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On the Laurent polynomial rings

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ABSTRACT : We describe some properties of the Laurent polynomial rings in a finite number of indeterminates over a commutative unitary ring. We study some sub-rings of the Laurent polynomial rings. We finally obtain two cancellation properties.

A ring $R = A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$, where A is a commutative unitary ring and X_1, \dots, X_n are indeterminates over A , is called a Laurent polynomial ring (sometimes it is called a torus extension). A Laurent polynomial ring can be realized as the ring of quotients of an usual polynomial ring with respect to the multiplicative system generated by the indeterminates X_1, \dots, X_n . Therefore it is natural to study what properties of the polynomial rings can be extended to the Laurent polynomial rings. We also shall obtain some cancellation properties of these rings.

1. Lemma. Let A be a domain and X_1, \dots, X_n indeterminates over A . If $f \in A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ is invertible in $A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ then there are $a \in A$ invertible in A and $n_1, \dots, n_n \in \mathbb{Z}$ such that $f = aX_1^{n_1} \dots X_n^{n_n}$.

Proof. Let's first remark that it suffices to prove the lemma for $n = 1$. Let $X = X_1$ and let $f, g \in A[X, X^{-1}]$ be such that $fg = 1$. Since 1 is homogeneous in $A[X, X^{-1}]$ it follows that f, g are homogeneous. Therefore $f = aX^m$, $g = bX^n$, where $a, b \in A$, $m, n \in \mathbb{Z}$. Hence $ab = 1$ and $n = -m$.

Remark. The proof of the lemma 1 is standard. We included it here because it is useful for the rest of the paper.

2. Proposition. Let A be a domain and X an indeterminate over A . Then $\text{Aut}_A(A[X, X^{-1}]) = \{t: A[X, X^{-1}] \longrightarrow A[X, X^{-1}] ; t(X) = aX^m, \text{ where } a \text{ is invertible in } A \text{ and } m \in \{-1, 1\}\}$.

Proof. Let $t \in \text{Aut}_A(A[X, X^{-1}])$ and let $s = t^{-1}$. We may suppose that

$$t(X) = \frac{a_0 + \dots + a_n X^n}{X^u}, \quad s(X) = \frac{b_0 + \dots + b_n X^n}{X^v},$$

where $a_j, b_j \in A$, $n, n, u, v \in \mathbb{N}$, $a_n \neq 0$, $b_n \neq 0$. Therefore

$$X \cdot t(X)^v = b_0 + b_1 t(X) + \dots + b_n t(X)^n \quad (1)$$

If $\deg t(X) > 0$ from (1) it follows that $\deg s \cdot \deg t = 1$. Therefore $\deg s = \deg t = 1$. If $\text{ord } t = 1$ then $t(X) = aX$ and from (1) it follows that $s(X) = bX$ and $ab = 1$ ($a, b \in A$). If $\text{ord } t < 0$ from (1) it would follow that $1 = (n-v)\text{ord } t$, which is impossible.

If $\deg t < 0$ from (1) it follows that $1 + v \cdot \text{ord } t = n \cdot \text{ord } t$. Therefore $1 = (n-v) \cdot \text{ord } t$, hence $\text{ord } t = -1$. Since $\deg t < 0$ it follows that $\text{ord } t = \deg t = -1$. Then $t(X) = \frac{a}{X}$, where $a \in A \setminus \{0\}$. From $t(s(X)) = X$ it follows that a is invertible in A .

3. Lemma. Let A be a commutative unitary ring, X an indeterminate over A and $Y \in A[X, X^{-1}]$ such that $\frac{A[X, X^{-1}]}{K(A[X, X^{-1}])} = \frac{A[Y, Y^{-1}]}{K(A[Y, Y^{-1}])}$. Then $A[X, X^{-1}] = A[Y, Y^{-1}]$ and $t \in \text{Aut}_A(A[X, X^{-1}])$, where $t(X) = Y$.

Proof. Let $a, b \in A[Y, Y^{-1}]$, $u_j, v_j \in A[X, X^{-1}]$, $c_j, d_j \in K(A)$ such that

$$X = a(Y) + \sum_{j=1}^r c_j u_j(X),$$

$$X^{-1} = b(Y) + \sum_{j=1}^t d_j v_j(X).$$

Let \underline{n} be the ideal generated by $c_1, \dots, c_g, d_1, \dots, d_t$ in A . Since \underline{n} is a nilpotent ideal and $A[X, X^{-1}] \subset A[Y, Y^{-1}] + \underline{n}A[X, X^{-1}]$ it follows that $A[X, X^{-1}] = A[Y, Y^{-1}]$.

