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AN INTRODUCTION TO THE APPLICATION OF RELAXATION METHOD IN NUMERICAL WEATHER PREDICTION *

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ABSTRACT

This paper is intended for workers in the field of numerical weather prediction to acquire experience and gain insight on the use of the relaxation method. Two approaches were carried out, one by explaining the method using hand calculations as applied to a given problem and the second one was the discussion of how the calculations could be carried out on a digital computer.

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I. INTRODUCTION

The historical development of numerical weather prediction can be traced back from the initial attempt and effort made by L.F. Richardson, a British mathematical physicist, in 1911, when he manually carried out some calculations to predict the values of weather parameters at equally spaced mesh or grid points in his forecast domain. He fully laid out the basic plans or procedures that are now followed in numerical weather prediction. To mention, the initial concepts are based on physical laws which at that time, contemporary fluid-mechanists seemed to accept as the principles needed to predict the future states of the atmosphere, just in a similar manner to the laws predicting the orbits of the planets. Presently, along these lines numerical weather predictions are being pursued but with modifications and improvements introduced by the advent of sophisticated computers, applications of remote sensing techniques, results of recent investigations, etc. But the procedures of representing the differential equations in terms of its equivalent difference equations remain the same leading to the calculations of desired parameters at each mesh or grid point.

II. OBJECTIVE AND SIGNIFICANCE OF THIS WORK IN NUMERICAL WEATHER PREDICTION

In a numerical weather prediction scheme, the forecast domain is subdivided into equally spaced squares and the intersection points or grid points are the points where the weather parameters are evaluated. Thus, the number of grid points corresponds to the number of equations to be evaluated. The suggested and appropriate method to be used here is the so called relaxation method. It is the aim of the work to explore and present in detail, the application of the relaxation techniques in problems where the number of unknowns to be evaluated is large. To carry out this objective, two approaches will be made, one by demonstrating the method through the use of hand calculations in the solution of a system of linear equations and secondly, the use of a digital computer is solving a Poisson-type of differential equation in its equivalent difference form.

This presentation is significant in a way that it is designed for beginners in numerical weather modelling and also for those with limited experience in handling a system of equations of n unknowns (n being >> 50) arising in any given problem. Likewise this also leads them to exposure and introduction to such mathematical tools or techniques that are not usuallly taken up in under-graduate courses. But the most important aspect is, it serves them an essential backgrounder in their chosen field of activities.

III. METHODOLOGY

As planned, there are two approaches to be carried out, Case I will be the application of the relaxation method using hand or desk calculators in the computation process of a system of linear equations with three unknowns. Operation tables with explanatory notes are presented to facilitate the discussion of the sample problem. The so called Gauss-Seidel method will be introduced and applied to the same system of equations. The second approach, Case II, will be a discussion of how the application of the relaxation method will be applied to a problem with the use of digital computers.

Case I: A system of linear equations in three dimensions

Given the equations in three unknowns x_1, x_2 and x_3

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= k_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= k_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= k_3 \quad .
\end{aligned}
\tag{1}
$$

Let us introduce the residuals r_1, r_2 and r_3 and define them by the relations

$$
\begin{aligned}
r_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 - k_1 \\
r_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 - k_2 \\
r_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 - k_3 \quad .
\end{aligned}
\tag{2}
$$

The aim here is to have a better approximation of x_1, x_2 and x_3 as the residuals r_1, r_2 and r_3 become smaller and smaller, eventually negligible or zero. From this point, let us construct an operation table as in Fig.1. This figure shows the effect on r_1, r_2 and r_3 when x_1 is given a change by 1 unit, keeping the other unknowns constant. Let us call this operator R_1 and similarly we have R_2 and R_3 for r_2 and r_3, respectively.

	x_1	x_2	x_3	r_1	r_2	r_3
R_1	1	0	0	a_{11}	a_{21}	a_{31}
R_2	0	1	0	a_{12}	a_{22}	a_{32}
R_3	0	0	1	a_{13}	a_{23}	a_{33}

Fig.1

For further explanation, consider the operator R_2, it increases x_2 by 1, keeping x_1 and x_3 constant. This will have effect on increasing r_1, r_2, r_3 by 1 a_{11} 1 a_{12} and 1 a_{13}, respectively. Note also that the operation table in Fig.1 consists of a unit matrix I and A^T, the tranpose of matrix A of the coefficient in (1).

Example I(a): solve this system of equation by relaxation technique with the help of operation table.

