RELATIVISTIC ELECTROMAGNETIC INSTABILITIES NEAR Eelectron CYCLOTRON FREQUENCY AND HARMONICS

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I. INTRODUCTION

Electrons having highly anisotropic, relativistic distribution functions are frequently created in intensely heated plasmas using electron-cyclotron resonance heating, and characterize intense, relativistic electron beams used in devices for generating electromagnetic energy. In this paper we present the absolute versus convective nature and propagation characteristics of instabilities that are driven by beam-type electron distributions representing such highly anisotropic electron plasmas. We study instabilities that propagate along and perpendicular to the steady, uniform magnetic field in which the plasma exists.

II. SPACE-TIME PROPAGATION OF INSTABILITIES

The space-time evolution of linear instabilities in a plasma can be determined from an analysis of the Green's function for the plasma. Thus, the pinch-point singularities that determine the time-asymptotic behavior of the Green's function allow one to distinguish between absolute and convective instabilities, and to determine the time-asymptotic pulse shape of the instability [1,2,3]. In particular, we recall that these analyses point to the fact that the propagation of instabilities cannot be studied by the usual means of geometric optics in stable plasmas where one relies on local solutions of the dispersion relation, e.g. k(\omega), and the concept of group velocity.

Previous analyses [1,2,3] for determining the time-asymptotic pulse shape of instabilities were, strictly speaking, applicable only to nonrelativistic propagation (e.g., electrostatic modes). We have recently generalized the pinch-point analysis so that time asymptotic pulse shapes can be obtained for relativistic-electromagnetic instabilities in three-dimensions [4,5]. Here, we summarize our results; the detailed theory is presented elsewhere [6].

Let the dispersion relation, in the laboratory frame of reference, be $D(q,\omega) = 0$. The time-asymptotic, unstable pulse shape is determined by the pinch-points as seen by an observer moving with a velocity $\vec{V}$ relative to the laboratory frame. For such an observer, the Fourier-Laplace transform of the Green’s function is given by $D_q^{-1}(\vec{k}',\omega',\vec{V}) = D^{-1}[\vec{k}(\vec{k}',\omega',\vec{V}); \omega(\vec{k}',\omega',\vec{V})]$ where the unprimed and primed quantities are related by well-known relativistic transformations [7]. The pinch-points ($\vec{k}_p, \omega_p$) are the proper solutions of $D_q = 0$ and $D_q = 0$, and the time-asymptotic pulse shape is determined by $\max(\omega_p)/\gamma$. Work supported in part by DOE Contract No. DE-AC02-76ER01358, Air Force Contract F33615-81-C-1426, and NSF Grant ECS 82-13430.
where \((\text{max } \omega''_{i})\) is the maximum imaginary part among the unstable pinch-point frequencies in the observer's frame and \(\nu_{s} = (1 - V^2/c^2)^{-1/2}\).

In the following sections we shall make use of these results to illustrate the build-up and propagation properties of instabilities in plasmas having highly anisotropic electron distribution functions.

III. RELATIVISTIC DISPERSION RELATIONS

We assume an infinite, homogeneous plasma in a uniform magnetic field, \(B_0\), with the ions forming a stationary, cold, neutralizing background. In the unperturbed state the electron distribution function can be taken as \(f_0(p) = f_0(p_{\perp}, p_{\parallel})\) where \(p_{\perp}(p_{\parallel})\) is the magnitude of the momentum perpendicular (parallel) to \(B_0\). The perturbed electron distribution function is determined by the linearized, relativistic Vlasov equation:

\[
\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m_e} \cdot \frac{\partial f}{\partial \mathbf{p}} - \epsilon \frac{\mathbf{p} \times \mathbf{B}_0}{(m_e^2 + p^2/c^2)^{1/2}} \cdot \frac{\partial f}{\partial \mathbf{p}} = \epsilon \left( \frac{\partial f_0}{\partial \mathbf{p}} + \frac{\mathbf{p} \times \mathbf{B}_1}{(m_e^2 + p^2/c^2)^{1/2}} \right) \cdot \frac{\partial f_0}{\partial \mathbf{p}}
\]

The dispersion relation is obtained by using equation (1) with the complete set of Maxwell's equations. For the two cases we choose to illustrate, namely \(k \parallel \mathbf{B}_0\) and \(k \perp \mathbf{B}_0\), and beam-type, anisotropic distribution functions, the dispersion relation can be written as follows:

Case 1, \(k_{\perp} = 0\).