It remains to prove that Y is an indeterminate over A . Let $a_0, \dots, a_m \in A$ be such that $a_0 + a_1 Y + \dots + a_m Y^m = 0$. Let B be the subring of A generated by $1_A, a_0, \dots, a_m$, the coefficients of Y in $A[X, X^{-1}]$ and the coefficients of X in $A[Y, Y^{-1}]$. Let $t: B[X, X^{-1}] \longrightarrow B[X, X^{-1}]$ be the B -morphism defined by $t(X) = Y$. Since B is noetherian then $B[X, X^{-1}]$ is also noetherian and the surjection t is also an injection.

4. Proposition. Let A be a commutative unitary ring and X an indeterminate over A . If $t \in \text{Aut}_A(A[X, X^{-1}])$ then

$$t(X) = \sum_{i=-n}^n a_i X^i, \text{ where } a_i \in \underline{N}(A) \text{ for } i \neq s, a_s \text{ is invertible}$$

in A and $s = \pm 1$.

Proof. Let $A^* = \frac{A}{\underline{N}(A)}$ and let's consider the canonical map $u: \text{Aut}_A(A[X, X^{-1}]) \longrightarrow \text{Aut}_{A^*}(A^*[X, X^{-1}])$. If

$s \in \text{Aut}_{A^*}(A^*[X, X^{-1}])$ let $y \in A[X, X^{-1}]$ be such that $y \bmod \underline{N}(A) = s(X)$. Therefore $\frac{A[X, X^{-1}]}{\underline{N}(A[X, X^{-1}])} = \frac{A[Y, Y^{-1}]}{\underline{N}(A[Y, Y^{-1}])}$. It follows that

there is $g \in \text{Aut}_A(A[X, X^{-1}])$ such that $g(X) = y$. Then $u(g) = s$, so the map u is onto.

Let's remark that $\ker u = \{n \in \text{Aut}_A(A[X, X^{-1}]); n(X) = a_{-n} X^{-n} + \dots + a_0 + a_1 X + \dots + a_n X^n, a_i \in \underline{N}(A)\}$. Since u is onto

it follows that $\text{Aut}_{A^*}(A^*[X, X^{-1}]) \cong \frac{\text{Aut}_A(A[X, X^{-1}])}{\ker u}$. Therefore,

if $t \in \text{Aut}_A(A[X, X^{-1}])$ then $t(X) = g(X) + n(X)$, where $g \in \text{Aut}_{A^*}(A^*[X, X^{-1}])$ and $n(X) = \sum_{i=-m}^n a_i X^i$, $a_i \in \underline{N}(A)$ ($i = -m, -m+1, \dots, n$).

Remark. The proposition 4 shows that the structure of $\text{Aut}_A(A[X, X^{-1}])$ is not far from that of $\text{Aut}_A(A[X])$ (cf. with [3]).

5. Lemma. Let A, B be domains, $K = Q(A)$, $A \subset B$. If there is $f \in B \setminus A$ such that $B \subset K[f, f^{-1}]$ then f is invertible in B or f is irreducible in B .

Proof. Let's suppose that f is not invertible in B .

If there are $g, h \in B$ such that

$$f = g \cdot h \quad (1)$$

then $g = \frac{g_1}{f^u}$, $h = \frac{h_1}{f^v}$, where $g_1, h_1 \in K[f]$, $u, v \in \underline{N}$.

From (1) it follows that

$$g_1 h_1 = f^{1+u+v} \quad (2)$$

Since f is irreducible in $K[f]$ there are $m, n \in \underline{N}$ such that $g_1 = f^m$, $h_1 = f^n$ (modulo the multiplication by an invertible element). Therefore $g = f^{m-u}$, $h = f^{n-v}$. Since $f^{-1} \notin B$ it follows that $m-u \geq 0$, $n-v \geq 0$. From (1) we deduce that $m-u = 0$ or $n-v = 0$, i.e. g or h is invertible in B . Therefore f is irreducible in B .

6. Proposition. Let A, B be domains, $K = Q(A)$ and X_1, \dots, X_n indeterminates over A such that $A \subset B \subset A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$. If B is a GCD-domain and there is $f \in B \setminus A$ such that $B \subset$

$K[f, f^{-1}]$ and $af^{-1} \notin B$ for every $a \in A \setminus \{0\}$ then $B = A[f]$.

Proof. From the lemma 5 it follows that f is irreducible in B .

Let $b \in B \setminus K[f, f^{-1}]$. Then there are $a, a_0, \dots, a_m \in A$, $a \neq 0$, $t, m \in \mathbb{N}$ such that

$$abf^t = a_0 + a_1f + \dots + a_mf^m \quad (1)$$

If $t > 0$ then f divides a_0 in B . Therefore there is $h \in B \setminus K[f, f^{-1}]$ such that

$$a_0 = f \cdot h \quad (2)$$

Since we may suppose that $b \neq 0$, we may assume that $a_0 \neq 0$ if $t > 0$. Therefore from (2) it would follow that $h = a_0f^{-1} \in B$, in contradiction with the hypothesis. Therefore $t = 0$ and the relation (1) becomes

$$ab = a_0 + a_1f + \dots + a_mf^m \quad (3)$$

We shall prove by induction over m that in (1), a divides a_0, a_1, \dots, a_m .