Given

$$\begin{aligned} 10X_1 + 2X_2 - X_3 &= 58 \\ X_1 + 7X_2 + 2X_3 &= -4 \\ 2X_1 + X_2 + 20X_3 &= 73 \end{aligned} \qquad (3)$$

Let us apply the operation table.

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	x_1	x_2	x_3	r_1	r_2	r_3
R_1	1	0	0	10	1	2
R_2	0	1	0	2	7	1
R_3	0	0	1	- 1	2	20

Fig.2

Actual Relaxation Table

Lines	x_1	x_2	x_3	r_1	r_2	r_3
1	0	0	0	-58	4	-73
2	0	0	4	-62	12	7
3	6	0	0	- 2	18	19
4	0	0	-1	- i	16	- 1
5	0	-2	0	- 5	2	- 3
6	6	-2	3	- 5	2	- 3
7	0	0	0	-50	20	30
8	5	0	0	0	25	-20
9	0	-4	0	- 8	- 3	-24
10	0	0	1	- 9	- 1	- 4
11	1	0	0	1	0	2
12	6	-4	1	1	0	- 2

Fig.3

Explanatory notes line by line

(1) This line is obtained by letting $x_1 = x_2 = x_3 = 0$ and calculating r_1, r_2 and r_3 or any convenient way of obtaining residuals.

(2) Note that r_3 has the largest residual, let us try to liquidate it first by applying a multiple of the operator R_3, i.e. $4\ R_3$. We can deduce that this adds -4, 8, 80 to r_1, r_2 and r_3, respectively, to obtain the values of residuals r_1, r_2 and r_3 in line (2).

(3),(4) and (5) These lines are obtained similarly as above by the use of multiple operators, $6\ R_1$, $-R_3$ and $-2\ R_2$, respectively.

(6) This line is obtained by adding increments of x_1, x_2 and x_3 in lines (2), (3), (4) and (5) to the initial approximation of r_1, r_2 and r_3 in line 1. This step is known as a "stack-taking' stage where the residuals r_1, r_2 and r_3 are obtained by substituting $x_1 = 6$, $x_2 = -2$ and $x_3 = 3$ and as a check obtain the same r_1, r_2 and r_3 in line (5).

(7) To avoid decimal fraction in the operation table, let us multiply line (6) by 10 to obtain line (7) where the units represent the first decimal place.

(8),(9),(10) and (11) These lines are obtained by applying the R operators, such as $+5\ R_1$, $-4\ R_2$, $1\ R_3$ and $1\ R_1$, respectively.

(12) This line is the "stack-taking" line, where the increments of x_1, x_2 and x_3 are added from lines (8), (9), (10) and (11) to get the current values of x_1, x_2 and x_3, respectively. We obtained for x_1, x_2 and x_3, 6.6, -2.4 and 3.1 values, respectively.

Case I(b):

For the same problem, the technique of liquidating residuals in turn will be applied. This is referred to as the Gauss-Seidel method. Consider the three unknowns

$$X^m = \left(X_1^m,\ X_2^m,\ X_3^m\right), \qquad (4)$$

(where m is the number of iteration)

be the solutions of the system of equations after m^{th} iteration or relaxation. Let the solution be discussed first in the matrix form:

$$\begin{aligned} q_{11}X_1^{m+1} + q_{12}X_2^m + q_{13}X_3^m &= k_1 \\ q_{21}X_1^{m+1} + q_{22}X_2^{m+1} + q_{23}X_3^m &= k_2 \\ q_{31}X_1^{m+1} + q_{32}X_2^{m+1} + q_{33}X_3^{m+1} &= k_3 \end{aligned} \qquad (5)$$

It follows that this system can be solved by forward substitution such as

$$\begin{aligned} X_1^{m+1} &= \left(k_1 - q_{12}X_2^m - q_{13}X_3^m\right)\Big/q_{11} \\ X_2^{m+1} &= \left(k_2 - q_{21}X_1^{m+1} - q_{23}X_3^m\right)\Big/q_{22} \\ X_3^{m+1} &= \left(k_3 - q_{31}X_1^{m+1} - q_{32}X_2^{m+1}\right)\Big/q_{33}\ . \end{aligned} \qquad (6)$$

IF [A] is the matrix of the coefficients (5) and then defining L and U matrices as

$$[L] = \begin{pmatrix} q_{11} & 0 & 0 \\ q_{21} & q_{22} & 0 \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \qquad (7)$$

$$[U] = \begin{pmatrix} 0 & q_{12} & q_{31} \\ 0 & 0 & q_{32} \\ 0 & 0 & 0 \end{pmatrix}$$

we can write

$$[A] = [L] + [U] \qquad (8)$$

and thus we have

$$L X^{m+1} + U X^m = K$$

$$X^{m+1} = L^{-1}\left(K - UX^m\right) \qquad (9)$$

This is known as the Gauss-Seidel method represented in the matrix notation. The residuals in this method are liquidated <u>in turn</u>.