For this case we choose:

\[
f_0(p_{\perp}, p_{\parallel}) = \frac{1}{2\pi p_{\perp}^0} \delta(p_{\perp} - p_{\perp}^0) \delta(p_{\parallel} - p_{\parallel}^0)
\]

For right-circularly polarized waves, we then find:

\[
\frac{c^2 k^2}{\omega^2} = 1 - \chi_r(k_{\parallel}, \omega)
\]

where

\[
\chi_r = \frac{\omega_p^2}{\omega^2} \left[ \frac{\omega - k_{\parallel} v_{\perp}^0}{\omega - k_{\parallel} v_{\perp}^0 - \omega_e} + \frac{v_{\perp}^2}{c^2} \frac{c^2 k_{\parallel}^2 - \omega^2}{2 c^2 (\omega - k_{\parallel} v_{\perp}^0 - \omega_e)^2} \right]
\]

where \(m = \gamma m_0, \omega_e^2 = (n e^2/\epsilon_0 n_0 \gamma) \equiv \omega_{ce}^2/\gamma, \omega_s = (eB_0/\gamma m_0) \equiv \omega_{cs}/\gamma, \) and \(\gamma^2 = (1 + (p^2/m_0^2c^2))\) with \(n_0\) being the density of the electrons and \(v_0 = (p_0/\gamma m_0)\).

Case 2, \(k_{\parallel} = 0\).

Here we consider the extraordinary waves propagating across the magnetic field. These waves are elliptically polarized, and for a beam-type distribution of the form:

\[
f_0(p_{\perp}, p_{\parallel}) = \frac{1}{2\pi p_{\perp}^0} \delta(p_{\perp} - p_{\perp}^0) \delta(p_{\parallel})
\]

the dispersion relation is:

\[
\frac{c^2 k^2}{\omega^2} = K_{yy} + \frac{K_{yy}^2}{K_{zz}}
\]

where,

\[
K_{zz} = 1 - \frac{\omega_p^2}{\omega} \sum_{n=-\infty}^{\infty} n^2 \left[ \frac{v_{\parallel}^2}{c^2} \frac{\omega}{\epsilon_2 (\omega - n \omega_e)^2} + \frac{2 J_n(\xi)J_n'(\xi)}{\xi (\omega - n \omega_e)} \right]
\]

\[
K_{yy} = -i \frac{\omega_p^2}{\omega} \sum_{n=-\infty}^{\infty} n \left[ \frac{J_n(\xi)J_n'(\xi)}{\omega - n \omega_e} \left( 1 - \frac{v_{\parallel}^2}{c^2} \frac{\omega}{(\omega - n \omega_e)} \right) + \frac{\xi}{\omega - n \omega_e} \left( J_n^2(\xi) + J_n(\xi)J_n'(\xi) \right) \right]
\]

\[
K_{yy} = 1 - \frac{\omega_p^2}{\omega} \sum_{n=-\infty}^{\infty} \left[ \frac{J_n^2(\xi)}{\omega - n \omega_e} \left( 2 - \frac{v_{\parallel}^2}{c^2} \frac{\omega}{(\omega - n \omega_e)} \right) + \frac{2 \xi}{(\omega - n \omega_e)} J_n(\xi)J_n'(\xi) \right]
\]
Figure 1. Dispersion relation for $k_z = 0$; real-$\omega$ ($\omega_r$, solid line) and imaginary-$\omega$ ($\omega_i$, dashed line) versus real-$k$.

Figure 2. Pulse shapes (symmetric about $V = 0$) for $k_z = 0$. 1 and 2 are the Whistler instability; 3 is the relativistic instability.
Figure 3. Dispersion relation for \( k_{||} = 0 \) and different densities; real-\( \omega \) (\( \omega_r \), solid line) and imaginary-\( \omega \) (\( \omega_i \), dashed line) versus real-\( k \).

Figure 4. Pulse shape (symmetric about \( V = 0 \)) for instability near \( u_c \) (\( k_{||} = 0, \omega_p/\omega_c = 0.1 \)).
$J_n$ is the Bessel function of n-th order. $\xi = (k_p \omega / m_0 \omega_0)$ and the prime indicates a derivative with respect to the argument.

IV. RESULTS AND DISCUSSION

Case 1, $k_p = 0$

We study this case for the following parameters: $\omega_p / \omega_0 = 0.2$, $v_{\perp 0} / c = 0.2$, $v_{00} / c = 0$. The dispersion relation is numerically solved for $\omega$ as a function of real $k$. The real (solid line) and imaginary (dashed line) parts of $\omega / \omega_0$ are plotted in Fig. 1. The scale on the right-hand side is for the imaginary part. The relativistic instability arises from the coupling of the fast electromagnetic wave to the relativistic negative energy branch (indicated by the – sign on the graph) while the Whistler instability is due to the Whistler branch coupling to the negative energy branch [8]. The relativistic instability has no counterpart in the case when we solve the non-relativistic Vlasov equation as the fast electromagnetic wave cannot couple to the non-relativistic negative energy branch.