If $m = 0$ then $ab = a_0$, therefore $b \in K \cap \Delta[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] = A$.

Let's prove that " $m \leq s$ " implies " $m = s+1$ ".

Since $t = 0$ it follows that $b \in K[f]$. But $K[f] = K[f+a]$ for every $a \in K$, so we may suppose that f is without constant term. Then there is $c \in A$ such that

$$a_0 = ac \quad (4)$$

It follows that

$$a(b - c) = f(a_1 + \dots + a_mf^{m-1}) \quad (5)$$

Since x is irreducible in B and B is a GCD-domain then there is $d \in B$ such that

$$b - c = df \quad (6)$$

Then

$$da = a_1 + \dots + a_n f^{n-1} \quad (7)$$

By the induction it follows that $a_i = ac_i$, where $c_1, \dots, c_n \in A$. Therefore

$$a(b - c - c_1 f - \dots - c_n f^n) = 0 \quad (8)$$

Since $a \neq 0$ and B is a domain it follows that $b = c + c_1 f + \dots + c_n f^n \in A[f]$. We deduce that $B = A[f]$.

Remark. If $A \subset B \subset A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$, $B \subset K[f, f^{-1}]$ and there is $a \in A \setminus \{0\}$ such that $af^{-1} \in B$ then it is possible that $B \neq A[f]$. For instance, let $A = \mathbb{Z}$, $B = \mathbb{Z}[\frac{2X_1}{X_2}, \frac{X_2}{X_1}]$, $f = \frac{2X_1}{X_2}$. Then $\mathbb{Z} \subset B \subset \mathbb{Z}[X_1, X_1^{-1}, X_2, X_2^{-1}]$, $B \subset \mathbb{Q}[f, f^{-1}]$, $f^{-1} = \frac{X_2}{2X_1} \in B$, $2f^{-1} = \frac{X_2}{X_1} \in B$ and $B \neq A[f] = \mathbb{Z}[\frac{2X_1}{X_2}]$.

7. Proposition. Let A, B be domains, $K = \mathbb{Q}(A)$ and X an indeterminate over A such that $A \subset B \subset A[X, X^{-1}]$. If B is a GCD-domain and there is $f \in B \setminus A$ such that $B \subset K[f, f^{-1}]$ then $B = A[f]$ or $B = A[aX^m, bX^{-m}]$, where $a, b \in A \setminus \{0\}$, $m \in \mathbb{N}^*$.

Proof. If $f^{-1} \in B$ then $f = aX^m$, where a is invertible in A and $m \in \mathbb{Z}$. Since $f \notin A$ we have $m \in \mathbb{N}^*$. Therefore

$$A[aX^m, a^{-1}X^{-m}] \subset B \subset K[aX^m, a^{-1}X^{-m}].$$

Since $B \subset A[X, X^{-1}]$ it follows that $B = A[aX^m, a^{-1}X^{-m}]$.

If $f^{-1} \notin B$ from the lemma 5 it follows that f is irreducible in B . If $af^{-1} \in B$ for every $a \in A \setminus \{0\}$ then $B = A[f]$

by proposition 6. Let's suppose that there is $a \in A \setminus \{0\}$ such that $af^{-1} \in B$. We reexamine under this hypothesis the relations (1) and (2) from the proof of the proposition 6. If $B \neq A[f]$ then we may suppose that $a_0 \neq 0$, $h \in B$ and

$$a_0 = f.h \quad (2)$$

Since a_0 is homogeneous it follows that f, h are homogeneous in $A[X, X^{-1}]$, hence $f = cX^s$, $h = dX^s$, where $c, d \in A \setminus \{0\}$, $cd = a_0$, $s \in \mathbb{Z} \setminus \{0\}$. The relation (1) from the proof of the proposition 6 becomes

$$b = \frac{1}{a} \sum_{i=0}^m a_i f^{i-t} = \sum_{i=0}^m \frac{a_i c^{i-t}}{a} X^{s(i-t)} \quad (1')$$

But from $b \in A[X, X^{-1}]$ it follows that $\frac{a_i c^{i-t}}{a} \in A$ for every $i = 0, 1, \dots, m$. Therefore we may suppose that there are $c, d \in A \setminus \{0\}$, $s \in \mathbb{N}^*$ such that

$$A[cX^s, dX^s] \subset B \subset A[X^s, X^{-s}] \quad (*)$$

Let $e_1 X^s, e_2 X^s \in B \setminus \{0\}$. Then $(e_1, e_2) X^s \in B \setminus \{0\}$, where (e_1, e_2) is the greatest common divisor of e_1 and e_2 in B .

Let $M = \{b \in B; b = e_1 X^s, e_1 \in A \setminus \{0\}\}$, $N = \{b \in B; b = f_j X^{-s}, f_j \in A \setminus \{0\}\}$.

