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ASPECTS OF THREE FIELD APPROXIMATIONS DARWIN, FROZEN, EMPULSE

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ABSTRACT

The traditional approach used to study high energy beam propagation relies on the frozen field approximation. A minor modification of the frozen field approximation yields the set of equations applied to the analysis of the hose instability. These models are contrasted with the Darwin field approximation. A statement is made of the Darwin model equations relevant to the analysis of the hose instability.

INTRODUCTION

Traditionally a highly relativistic charged particle beam has been studied using the frozen field approximation. This approximation assumes the field pattern remains fixed in a frame moving along with the beam. With minor modification a set of equations has been derived in earlier work from the frozen field approximation to study hose instability [1]. These equations are referred to as EMPULSE equations.

Recently, accelerator injector designs have been evaluated using the Darwin field approximation [2,3]. The Darwin approximation is thus used to model charged particles from rest all the way up to relativistic velocities. Since the Darwin approximation has been applied in the relativistic regime it is natural to ascertain its relationship to the frozen field approximation.

The relationship between these models is found by starting with the Darwin model and then assuming a fixed field pattern in the beam frame. The resulting frozen Darwin model is compared to the conventional frozen field approximation and the EMPULSE equations. Both the frozen Darwin and EMPULSE equations neglect radiation, however, they satisfy slightly different charge conservation equations. A noteworthy attribute of the frozen Darwin model is that it does not introduce a preferred axis. In other work this property of the frozen Darwin model is under investigation. It remains to be shown no forces result when an equilibrium beam propagates at an angle to a reference axis. This point must be resolved before applying the frozen Darwin model to the analysis of non-axisymmetric beams.

In section one the usual assumptions of high energy beam dynamics are stated. The frozen field approximation is presented in section two along with the EMPULSE equations. In section three the Darwin approximation is cast in the frozen field form. The terms differing from the frozen field and EMPULSE equations are then enumerated. General considerations concerning the use of the frozen Darwin model for non-axisymmetric beams are discussed in section four.

I. ASSUMPTIONS OF HIGH ENERGY BEAM DYNAMICS

The particular beam dynamics which are allowed validate the use of the frozen field approximation. Thus, comparisons of models in what follows are only valid in this limit. The term high energy is taken to mean $\gamma >> 1$ where $\gamma = (1-\beta^2)^{-1/2}$ and $\beta = v/c$. Along with the notion of high energy, generally terms of order γ^{-2} are neglected. This then implies $v_z = c$, where c is the speed of light.

The particle motion is assumed to be paraxial so $v_\perp^2 \ll v_z^2$. Consistent with this assumption the assumed ordering is

$$\nabla_\perp / \frac{\partial}{\partial z})_x \gg 1 \qquad (1)$$

$$\frac{\partial}{\partial x})_z / \frac{\partial}{\partial z})_x \gg 1. \qquad (2)$$

The consequence of Eq. (1) and Eq. (2) is $\lambda_\beta \gg a$ and $\delta x \sim a$, where the beam radial scale is a, the axial scale is λ_β and $x = ct - z$. The coordinate x takes the place of time for a highly relativistic beam and may be considered as the distance back from the beam head.

II. THE FROZEN FIELD APPROXIMATION AND EMPULSE EQUATION

The frozen field approximation is derived from Maxwell's equations using the ordering in Eq. (2). Maxwell's equations without approximation are below

$$\nabla \times \vec{B} = 4\pi c^{-1} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \qquad (3a)$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \qquad (3b)$$

$$\nabla \cdot \vec{B} = 0 \qquad (3c)$$

$$\nabla \cdot \vec{E} = 4\pi\rho \quad . \qquad (3d)$$

In Eq. (3) $\vec{B}$ is the magnetic induction, $\vec{E}$ is the electric field, $\vec{J}$ is current density and ρ is charge density.

