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QUANTUM OSP-INVARIANT NON-LINEAR SCHRÖDINGER EQUATION *

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ABSTRACT

The generalizations of the non-linear Schrödinger equation (NS) associated with the orthosymplectic superalgebras are formulated. The simplest osp(1/2)-NS model is solved by the quantum inverse scattering method on a finite interval under periodic boundary conditions as well as on the whole line in the case of a finite number of excitations.

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The well-known completely integrable field theory models: nonlinear Schrödinger equation and sine-Gordon equation were used to develop quantum analogue of the inverse scattering method (cf.reviews [1-3]). It was proved that the supersymmetric generalization of the sine-Gordon equation is an integrable model in the classical case [4] as well as in quantum one [3] . Generalizations of the nonlinear Schrödinger equation (NS) on multicomponent case have been made by different approaches, in particular using simple Lie algebras (cf. [5]). NS corresponding to the superalgebras sl(m/n) has commuting as well as anticommuting fields [6] . The aim of the present letter is to study NS with two components: $q(x,t)$-Bose-field and $\psi(x,t)$-Fermi-field

$$i\partial_t q = -\partial_x^2 q + 2\alpha q^+ q q + \alpha \psi^+ \psi q - i(\alpha)^{1/2} \psi \partial_x \psi, \quad (1)$$

$$i\partial_t \psi = -2\partial_x^2 \psi + \alpha q^+ q \psi - i(\alpha)^{1/2} (2q\partial_x \psi^+ + \psi^+ \partial_x q) \quad (2)$$

where + denotes conjugation in a normed infinite dimensional Grassmann algebra $G = G_0 \oplus G_1$, $q \in G_0$, $\psi \in G_1$. Eqs(1), (2) can be solved by the inverse scattering method(ISM) using an auxiliary linear problem connected with the orthosymplectic superalgebra osp (1/2).

Many properties of the superalgebra osp(1/2) are similar to that of sl(2) [7] . There are additionally two odd generators $V_\pm$ which add to the set of three (even) generators of sl(2): h, $X^\pm$

$$[h, X^\pm] = \pm X^\pm, \quad [X^+, X^-] = 2h, \quad (3)$$

$$[h, V_\pm] = \pm\frac{1}{2}V_\pm, \quad [X^\pm, V_\mp] = V_\pm, \quad [X^\pm, V_\pm] = 0, \tag{4}$$

$$\{V_\pm, V_\pm\} = \pm\frac{1}{2}X^\pm, \qquad \{V_+, V_-\} = -\frac{1}{2}h, \tag{5}$$

where $[\ ,\ ]$ commutator, $\{\ ,\ \}$ anticommutator.

The Casimir operator is

$$C_2 = h^2 + \frac{1}{2}(X^+X^- + X^-X^+) + [V_+, V_-]. \tag{6}$$

The nontrivial irreducible representation of lowest dimension is [7]

$$h = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X^+ = (X^-)^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

$$V_+ = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_- = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{8}$$

The finite dimensional irreducible representations of osp(1/2) are parametrized by a half integer number s. The dimension of the irreducible representation V(s) is $\dim V(s) = 4s+1$ and $C_2(s) = s(s+1/2)$ [7].

According to the modern Hamiltonian version of ISM [2], to a given simple Lie (super)algebra g there corresponds the r-matrix which satisfies the classical Yang-Baxter equation [3]

$$[r^{12}(\lambda), r^{13}(\lambda+\nu) + r^{23}(\nu)] + [r^{13}(\lambda+\nu), r^{23}(\nu)] = 0 \tag{9}$$

The solution $r^{ab}(\lambda)$ lies in the tensor product of two copies of g: $\quad r^{ab}(\lambda) \in g_a \otimes g_b, \quad a, b = 1, 2, 3.$

For g=osp(1/2) it is given by

$$r(\lambda) = \frac{2\alpha}{\lambda}\left(\tilde{h}\tilde{\tilde{h}} + \frac{1}{2}(\tilde{X}^+\tilde{\tilde{X}}^- + \tilde{X}^-\tilde{\tilde{X}}^+) + (\tilde{V}_+\tilde{\tilde{V}}_- - \tilde{V}_-\tilde{\tilde{V}}_+)\right) \tag{10}$$

where we use the notations $\tilde{X} = X \otimes I$, $\tilde{\tilde{X}} = I \otimes X$. Let us underline that the tensor product is the graded one. The nonzero entries of the r-matrix (10) in the fundamental representation are

$$r(\lambda) = \frac{\alpha}{2\lambda}\left|\begin{array}{ccc|ccc|ccc} 1 & & & & & & & & \\ & & & 1 & & & & & \\ & & -1 & & -1 & & 2 & & \\ \hline & 1 & & & & & & & \\ & & 1 & & & & -1 & & \\ & & & & & & & 1 & \\ \hline & & 2 & & 1 & & -1 & & \\ & & & & & 1 & & & \\ & & & & & & & & 1 \end{array}\right| \tag{11}$$