From the above it follows that there are $e, f \in A$ such that $M \subset A[eX^s]$, $N \subset A[fX^{-s}]$. From (*) it follows that

$$A[eX^s, fX^{-s}] \subset B \subset A[eX^s, fX^{-s}] \quad (**)$$

Hence $B = A[eX^s, fX^{-s}]$.

8. Proposition. Let k be a commutative field, X_1, \dots, X_n indeterminates over k and A a domain such that $k \subset A \subset k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$. If $\dim A = 1$ then there is an indeterminate Y

over k such that $k \subset A \subset k[X, X^{-1}]$.

Proof. Let $P = k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ and let $\mathfrak{p} = (X_1^{-1}, \dots, X_n^{-1})R \cap A$. Because $(X_1^{-1}, \dots, X_n^{-1})R$ is a maximal ideal in R it follows that $\mathfrak{p} \in \text{Spec } A$. The rest of the proof is similar to that of 5.6 from [1].

9. Corollary. Let k be a commutative field, X_1, \dots, X_n indeterminates over k and A a GCD-domain such that $k \subset A \subset k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$. If $\dim A = 1$ then A is isomorphic to $k[X_1]$ or to $k[X_1, X_1^{-1}]$.

Proof. By proposition 8 there is an indeterminate X over k such that $k \subset A \subset k[X, X^{-1}]$. From proposition 7 it follows that $A = k[X]$ or $A = k[aX^s, bX^{-s}]$, where $a, b \in k \setminus \{0\}$, $s \in \mathbb{N}^*$. Since $k[X^s, X^{-s}] = k[aX^s, bX^{-s}]$ it follows that A is isomorphic to $k[X_1]$ or to $k[X_1, X_1^{-1}]$.

Remark. If k is algebraically closed, A a normal affine ring of dimension 1 and $n = 1$ our corollary 9 follows also from 4.1 of [8].

10. Proposition. Let A be a domain, X an indeterminate over A and B an UFD such that $A \subset B \subset A[X, X^{-1}]$. Then $B = A$ or B is isomorphic to $A[X]$ or to $A[cX, dX^{-1}]$, where $c, d \in A \setminus \{0\}$.

Proof. Let's suppose that $A \subsetneq B$. Let $K = Q(A)$. Therefore

$$K \subset K \otimes_A B \subset K[X, X^{-1}].$$

Since $K \otimes_A B$ is a GCD-domain from the corollary 9 it follows that there is $f \in K \otimes_A B \setminus K$ such that $K \otimes_A B = K[f]$ or $K \otimes_A B = K[f, f^{-1}]$.

$$\text{Let } S = A \setminus \{0\}. \text{ Then } K = S^{-1}A.$$

Let's suppose that $K \otimes_A B = K[f]$. Because $K[f] = K[sf]$ for every $s \in S$ we may suppose that $f \in A[X, X^{-1}]$. Therefore $f \in A[X, X^{-1}] \cap (K \otimes_A B) = A[X, X^{-1}] \cap S^{-1}B = B$, hence $f \in B$. Since $K \otimes_A B = K[f]$ it follows that $B = K[f]$. If $f^{-1} \notin B$ it follows that f is irreducible in B . From the proof of the proposition 6 it follows that $B = A[f]$, hence B is isomorphic to $A[X]$.

If $f^{-1} \in B$ then $B \subset K[f, f^{-1}]$.

Let's consider now the case $K \otimes_A B = K[f, f^{-1}]$. Because $K[f, f^{-1}] = K[sf, (sf)^{-1}]$ for every $s \in S$, we may suppose that $f \in A[X, X^{-1}]$. Therefore $f \in A[X, X^{-1}] \cap (K \otimes_A B) = B$, hence $f \in B$. Since $K \otimes_A B = K[f, f^{-1}]$ then $B \subset K[f, f^{-1}]$. Therefore $B = A[f]$ (and B is isomorphic to $A[X]$) or there are $c, d \in A \setminus \{0\}$, $m \in \mathbb{N}^*$ such that $B = A[cX^m, dX^{-m}]$, hence B is isomorphic to $A[cX, dX^{-1}]$.

11. Corollary. Let A be a domain, X an indeterminate over A and B a noetherian UFD such that $A \subset B \subset A[X, X^{-1}]$. Then A is a UFD.

Proof. It suffices to remark that B is a graduate

ring, $B = \sum_{i \in \mathbb{Z}} B_i$, where $B_0 = A$.

12. Proposition. Let A be a domain, X_1, \dots, X_n indeterminates over A and B a UFD. If $A \subset B \subset A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ and $\text{tr.deg.}_A B \leq 1$ then $B = A$ or B is isomorphic to $A[X]$ or to $A[cX, dX^{-1}]$, where X is an indeterminate over A , $c, d \in A \setminus \{0\}$.

Proof.

Let $K = Q(A)$. Then

$K \subset K \otimes_A B \subset K[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$.