The precise change of variable relationship,

$$\left.\frac{\partial}{\partial z}\right)_t = \left.\frac{\partial}{\partial z}\right)_x - \left.\frac{\partial}{\partial x}\right)_z \tag{4a}$$

combined with Eq. (2) yields

$$\left.\frac{\partial}{\partial z}\right)_t = - \left.\frac{\partial}{\partial x}\right)_z \quad . \tag{4b}$$

Equation (4b) is the approximation used to derive the frozen field model. Generally, beam analysis proceeds in cylindrical coordinates. Thus, the frozen field approximation is written out in cylindrical coordinates by using Eq. (4b) in Eq. (3).

$$\frac{1}{r}\frac{\partial B_z}{\partial \theta} + \frac{\partial B_\theta}{\partial x} = 4\pi c^{-1} J_r + \frac{\partial E_r}{\partial x} \tag{5a}$$

$$-\frac{\partial B_r}{\partial x} - \frac{\partial B_z}{\partial r} = 4\pi c^{-1} J_\theta + \frac{\partial E_\theta}{\partial x} \tag{5b}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_z}{\partial \theta} = 4\pi c^{-1} J_z + \frac{\partial E_z}{\partial x} \tag{5c}$$

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} + \frac{\partial E_\theta}{\partial x} = -\frac{\partial B_r}{\partial x} \tag{5d}$$

$$-\frac{\partial E_r}{\partial x} - \frac{\partial E_z}{\partial r} = -\frac{\partial B_\theta}{\partial x} \tag{5e}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_z}{\partial \theta} = -\frac{\partial B_z}{\partial x} \tag{5f}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} - \frac{\partial B_z}{\partial x} = 0 \tag{5g}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} - \frac{\partial E_z}{\partial x} = 4\pi\rho \quad . \tag{5h}$$

The frozen field approximation to the charge continuity equation is derived from the "frozen" divergence of Eq. (5a), (5b) and (5c).

$$\frac{1}{r}\frac{\partial\, rJ_r}{\partial r} + \frac{1}{r}\frac{\partial J_\theta}{\partial \theta} - \frac{\partial}{\partial x} J_z + c\frac{\partial \rho}{\partial x} = 0 \tag{5i}$$

The only difference between Eq. (5i) and the true continuity equation is the neglect of the $\partial J_z/\partial z)_x$ term.

The primary constituent of the transverse beam motion is the pinch force per charge, $E_r - B_\theta$. Assuming axisymmetry and ignoring all field components except B_θ, E_r and E_z, Eq. (5) can be written in a way which shows the role of the pinch force. From Eq. (5a) and (5e),

$$\frac{\partial}{\partial x}(E_r - B_\theta) = -4\pi c^{-1} J_r \tag{6a}$$

$$-\frac{\partial E_z}{\partial r} - \frac{\partial}{\partial x}(E_r - B_\theta) = 0 \quad . \tag{6b}$$

The x variation of the pinch force can then be viewed as caused by the radial current or radial variation of E_z.

The axisymmetric frozen field model is derived from Eq. (5) and Eq. (6a) and (6b). The equation for B_θ is obtained from Eq. (5c).

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = 4\pi c^{-1} J_z + \frac{\partial E_z}{\partial x} \tag{6c}$$

Equations (6a) and (6b) are combined to yield an equation for E_z.

$$\frac{\partial E_z}{\partial r} = 4\pi c^{-1} J_r \ . \tag{6d}$$

The frozen field charge continuity equation comes from Eq. (5i).

$$\frac{1}{r}\frac{\partial}{\partial r} r J_r - \frac{\partial}{\partial x} J_z + c\frac{\partial \rho}{\partial x} = 0 \ . \tag{6e}$$

The EMPULSE equations corresponding to Eq. (6c) and (6d) have been derived elsewhere [1].

$$\nabla_\perp^2 (\mathscr{A}+\phi) = -4\pi c^{-1} J_z \tag{7a}$$

$$\nabla_\perp \frac{\partial \mathscr{A}}{\partial x} = -4\pi c^{-1} \vec{J}_\perp \tag{7b}$$

$$\nabla_\perp^2 \frac{\partial \mathscr{A}}{\partial x} = -4\pi c^{-1} \nabla_\perp \cdot \vec{J}_\perp \tag{7c}$$

where $\mathscr{A} = A_z - \phi$. In terms of fields these equations can be written as follows.

