underline{v} \cdot \sigma_2 \partial_x \underline{v}$ [7].

Remark

Expansions like (16) work also for complex potentials. Real ones correspond to selfadjoint Dirac operators, non real ones to the non selfadjoint case. For the nonlinear Schrödinger equation do real potentials correspond to repulsion, pure imaginary one to attraction [9,10].

Lax Pairs in Terms of Kac-Moody Generators

Recently it has been realized that infinite dimensional integrable systems are related to infinite dimensional Lie algebras [11,12]. For certain systems such algebras have been identified as describing the symmetry of the system [13,14]. Especially the connection of Toda lattice systems to such algebras has been worked out.

Here we show that an expansion of Lax pairs in terms of generators of a Kac-Moody algebra leads to the hierarchy (8,9) too. We start from A_1 and A_2 expressed in terms of an element $g(t,x)$ belonging to the Kac-Moody group

$$A_1 = -\,g(\partial_x\, g^{-1})\ , \qquad A_2 = -\,g(\partial_t\, g^{-1})\ . \tag{20}$$

Consistency for (20) implies that

$$[\partial_x - A_1,\ \partial_t - A_2] = 0\ . \tag{21}$$

Next we expand A_1 and A_2 in terms of generators. We identify $\lambda^n H$, $\lambda^n E^1$ and $\lambda^n E^2$ in Eqs. (13) and (16) with generators H_n, E^1_n and E^2_n which belong to the algebra (for a review see [15])

$$\begin{aligned} &[H_m, H_n] = 0\ , && [E_m, E_n] = 0\ , \\ &[E^1_m, E^2_n] = 2i\, H_{m+n}\ , && [H_m, E^{1,2}_m] = \pm 2i\, E^{1,2}_{n+m}\ , \end{aligned} \tag{22}$$

and expand A_1 and A_2 like

$$\begin{aligned} A_1 &= iH_1 + i\,l\,a_1 - iH_0 c_1 - \underline{E}_0 \cdot \underline{v} \\ A_2 &= i\,l\,a_2^{(0)} - \sum_{n=0}^{N} \left(iH_n c_2^{(n)} + \underline{E}_n \cdot \underline{b}^{(n)} \right) \end{aligned} \tag{23}$$

Inserting (23) into (21) yields again systems (8,9).

Remark

Since in expansion Eq. (23) only generators H_n and $\underline{E}_n$ with $n \geq 0$ occur, one obtains the same system whether one uses a central extension of the algebra or not.

Remark

Gauge transformations can now be expressed like

$$\begin{aligned} \partial_{\substack{x\\t}} - A_{\substack{1\\2}}(t,x) &\to g_0(t,x)\left(\partial_{\substack{x\\t}} - A_{\substack{1\\2}}(t,x)\right) g_0^{-1}(t,x)) \\ g_0(t,x) &= \exp(i\,l\,\Lambda(t,x) - iH_0\,\chi(t,x)) \end{aligned} \tag{24}$$

where $\{g_0\}$ form an abelian subgroup of the Kac-Moody group.

Remark

Recently it has been realized that various realizations of elements of Kac-Moody algebras may be also of use for solving nonlinear equations [11,12]. A particular simple realization starts from creation and annihilation operators A_n^r, $A_n^{r\dagger}$, $r = 1,2$, $n \in Z$,

$$[A_m^r, A_n^s] = 0\,, \qquad [A_m^{r\dagger}, A_n^s] = \delta^{rs}\,\delta_{nm}\,, \tag{25}$$

where finally

$$H_m = \sum_q \underline{A}^{\dagger}_{q-m}\, H\, \underline{A}_q\,, \qquad E_m^{\substack{1\\2}} = \sum_q \underline{A}^{\dagger}_{q-m}\, E^{\substack{1\\2}}\, \underline{A}_q\,, \tag{26}$$

fulfill the algebra (22). For the quantum mechanical vacuum $|0\rangle$ defined by $A_q^r|0\rangle = 0 \;\forall\; q$ H_m, E_m^1 and E_m^2 annihilate it for all m; in addition A_1 and A_2 leave invariant the one particle subspace.

Remark

More interesting is the realization where one takes the filled Dirac sea vacuum $|0\rangle$ with $A_m^r|0\rangle = 0$ for $m \geq 0$ and $A_m^{r\dagger}|0\rangle = 0$ for $m < 0$; normal ordering of operators of (26) yields a central extension of the algebra (22) [12]. Since A_1 and A_2 again leave the one particle subspace of the Hilbert space, built on $|0\rangle$, invariant, we may write

$$\partial_x|\psi\rangle = A_1|\psi\rangle \quad \text{and} \quad \partial_t|\psi\rangle = A_2|\psi\rangle \tag{27}$$

and expand

$$|\psi\rangle = \sum_{n\geq 0} \underline{\phi}_n^+(t,x)\, \underline{A}_n^\dagger|0\rangle + \sum_{n<0} \underline{\phi}_n^-(t,x)\, \underline{A}_n|0\rangle\ . \tag{28}$$

Particle and antiparticle subspaces are separately left invariant so we may consider only the former. From (27) we get

$$\partial_x \underline{\phi}_n^+ = iH\, \underline{\phi}_{n+1}^+ + (ia_1 - iHc_1 - \underline{E}\cdot\underline{v})\, \underline{\phi}_n^+\ , \quad \forall\, n \geq 0\ . \tag{29}$$

Therefore if we put $\underline{\phi}_0^+$ equal to a spinor $\underline{\phi}(\lambda;t,x)$ which satisfies (13) we get

$$\underline{\phi}_n^+ = \lambda^n\, \underline{\phi} \qquad \forall\, n \geq 0\ . \tag{30}$$

With that choice we obtain for A_1 and A_2

$$\begin{aligned} A_1|\psi\rangle &= \sum_{n\geq 0} (X\, \underline{\phi}_n^+)\, \underline{A}_n^\dagger|0\rangle \\ A_2|\psi\rangle &= \sum_{n\geq 0} (T\, \underline{\phi}_n^+)\, \underline{A}_n^\dagger|0\rangle\ . \end{aligned} \tag{31}$$

Conclusion

In this letter we indicated three ways to obtain a hierarchy of nonlinear equations which generalize well-known systems of the literature. The most promising approach seems to us to be the connection to infinite dimensional Lie algebras and we expect to obtain further consequences from it.

Let us finally mention, that besides the conventional Hamiltonian formulation [9,10] where an ultralocal symplectic form is taken, there exists also an approach, where a non-ultralocal form is taken and the evolution equations (9) for the N-th system are given by

$$\begin{aligned} \partial_t c_1 &= \{c_1, H_N\}_{P.B.} = \partial_x \frac{\partial H_N}{\partial c_1} \\ D_t \underline{v} &= \{\underline{v}, H_N\}_{P.B.} = D_x \frac{\partial H_N}{\partial \underline{v}} \,. \end{aligned} \tag{32}$$

It is not difficult to construct the appropriate Hamiltonian H_N. In a gauge $c_2^{(o)} = 0$ the covariant derivate D_t becomes the ordinary one. Further elaboration on these systems will be published elsewhere.

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