If $\text{tr.deg.}_A B = 0$ then $\dim(K \otimes_A B) \leq \text{tr.deg.}_K(K \otimes_A B) = 0$.

Hence $B \subset K$, therefore $B = A$.

Let's suppose that $\text{tr.deg.}_A B = 1$. Then $\dim(K \otimes_A B) \leq 1$. If $\dim(K \otimes_A B) = 0$ then $B = A$. Therefore we may suppose that $\dim(K \otimes_A B) = 1$. From the corollary it follows that $K \otimes_A B$ is isomorphic to $K[X_1]$ or to $K[X_1, X_1^{-1}]$. Then there is an indeterminate Y over K such that $K \otimes_A B = K[Y]$ or $K \otimes_A B = K[Y, Y^{-1}]$.

If $K \otimes_A B = K[Y]$ then $B \subset K[Y]$, hence B is isomorphic to $B[Y]$.

If $K \otimes_A B = K[Y, Y^{-1}]$ then $B \subset K[Y, Y^{-1}]$, hence B is isomorphic to $A[Y]$ or to $A[cY, dY^{-1}]$, where $c, d \in A \setminus \{0\}$.

13. Proposition. Let A be a UFD, $A \subset B$ a ring extension, X, Y, Z indeterminates over A and T an indeterminate over B . If $B[T, T^{-1}] = A[X, X^{-1}, Y, Y^{-1}, Z, Z^{-1}]$ and $A[X, X^{-1}] \subset B$ then B is isomorphic to one of the following rings: $A[X, X^{-1}]$, $A[X, X^{-1}, Y]$, $A[X, X^{-1}, cY, dY^{-1}]$, where $c, d \in A \setminus \{0\}$.

Proof. Since A is a UFD it follows that $B[T, T^{-1}] = A[X, X^{-1}, Y, Y^{-1}, Z, Z^{-1}]$ is a UFD, hence B is a UFD.

We may suppose that $Z \notin B$. Indeed, if $Y, Z \in B$ then $B[T, T^{-1}] = A[X, X^{-1}, Y, Z][Y^{-1}, Z^{-1}] \subset B[Y^{-1}, Z^{-1}] = S^{-1}B$ (1) where S is the multiplicative system generated by Y, Z .

From (1) it would follow that $1 + \dim B < \dim B[T, T^{-1}] \leq \dim S^{-1}B \leq \dim B$, a contradiction.

Since $Z \notin B$ it follows that

$$B \cap (Z - 1)B[T, T^{-1}] = (0) \quad (2)$$

Indeed, let $u \in B \cap (Z - 1)B[T, T^{-1}]$. Because $Z - 1 \in B[T, T^{-1}] \setminus \{0\}$ there is $h(T) \in B[T, T^{-1}] \setminus \{0\}$ such that $Z - 1 = h(T)$. Therefore there is $g(T) \in B[T, T^{-1}]$ such that $u = h(T)g(T)$. If

$u \neq 0$ then $g(T) \neq 0$. Since $u \in B$ it would follow that $h(T) = b_1 T^m$, $g(T) = b_2 T^{-m}$, where $b_1, b_2 \in B \setminus \{0\}$, $m \in \mathbb{Z}$. Therefore $Z^{-1} = b_1 T^m$. But Z is homogeneous in $B[T, T^{-1}]$. Therefore $m = 0$, hence $Z \in B$, a contradiction. Therefore $u = 0$ and (2) is proved.

It follows that we have the following ring extension

$$B \subset \frac{B[T, T^{-1}]}{(Z-1)B[T, T^{-1}]} = \frac{A[X, X^{-1}, Y, Y^{-1}, Z, Z^{-1}]}{(Z-1)A[X, X^{-1}, Y, Y^{-1}, Z, Z^{-1}]} = A[X, X^{-1}, Y, Y^{-1}].$$

Therefore

$$A[X, X^{-1}] \subset B \subset A[X, X^{-1}][Y, Y^{-1}] \quad (3)$$

Since B is a UFD the proposition results from (3) and the proposition 10.

Remarks. 1) If in the proposition 13 the ring A is noetherian then B is isomorphic to $A[Y, X, X^{-1}]$ or to $A[X, X^{-1}, cY, dY^{-1}]$, where $c, d \in A \setminus \{0\}$.

ii) The proposition 13 is similar to an analogous result on polynomial rings. ([9]).

Definition. A commutative unitary ring A is called strongly n -torus invariant if from the equality of Laurent polynomial rings $A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] = B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}]$ it follows that $A = B$.

We shall give two sufficient conditions for the strongly torus invariance.

For the first result we need the following lemma.