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = 4\pi c^{-1} J_z \tag{7d}$$

$$\frac{\partial E_z}{\partial r} = 4\pi c^{-1} J_r \tag{7e}$$

$$\frac{c}{4\pi}\frac{\partial}{\partial x}\nabla_\perp \cdot \vec{E}_\perp + \nabla_\perp \cdot \vec{J}_\perp - \frac{\partial J_z}{\partial x} = 0 \ . \tag{7f}$$

Comparing Eq. (6c) and (6d) with Eq. (7d) and (7e) it can be seen the EMPULSE equations only differ from the frozen field approximation by the $\partial E_z/\partial x$

term of Eq. (6c). This means the EMPULSE equations do not have the displacement current. The divergence of Eq. (3a) yields,

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (8)$$

which is not derivable just from Eq. (7d) which corresponds to the z component of Eq. (3a). Instead Eq. (7f) which is the frozen field equivalent of Eq. (8) without $\partial E_z/\partial x$ is derivable from Eq. (7d), Eq. (7e) and Eq. (5h) dropping $\partial E_z/\partial x$. It is necessary to drop $\partial E_z/\partial x$ in Eq. (5h) to recover the continuity equation since this term is omitted in Eq. (7d). Also in Eq. (7e) it is understood J_r does not include the beam J_r since this contribution is neglected in the paraxial approximation. The EMPULSE charge continuity equation then differs from the frozen field by the $\partial E_z/\partial x$ term omission.

III. DARWIN FIELD APPROXIMATION

The Darwin field approximation [4,5] provides a set of field equations consistent with a Lagrangian correct to order β^2. There are two ways of understanding how this approximation impacts Maxwell's equations.

First the part of the Lagrangian, L related to fields consist of a sum of an electrostatic scalar potential ϕ and a vector potential $\vec{A}$.

$$L \sim \phi + \vec{\beta} \cdot \vec{A} \qquad (9)$$

In general, there are relativistic corrections to both ϕ and $\vec{A}$. In the Coulomb gauge, $\nabla \cdot \vec{A} = 0$ and the potential ϕ is known to all orders in β. Thus, in this gauge L only has relativistic corrections from $\vec{A}$. The Coulomb gauge infinite media, open boundary solution for $\vec{A}$ scales like β since $\vec{J}$ scales like the velocity.

$$\vec{A}\,(r,t) = \frac{1}{c}\int \frac{\vec{J}\,(r',\ t - |r-r'|/c)}{|r - r'|}\, d^3r' \tag{10}$$

In Eq. (10) $\vec{J}$ represents the right hand side of Eq. (3a). From Eq. (10) it can be seen the effect of relativity is contained in the evalution of $\vec{J}$ at a retarded time. This means the $\vec{A}$ required to cause the Lagrangian to be correct to order β^2 is just the unretarded function. To see what is neglected Eq. (10) can be expanded about the unretarded solution.

$$\begin{aligned}\vec{A}\,(r,t) &= \frac{1}{c}\int \frac{\vec{J}\,(r',t)}{|r - r'|}\, d^3r' \\ &\quad - \frac{1}{c^2}\int \frac{\partial\vec{J}}{\partial t}\, d^3r + \ldots \\ &= \vec{A}_{\text{unretarded}} - \frac{1}{c^2}\int \frac{\partial\vec{J}}{\partial t}\, d^3r' \quad .\end{aligned} \tag{11}$$

The first neglected term in Eq. (11) scales like a wave number or inverse distance. This indicates the Darwin approximation is restricted by the allowed current variation.