Example 2: Let us solve the previous problem using this method. Given

$$\begin{aligned} 10X_1 + 2X_2 - X_3 &= 58 \\ X_1 + 7X_2 + 2X_3 &= -4 \\ 2X_1 + X_2 + 20X_3 &= 73 \end{aligned} \tag{10}$$

we have

$$\begin{aligned} X_1^{m+1} &= \left(58 - 2X_2^{m} + X_3^{m}\right)/10 \\ X_2^{m+1} &= \left(-4 - X_1^{m+1} - 2X_3^{m}\right)/7 \\ X_3^{m+1} &= \left(73 - 2X_1^{m+1} - X_2^{m+1}\right)/20 \end{aligned} \tag{11}$$

Construct an operation table

m	1	2	3	4
x_1^m	0	5.80	6.329	6.55
x_2^m	0	-1.371	-2.204	-2.40
x_3^m	0	2.55	3.127	+3.11

<u>Fig.4</u>

After 4th iteration or relaxation the values of x are

$x_1 = 6.55$ $\quad x_2 = -2.4$ and $x_3 = +3.11$.

<u>Case II</u>: Application of the relaxation technique to Poisson-type of equation using digital computers

The digital computer methods are obviously important, at this point when the n values to be evaluated is from 50 or more. But beginners are advised to seek first experience and gain insight of the technique as in the two previous examples.

Let us now consider a typical differential equation usually encountered in numerical weather prediction. Consider a Poisson-type of equations in two dimensions

$$\nabla^2 \phi = -\kappa \quad , \tag{12}$$

where

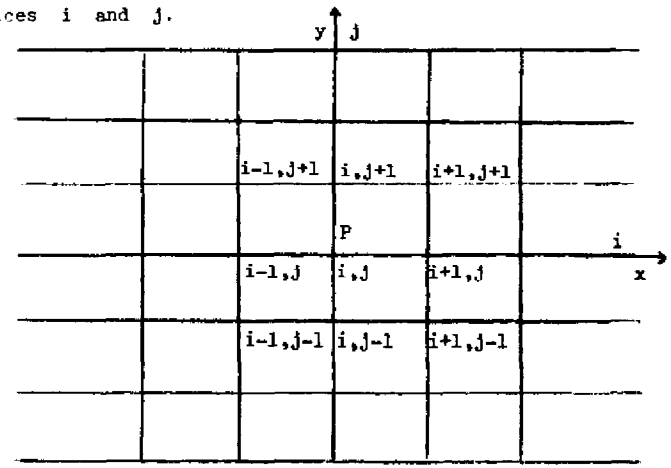
$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and κ is a constant while ϕ is a known function in x and y. Let i and j be indices along the x and y axes, respectively denoting the positions such as P(i,j).

$x = i\,\Delta x \qquad i = 0, \pm1, \pm2, \pm3 \ldots\ldots$

$y = j\,\Delta y \qquad j = 0, \pm1, \pm2, \pm3 \ldots\ldots$

Let us divide the xy plane into a network of families of parallel lines to the axes (see Fig.5) of interval $\Delta x = \Delta y = 1$ unit. Note that the intersections of the parallel lines are referred to as mesh or grid points and can be identified by the indices i and j.



<u>Fig.5</u>

Transforming Eq.(12) to its equivalent difference equation

$$\phi_{xx} \approx \frac{\partial^2}{\partial x^2}\phi \qquad \phi_{yy} \approx \frac{\partial^2}{\partial y^2}\phi$$

$$\phi_{xx} = \phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j} \tag{13}$$

$$\phi_{yy} = \phi_{i,j+1} - 2\phi_{ij} + \phi_{i,j-1}$$

we have

$$\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} + K = 0 \,. \tag{14}$$

Consider the domain of integration in the xy plane as in Fig.5 with the boundary condition specifying the initial values of ϕ along the boundaries. There are ϕ_{ij} values to be determined in the interior of the domains. IF the region is a rectangle and we have

$$0 \leq i \leq N \qquad 0 \leq j \leq M$$

and IF i ranges from 0 to N and j from zero to M, then there are (N-1)(M-1) interior points in the domain. At these interior points, values of ϕ are to be calculated. Therefore, there are exactly (N-1)(M-1) unknowns to be evaluated. Accordingly, the solutions of Eq.(14) may be considered as the solution of a simultaneous system of (N-1)(M-1) linear equations. As previously shown, the Gauss-Seidel method can be applied here but using hand calculations are impractical.