14.Lemma. Let A, B be commutative domains, X_1, \dots, X_n indeterminates over A and Y_1, \dots, Y_n indeterminates over B . If $A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] = B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}]$ then there are ho-

homogeneous forms $u_{ij} \in A[X_1, X_1^{-1}, \dots, X_i, X_i^{-1}, \dots, X_n, X_n^{-1}]$, $v_{ij} \in B[Y_1, Y_1^{-1}, \dots, Y_j, Y_j^{-1}, \dots, Y_n, Y_n^{-1}]$ and $e_{ij}, f_{ij} \in \mathbb{Z}$ such that $X_i = v_{ij} Y_j^{e_{ij}}$, $Y_j = u_{ij} X_i^{f_{ij}}$.

Proof. Let $A_1 = A[X_1, X_1^{-1}, \dots, X_i, X_i^{-1}, \dots, X_n, X_n^{-1}]$, $B_j = B[Y_1, Y_1^{-1}, \dots, Y_j, Y_j^{-1}, \dots, Y_n, Y_n^{-1}]$. Then $A_1[X_i, X_i^{-1}] = B_j[Y_j, Y_j^{-1}]$ and we apply lemma 1.

15. Proposition. Let A, B be commutative domains, X_1, \dots, X_n be indeterminates over A and Y_1, \dots, Y_n be indeterminates over B. If $Q(A) \subseteq Q(B)$ and $A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] = B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}]$ then $A = B$.

Proof. Let's first remark like in the proof of 3.1 of [8] that from the hypothesis it follows that $A \subseteq B$. Indeed, let $x \in A$, hence $x \in Q(B)$. Then $x = \frac{b}{b'}$, where $b, b' \in B$, $b' \neq 0$. Since b, b' are homogeneous of degree zero in $B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}]$ we have $\deg_{Y_1, \dots, Y_n} x = 0$, hence $x \in B$. Therefore $A \subseteq B$.

Since $A \subseteq B$ we have $B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}] = A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] \subseteq B[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] \subseteq B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}]$, because $X_i, X_i^{-1}, \dots, X_n, X_n^{-1} \in B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}]$ by the hypothesis. Therefore

$$\begin{aligned} B[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] &= B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}], \text{ hence} \\ A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] &= B[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] \\ &= B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}] \end{aligned} \quad (1)$$

We shall prove by induction over n that $A = B$.

If $n = 1$ let $b \in B$, hence $b = \sum e_i Y_1^{e_i}$. Since Y_1 is

algebraically independent over B and $b, a_1 \in B$ it follows that $b = a_0$, hence $b \in A$. Therefore $B \subset A$, i.e. $B = A$.

Let's suppose the result proved for $n-1$ indeterminates and let's prove it for n indeterminates. Because $A \subset B$ we have

$$A[X_1, X_1^{-1}, \dots, X_{n-1}, X_{n-1}^{-1}] \subset B[X_1, X_1^{-1}, \dots, X_{n-1}, X_{n-1}^{-1}].$$

Since $A[X_1, X_1^{-1}, \dots, X_{n-1}, X_{n-1}^{-1}][X_n, X_n^{-1}] = B[X_1, X_1^{-1}, \dots, X_{n-1}, X_{n-1}^{-1}][X_n, X_n^{-1}]$ it follows that $A[X_1, X_1^{-1}, \dots, X_{n-1}, X_{n-1}^{-1}] = B[X_1, X_1^{-1}, \dots, X_{n-1}, X_{n-1}^{-1}]$, hence by induction $A = B$.

16. Corollary. Let A be a domain and A' its integral closure. If A' is strongly n -torus invariant then A is strongly n -torus invariant.

Remark. For $n = 1$ our proposition 15 and corollary 16 are proved in 3.1 and 3.2 from [8]. Our proof of the proposition 15 is diverse from that of 3.1 from [8]. The corollary 16 can be deduced from the proposition 15 like 3.2 is deduced from 3.1 in [8].

17. Theorem. Let k be a commutative infinite field and A a k -algebra which is a domain. If $\text{card}(\text{Aut}_k(A)) < \text{card}(k)$ then A is strongly n -torus invariant.

Proof. Let $R = A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] = B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}]$. For $m \in k \setminus \{0\}$ we define $F_m: R \longrightarrow R$ as follows:

$$F_m(Y_j) = mY_j \quad (j = 1, 2, \dots, n), \quad F_m(b) = b, \quad b \in B.$$

Let's remark that $F_m \in \text{Aut}(R)$. We also have

$$F_m(X_1) = F_m(v_{1j} Y_j^{e_{1j}}) = m^{e_{1j}} \cdot Y_j^{e_{1j}} \cdot F_m(v_{1j}).$$

Since v_{1j} is homogeneous in $Y_1, \dots, Y_{j-1}, Y_{j+1}, \dots, Y_n$ it follows that we have $F_m(v_{1j}) = m^{\deg v_{1j}} \cdot v_{1j}$. Therefore

$F_m(X_i) = m^{e_{ij} + \deg v_{ij}} \cdot v_{ij} X_j^{e_{ij}} = m^{\deg_Y X_i} X_i = m^{e_i} X_i$, where $e_i = \deg_{Y_1, \dots, Y_n} X_i$. It follows that

$$R = F_m(A)[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}].$$

Let $p: A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] \longrightarrow A$ be defined by $p(X_1) = \dots = p(X_n) = 1$, $p(a) = a$, $a \in A$ and let $i: A \hookrightarrow A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ be defined by $i(a) = a$, $a \in A$. We define $s_m: A \longrightarrow A$ by $s_m = p \circ F_m \circ i$. Therefore $s_m \in \text{End}(A)$.