The second means of seeing the implication of the Darwin approximation is to write out the Darwin field equations in terms of solenoidal (subscript t) and irrotational (subscript ℓ) components.

$$\nabla \times \vec{B} = 4\pi c^{-1} (\vec{J}_\ell + \vec{J}_t) + \frac{1}{c}\frac{\partial \vec{E}_\ell}{\partial t} \tag{12a}$$

$$\nabla \times \vec{E}_t = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{12b}$$

$$\nabla \cdot \vec{E}_\ell = 4\pi\rho \tag{12c}$$

$$\nabla \cdot \vec{B} = 0 \quad . \tag{12d}$$

The magnetic field is strictly solenoidal so $\vec{B}$ has an implied t subscript. Comparing with Maxwell's equations in Eq. (3) the only difference is the neglect of $\partial\vec{E}_t/\partial t$ in Eq. (12a). Retaining the irrotational part of the displacement current is just sufficient to maintain consistency with the continuity equation.

$$\nabla \cdot \vec{J}_\ell + \frac{\partial}{\partial t}(\nabla\cdot\vec{E}_\ell)/4\pi = 0 \quad . \tag{13}$$

Because the curl of any vector is solenoidal Eq. (12a) implies,

$$\vec{J} = -\frac{1}{4\pi}\frac{\partial \vec{E}_\ell}{\partial t} \tag{14}$$

and thus Eq. (12a) can be written,

$$\nabla \times \vec{B} = 4\pi c^{-1}\,\vec{J}_t \quad . \tag{15}$$

The curl of Eq. (15) yields an elliptic equation rather than a wave equation so radiation is absent from the Darwin approximation. Likewise, the curl of Eq. (12b) yields an elliptic equation for E_t rather than a wave equation.

$$\nabla^2 \vec{E}_t = 4\pi c^{-2} \frac{\partial \vec{J}_t}{\partial t} \tag{16}$$

Equation (16) shows $\vec{E}_t$ has the time derivative of $\vec{J}_t$ as a source. This means the viability of neglecting $\partial \vec{E}_t/\partial t$ in Eq. (12a) depends on the size of the time variation of $\vec{J}_t$. If the time variation is small, $\partial \vec{E}_t/\partial t$ is insignificant and the Darwin approximation is good. Thus, the Darwin approximation is precise when $\vec{J}_t$ is <u>constant</u> and there is no radiation. The magnitude of $\vec{J}_t$ may be large in this case. To illustrate this point consider the case of a highly relativistic beam with uniform charge density. Assume $\vec{J} = \hat{z}\, J_z(r)$ then $\nabla \cdot \vec{J} = 0$ and the current is solenoidal. Furthermore, $E_r = B_\theta$ and a solution of Eq. (12c) gives $E_{r\ell} = B_\theta$ so $E_{rt} = 0$. Thus, for this case the Darwin model provides the exact solution It is then clear the degree of approximation depends on the amount of current variation.

The Darwin equations may be compared with the frozen field Eq. (6) and the EMPULSE Eq. (7) equations by using Eq. (4b) in Eq. (12). This is then the frozen Darwin model.

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = 4\pi c^{-1}(J_{z\ell} + J_{zt}) + \frac{\partial}{\partial x} E_{z\ell} \tag{17a}$$

$$\frac{\partial}{\partial r}(E_{z\ell} + E_{zt}) = 4\pi c^{-1}(J_{r\ell} + J_{rt}) - \frac{\partial E_{rt}}{\partial x} \tag{17b}$$

$$\nabla_\perp \cdot \vec{J}_\perp - \frac{\partial J_z}{\partial x} + c\,\frac{\partial \rho}{\partial x} = 0 \quad . \tag{17c}$$

Comparing Eq. (6c) and (6d) with Eq. (17) it is evident the Darwin equations omit $\partial E_{zt}/\partial x$ in Eq. (17a) and subtract $\partial E_{rt}/\partial x$ in Eq. (17b). These terms are then the Darwin modification to the usual frozen field approximation. Since the Darwin model includes the complete charge continuity equation, the frozen Darwin Eq. (17c) is the same as Eq. (6e).