The relaxation or iterative method is generally applied to this kind of problem and the simplest to follow is a successive approximation or the so called simultaneous method of relaxation which will be discussed here.

Let us now consider Eq.(14) then, at a particular point (i,j). This equation will be used to find the values at the interior points of the defined domain, having stated the necessary boundary conditions. At this point, note that our problem is an outright boundary value problem. To carry out the

calculations of ϕ values at different P(i,j) grid points by relaxation method, let us begin by stating that this method recognizes at the start that an assumed functional value at any grid point will not satisfy the given difference equation, that is, there will be a residual at that given point. And all residuals must be computed for all points, before the relaxation process is performed.

Let the assumed value at P(i,j) be $Q^0_{i,j}$, as an initial guess solution, then we can rewrite Eq.(14) as

$$R^0_{ij} = \phi^0_{i+1,j} + \phi^0_{i-1,j} + \phi^0_{i,j+1} + \phi^0_{i,j-1} - 4\phi^0_{ij} + K \,. \tag{15}$$

If we alter the value of the function at this point by an amount $\Delta\phi^0_{i,j}$, then the residual will also be altered by $\Delta R^0_{i,j}$ and we can again rewrite Eq.(15)

$$R^0_{i,j} + \Delta R^0_{i,j} = \phi^0_{i+1,j} + \phi^0_{i-1,j} + \phi^0_{i,j+1} + \phi^0_{i,j-1} - 4(\phi^0_{ij} + \Delta\phi_{i,j}) + K \tag{16}$$

by subtraction, we have

$$\Delta R^0_{i,j} = -4\Delta\phi^0_{ij} \,.$$

It follows that if we wish to make the residual $R^0_{ij} = 0$, then we simply change the value of the function at a given point (i,j) by an amount

$$\Delta\phi^0_{i,j} = \frac{1}{4}R^0_{ij} \,.$$

In order to make R^0_{ij} equal to zero in Eq.(15), it is necessary to change the residuals $R_{i+,j}$, $R_{i-,j}$, $R_{i,j+1}$, $R_{i,j-1}$ by an amount of $\Delta\phi_{ij}$.

Let us make $\phi^1_{i,j}$ to denote the result of the change by $\Delta\phi_{i,j}$ then it follows that we can write

$$\phi'_{i,j} = \phi^0_{i,j} + \frac{1}{4}R^0_{i,j} \,. \tag{17}$$

This form of equation is describing a general recursion formula which can be applied to generate the values of ϕ^m_{ij} and the residuals at the interior grid points. We have

$$\phi_{i,j}^{m+1} = \phi_{i,j}^{m} + \frac{1}{4} R_{i,j}^{m} , \qquad (18)$$

where

$$R_{ij}^{m} = \nabla^2 \phi^m + K .$$

The equations above define an iterative procedure to approximate $^{m}_{ij}$ from an initial assumed value. This method is known as the simultaneous relaxation.

In recapitulating, from the obtained values at the interior points as well as their residuals, the aim of the method is to liquidate or reduce to zero (or nearly zero) these residuals as fast as possible by relaxing or altering the grid point values involved. The process may start at any region and jump all over the bounded region.

On a digital computer, this method can be carried out along this approach. The first step is to determine the largest residuals $R_1, R_2, \ldots, R_n$ where n is the number of interior points corresponding to numbers of equations or simply the product of (N-1) (M-1). Let us say R_3 is the largest, then a new value of Q_3 is obtained by solving Eq.(14) for Q_3 in order that R_3 becomes zero; calculate or readjust the other residuals for new values of Q_n. The scanning process (finding the largest residual) is programmed in the computer, although it is time consuming. However, there are new approaches using the computer to solve this problem and these are not covered up here. This is only an initial exposure of beginners for the use of the digital computer.

IV. CONCLUSION

The relaxation method is faster and flexible when used in solving a large number of unknowns. But, with the use of digital computers, programmersmust be keen enough for searching ways for the process to reach convergence as fast as possible. In here, a great deal of experience is needed both for the programmers and analysts. And the discussion of the method is aimed at newcomers in the field of numerical weather prediction, for them, to grasp and have insight to the problem as well as some actual experience in the computation technique.

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