We shall show that s_m is surjective. Let $a \in A$. Since $a \in A \subset R = F_m(A)[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ there are $a_{j_1} \dots a_{j_n} \in A$ such that

$$a = \sum_{\text{finite}} F_m(a_{j_1} \dots a_{j_n}) X_1^{i_1} \dots X_n^{j_n}. \text{ Then } a = p(a) = \sum p(F_m(a_{j_1} \dots a_{j_n})).$$

Let $a' = \sum a_{j_1} \dots a_{j_n} \in A$. It follows that $s_m(a') = p(F_m(a')) =$

$$\sum p(F_m(a_{j_1} \dots a_{j_n})) = a, \text{ hence } s_m \text{ is onto.}$$

The map s_m is also injective. For proving this let's remark that $\ker p = (X_1 - 1, \dots, X_n - 1)R$. But $F_m^{-1}(\ker p \cap F_m(A)) = (m^{-e_1} X_1 - 1, \dots, m^{-e_n} X_n - 1)R \cap A = (0)$, hence $\ker p \cap F_m(A) = (0)$. Therefore $\ker s_m = (0)$, i.e. s_m is injective.

Let $S = \{s_m ; m \in k \setminus \{0\}\}$. We shall prove that if $A \neq B$ then $\text{card } S > \text{card } k$, which is in contradiction with the hypothesis.

Let $u_{ij} = p(u_{ij}) \in A$, $v_{ij} = p(v_{ij}) \in A$. Since $X_i = v_{ij} X_j^{e_{ij}} = v_{ij} (u_{ij} X_i^{f_{ij}})^{e_{ij}} = v_{ij} u_{ij}^{e_{ij}} X_i^{f_{ij} e_{ij}}$ it follows that $1 = v_{ij}^{e_{ij}} u_{ij}^{e_{ij}}$, hence $u_{ij} \neq 0$, $v_{ij} \neq 0$.

Because $F_{\mathbb{N}}(u_{ij}) = F_{\mathbb{N}}(v_{ij}^{-f_{ij}} \cdot Y_j^{1-e_{ij} f_{ij}})$ \cdot $v_{ij}^{-e_{ij} f_{ij}} \cdot v_{ij}^{-f_{ij}} \cdot v_{ij}^{1-e_{ij} f_{ij}} \cdot Y_j^{1-e_{ij} f_{ij}} = v_{ij}^{1-e_{ij} f_{ij}} \cdot u_{ij}$ it follows that $s_{\mathbb{N}}(u_{ij}) = v_{ij}^{1-e_{ij} f_{ij}} \cdot u_{ij}$.

If there is (i,j) such that $1-e_{ij} f_{ij} \neq 0$ then card $S >$ card k , in contradiction with the hypothesis.

If $1-e_{ij} f_{ij} = 0$ for all (i,j) then $e_{ij} f_{ij} = 1$. Hence $X_j = u_{ij} X_i^{f_{ij}}$, where $f_{ij} = \pm 1$ and $f_{ij} = e_{ij}$. Therefore $R = A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] = B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}] = B[u_{11} X_1^{f_{11}}, (u_{11} X_1^{f_{11}})^{-1}, \dots, u_{nn} X_n^{f_{nn}}, (u_{nn} X_n^{f_{nn}})^{-1}]$.

If $A \neq B$ then from the proposition 15 it follows that there is $a \in A \setminus B$. Therefore there are $b_{j_1, \dots, j_n} \in B$ such that

$$a = \sum_{j_1, \dots, j_n} b_{j_1, \dots, j_n} (u_{11} X_1^{f_{11}})^{j_1} \dots (u_{nn} X_n^{f_{nn}})^{j_n} \tag{1}$$

Since $a \in A \setminus B$ we have $a \neq 0$, therefore $s_{\mathbb{N}}(a) \neq 0$ because $s_{\mathbb{N}}$ is injective. Therefore there is (j_1, \dots, j_n) such that $p(b_{j_1, \dots, j_n}) \neq 0$. Since $u_{11}^{j_1}, \dots, u_{nn}^{j_n} \neq 0$ and the field k is infinite from (1) and the expression of $s_{\mathbb{N}}(a)$ it follows that $\text{card}(S) >$ card k .

Therefore we must have $A = B$ and thus A is strongly n -torus invariant.

18. Corollary. Let k be a commutative field and A a

domain that contains k . If $\text{Aut}_k(A)$ is infinite then A is strongly n -torus invariant.