We obtain the nature of the instabilities by applying the pinch-point technique described above under 1. The Whistler instability is an absolute one with a pinch-point at $k_0 = \pm \infty$, $\omega_0 / \omega_0 = 0.08 + 0.02 i$. The relativistic instability is also absolute with $k_0 = 0$, $\omega_0 / \omega_0 = 0.99 + 0.02 i$. Clearly, the Whistler instability has the dominant growth rate. But, since its phase-velocity is less than $c$, it can be stabilized by an appropriate temperature spread along $B_0$ in $f_0$. This cannot be the case with the relativistic instability as its phase-velocity is greater than $c$. However, it can be stabilized by increasing the density of the plasma so that the fast electromagnetic wave decouples from the negative energy wave. The condition for thus stabilizing the relativistic instability is:

$$\left(\frac{\omega_p}{\omega_0}\right)^2 > \frac{1}{\gamma(2 - \beta^2_\perp)} \left[ (4\beta^2_\perp + 1)^{3/2} + (2\beta^4_\perp + 10\beta^2_\perp - 1) \right]$$

where $\beta_\perp = v_\perp / c$. To order $\beta^2_\perp$, the right-hand side reduces to $(2\beta^2_\perp / \gamma)$. For our parameters the relativistic instability is stabilized for $(\omega_p^2 / \omega_0) > 0.28$.

The two Whistler instabilities decouple for observer velocities, $|V| > 0$. However, for $|V| > 0$, the Whistler instability for one of the Whistler instabilities is dissolved by the relativistic instability so that there are only two absolute instabilities left after that. The pulse shapes are plotted in Fig. 2. The pulses labelled 1 and 2 are for the Whistler instabilities and that labelled 3 is for the relativistic instability.

Case 2, $k_p = 0$

In Fig. 3, we plot the dispersion relation for $v_{\perp 0} / c = 0.1$ and for $\omega_p / \omega_0 = 0.05, 0.1, 0.32$. The solid lines are the real part of $\omega / \omega_c$ while the dashed lines are the imaginary part of $\omega / \omega_c$. The scale for the imaginary part is indicated on the right-hand side. The instabilities occur at $\omega_c$ and its harmonics, where the negative energy mode couples to the positive energy, extraordinary, electromagnetic modes [9]. For low densities the imaginary part of $\omega$ is largest at $\omega_c$ while at higher density the dominant growth rate is at $2\omega_c$.

Carrying out the relativistic pulse shape analysis for $\omega_p / \omega_c = 0.1$ the instability at $\omega_c$ is found to be an absolute one with the pinch-point given by $k_0 / \omega_c = 1.005 + 5 \times 10^{-3} i$, $k_0 = 0$. The corresponding pulse shape is plotted in Fig. 4. The instability at $2\omega_c$ and higher harmonics is of the convective type. At $\omega_p / \omega_c = 0.21$ the instabilities at $\omega_c$ and $2\omega_c$ are both absolute with the pinch-points at $\omega_0 / \omega_c = 1.01 + 2.7 \times 10^{-3} i$, $(k_0 v_{\perp 0}) / \omega_c = 0.32 + 0.01 i$ and $\omega_0 / \omega_c = 2.001 + 8.3 \times 10^{-4} i$, $(k_0 v_{\perp 0}) / \omega_c = -0.218 + 0.002 i$, respectively.

In Fig. 5 we plot the pulse shapes for the instabilities at $\omega_c$ (solid line) and at $2\omega_c$ (dashed line with the scales at the right-hand side and above). From the graph one can see that the absolute instability at $\omega_c$ is showing tendencies towards becoming convective. Since the pulse-shapes are symmetric about $V = 0$, for higher densities the pulse splits up into two convective pieces moving in opposite directions. Thus, the plasma is globally unstable but the instability is not a normal mode of the plasma. At $\omega_p / \omega_c = 0.32$ the instability at $\omega_c$ is completely convective while the one at $2\omega_c$ remains absolute. It is interesting to note that for higher densities and harmonics the pulse edge moves out at a slower velocity than the pulse edge velocity of the $\omega_c$-instability at low densities. As the density of the anisotropic electrons is further increased so that $(\omega_p / \omega_c) \sim 1$ electrostatic instabilities with much larger growth rates will set in [9]. We shall describe their space-time evolution in later publications.
Figure 5. Pulse shapes (symmetric about $V=0$) for $k||=0$, $\omega_p/\omega_c=0.21$, and instabilities near $\omega_c$ (solid line) and near $2\omega_c$ (dashed line with the corresponding scales on top and right).

REFERENCES