VI. GENERAL CONSIDERATIONS

The frozen Darwin model is similar to the EMPULSE equations in that neither allows the emission of electromagnetic radiation, both conserve charge, and they differ only by small terms ($\partial \vec{E}_\ell/\partial x$). The Darwin approximation appears to be "more correct" because a portion of the displacement current is retained. It also does not introduce a preferred axis at an early stage, and therefore may serve as a superior model for computations which deviate significantly from axisymmetry or the paraxial limit. To date, all attempts to arrive at simplified field equations which conserve the centroid of an obliquely launched beam have failed, when deviations from the frozen field approximation were considered. The cause of this appears to be in the use of frame-dependent approximations to achieve simplification. The Darwin approximation, while it leads to a relatively complex set of equations, should bypass this problem. Work underway on the general field problem is based on the potential representations:

$$\vec{B} = \nabla \times \vec{A} \quad ,$$

$$\nabla \cdot \vec{A} = 0 \quad ,$$

$$\vec{E}_t = - \frac{\partial \vec{A}}{\partial ct} \quad ,$$

$$\vec{E}_\ell = - \nabla\phi \quad ,$$

$$\vec{J}_t = - \frac{c}{4\pi} \nabla^2 \vec{A} \quad ,$$

$$\vec{J}_\ell = \frac{1}{4\pi} \nabla \frac{\partial\phi}{\partial t} \quad ,$$

$$\nabla \cdot \vec{J} = - \frac{\partial\rho}{\partial t} \quad .$$

It is found that all components of $\vec{A}$ must be retained to represent a physical configuration. This is a complication compared with the EMPULSE equations, which contain only A_z and ϕ. A second complication is the appearance of second derivations in time which cannot be eliminated by a change of gauge.

V. SUMMARY

The frozen Darwin model has been compared with the frozen field approximation and the EMPULSE equations. The emphasis was on the axisymmetric equations for B_θ, E_z and the charge continuity equation. It was found the frozen Darwin B_θ equation is the same as the frozen field equation without $\partial E_{zt}/\partial x$. The frozen Darwin E_z equation differs from the frozen field equation by $\partial E_{rt}/\partial x$. The charge continuity equations are identical. The frozen Darwin B_θ equation contains the $\partial E_z /\partial x$ term absent from the EMPULSE, B_θ equation. The frozen Darwin E_z equation differs from the EMPULSE, E_z equation by $\partial E_{rt}/\partial x$. The frozen Darwin model also has a different charge continuity equation than the EMPULSE equations. This difference arise from the neglect of $\partial E_z/\partial x$ in the EMPULSE equations.

Concerning the use of the frozen Darwin model for non-axisymmetric analysis there are two major changes from previous work. First, the frozen Darwin model does not introduce a preferred axis. This is an important feature which relates to the forces experienced by a beam propagating at an angle to a reference axis. Second, from preliminary work it appears it is necessary to use A and A_z. The non-axisymmetric EMPULSE equations retain only A_z. Thus, there is an added degree of complexity in the frozen Darwin equations. For non-axisymmetric analysis no benefit was realized by writing equations in terms of solenoidal and irrotational concepts. Using potentials the impact of the frozen Darwin model is to introduce extra terms as discussed.

REFERENCES

[1] E. P. Lee, "The New Field Equations," UCID-17286, (1976).

[2] J. K. Boyd, D. W. Hewett, Bulletin Am. Phys. Soc. Vol. 30 No. 5, p. 1436 (1984).

[3] J. K. Boyd, G. J. Caporaso, A. G. Cole, "Numerical Simulation of an Accelerator Injector," IEEE Trans. Nuc. Sci. (to be published fall 1985).

[4] J. D. Jackson, Classical Electrodynamics, John Wiley New York, p. 593 (1975).

[5] C. W. Nielson, H. R. Lewis, Methods in Computational Physics, Vol. 16, p. 367, Academic Press, N.Y. (1976).

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