Remark. For $n = 1$ our corollary 18 follows from 3.8 of [8].

Another sufficient condition for the strongly torus invariance can be obtained with the help of the below defined pseudo-locally nilpotent derivations.

Definition. Let A be a ring. We call a pseudo-locally nilpotent derivation on A an infinite sequence of endomorphisms $D = (D_i)_{i \in \mathbb{Z}}$ defined on the additive group $(A, +)$ such that:

1) For every $a \in A$ there are $N(a), M(a) \in \mathbb{Z}$ such that $D_n(a) = 0$ if $n > N(a)$ or $n < M(a)$.

$$ii) D_n(ab) = \sum_{i+j=n} D_i(a)D_j(b) \text{ for every } a, b \in A, n \in \mathbb{Z}.$$

A pseudo-locally nilpotent derivation $D = (D_i)_{i \in \mathbb{Z}}$ is called trivially if $D_0 = \text{id}_A$ and $D_i = 0$ for $i \neq 0$.

Remark. The former definition is similar to that of the locally nilpotent derivations from [7].

19. Proposition. Let A be a commutative ring that contains an infinite field. If on A there are no pseudo-locally not trivially nilpotent derivations then A is strongly n -torus invariant for every n .

Proof. Let's suppose that there were a commutative ring $B \neq A$, X_1, \dots, X_p indeterminates over A and Y_1, \dots, Y_n indeterminates over B such that

$$A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] = B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}].$$

Let $a \in A \setminus B$. Then there is $f(Y_1, \dots, Y_n) \in B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}] \setminus B$ such that

$$a = f(Y_1, \dots, Y_n).$$

We may suppose that $f(Y_1, \dots, Y_n) = \sum_{j=m_1}^{m_2} g_j(Y_2, \dots, Y_n) Y_1^j$,

where $g_j(Y_2, \dots, Y_n) \in B[Y_2, Y_2^{-1}, \dots, Y_n, Y_n^{-1}]$, $m_1 \leq m_2$, $g_{m_1}(Y_2, \dots, Y_n) \neq 0$, $g_{m_2}(Y_2, \dots, Y_n) \neq 0$, m_1 or $m_2 \neq 0$.

Let T be a new indeterminate and let's consider the morphism $G: B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}] \longrightarrow B[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}, T, T^{-1}]$ defined as follows: $G(Y_1) = Y_1 T$, $G(Y_2) = Y_2, \dots, G(Y_n) = Y_n$. Then

$$G(a) = \sum_{j=m_1}^{m_2} g_j(Y_2, \dots, Y_n) Y_1^j T^j.$$

Let $g_j(Y_2, \dots, Y_n) Y_1^j = h_j(X_1, \dots, X_n) \in A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$. Then $h_{m_1}(X_1, \dots, X_n) \neq 0$, $h_{m_2}(X_1, \dots, X_n) \neq 0$.

If $m_2 \neq 0$ let $a_1, \dots, a_n \in A \setminus \{0\}$ be such that $h_{m_2}(a_1, \dots, a_n) \neq 0$ (this choice is possible because A contains an infinite field). If $m_2 = 0$ then $m_1 \neq 0$ (because $m_1 = m_2 = 0$ would imply $a \in B$) and let $a_1, \dots, a_n \in A \setminus \{0\}$ be such that $h_{m_1}(a_1, \dots, a_n) \neq 0$.

Let $t: A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}, T, T^{-1}] \longrightarrow A[T, T^{-1}]$

be the A -morphism defined as follows: $t(X_1) = a_1, \dots, t(X_n) = a_n$, $t(T) = T$. (We may suppose that a_1, \dots, a_n are non-zero elements from an infinite field included in A , Hence that a_1, \dots, a_n are invertible). Let $i: A \hookrightarrow A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}, T, T^{-1}]$ be

the canonical inclusion and let's consider $f: A \longrightarrow A[T, T^{-1}]$, where $f := t \circ G \circ i$.

Then f is a ring morphism and we have

$$f(a) = \sum_{j=m_1}^{m_2} h_j(a_1, \dots, a_n) T^j \in A[T, T^{-1}] \setminus A \text{ because } h_{m_2}(a_1, \dots, a_n) \neq 0 \text{ or } h_{m_1}(a_1, \dots, a_n) \neq 0 \text{ (if } m_2 \neq 0 \text{ or } m_2 = 0, m_1 \neq 0 \text{)}.$$

$$\text{Let } f(\alpha) = \sum_{\text{finite}} \alpha_i T^i. \text{ Let's consider } D_i(\alpha) := \alpha_i.$$

Since f is a ring morphism it follows that $(D_i)_{i \in \mathbb{Z}}$ is a pseudo locally nilpotent derivation on A . Because $f(a) \notin A$ it is not trivially. Therefore $B = A$, and A is strongly n -torus invariant for every n .

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