One can define a variety of integrable systems associated with the given r-matrix. The simplest one is the magnet with the auxiliary linear problem

$$-i\frac{d}{dx}f(x,\lambda) = \lambda\Big(\sum_j X_j S_j(x)\Big) f(x,\lambda), \tag{12}$$

where X_j, the set of generators (3-5) and the graded Poisson brackets of the functions $S_j(x)$ reproduce the commutation relations (3-5) of the corresponding generators. The appropriate gauge transformation $f(x,\lambda) = g(x)\varphi(x,\lambda)$ reduces (12) to the auxiliary linear problem of NS(1,2)

$$-i\frac{d}{dx}\varphi(x,\lambda) = L(x,\lambda)\varphi(x,\lambda),$$

$$L(x,\lambda) = \lambda h + (\alpha)^{1/2}(X^+q + X^-q^+ + V_+\psi + V_-\psi^+). \tag{13}$$

The Poisson brackets of the L-operator coefficient functions follow from the relation [2, 3]

$$i\{\tilde{L}(x,\lambda)\overset{\approx}{\otimes} L(y,\nu)\} = [r(\lambda-\nu), L(x,\lambda) + L(y,\nu)]\,\delta(x-y). \quad (14)$$

The nonzero brackets are given by

$$\{q(x), q^{+}(y)\} = i\,\delta(x-y),$$
$$\{\psi(x), \psi^{+}(y)\} = i\,\delta(x-y) \quad (15)$$

There is no problem to obtain the Lax representation for the system (1,2). Instead of that we use the Hamiltonian approach and connect the NS with the special functionals of the L-operator (13). They are the coefficients of the regularized determinant asymptotic expansion

$$\ln a_1(\lambda) = -i\sum_{n=1}^{\infty} J_{n-1}\,\lambda^{-n}$$

where $a_1(\lambda)$ is a diagonal element of the transition matrix $T(\lambda)$

$$T(\lambda) = \lim_{\ell\to\infty} e^{-i\lambda h\ell}\, P\left(\exp i\int_{-\ell}^{\ell} L(x,\lambda)\,dx\right) e^{-i\lambda h\ell} \quad (16)$$

The coefficients J_k, $k=0,1,2$ can be found from the recurrence relations which follow from (13) and the substitution $\varphi^{t}(x,\lambda) = (1, d(x,\lambda), b(x,\lambda))\exp(i\lambda x/2 + z(x,\lambda))$ [8]. As a result of (14) the Poisson brackets of the functionals J_k vanish. The first three coefficients are given by

$$N = \int dx\,(2q^{+}(x)q(x) + \psi^{+}(x)\psi(x)), \quad (17)$$

$$P = \int dx\,\left(-iq^{+}\frac{\partial}{\partial x}q - i\psi^{+}\frac{\partial}{\partial x}\psi\right), \quad (18)$$

$$H = \int dx\,\big(q^{+}_x q_x + 2\psi^{+}_x\psi_x + \alpha q^{+}q^{+}qq + \alpha q^{+}q\psi^{+}\psi - i(\alpha)^{1/2}(q\psi^{+}\psi^{+}_x + q^{+}\psi\psi_x)\big) \quad (19)$$

The system (1,2) is the Hamiltonian one

$$\dot{q}(x) = \{H, q(x)\}, \quad \dot{\psi}(x) = \{H, \psi(x)\}$$

with the Poisson brackets (15) and the Hamiltonian (19).

Further investigation of NS by ISM in classical theory crucially depends on boundary conditions: 1. decreasing $q(x), \psi(x) \to 0$, $|x|\to\infty$; 2. $q(x)$, $\psi(x) \to \mathrm{const} \neq 0$, $|x|\to\infty$; 3. periodic $q(x+\ell) = q(x)$, $\psi(x+\ell) = \psi(x)$ and can be done in a standard way [8].

The study of the system (1.2) in quantum theory is not so straightforward. The operator-valued fields $q(x)$, $\psi(x)$ satisfy the commutation relations

$$[q(x), q^{+}(y)]_{-} = \delta(x-y), \quad [\psi(x), \psi^{+}(y)]_{+} = \delta(x-y) \quad (20)$$

The applicability of the quantum inverse scattering method (QISM) [1-3] depends on the existence of the quantum R-matrix which can be calculated from the generalized Yang-Baxter equation [3]

$$R(\lambda-\nu)\left(\tilde{\mathcal{L}}_\Delta(x,\lambda)+\tilde{\tilde{\mathcal{L}}}_\Delta(\nu)+i\,\tilde{\mathcal{L}}_\Delta(\lambda)\tilde{\tilde{\mathcal{L}}}_\Delta(\nu)\right)=$$

$$=\left(\tilde{\mathcal{L}}_\Delta(\lambda)+\tilde{\tilde{\mathcal{L}}}_\Delta(\nu)+i\,\tilde{\tilde{\mathcal{L}}}_\Delta(\nu)\tilde{\mathcal{L}}(\lambda)\right)R(\lambda-\nu) \tag{21}$$

where $\mathcal{L}_\Delta(\lambda)=\int_\Delta L(x,\lambda)\,dx$, $\Delta\sim 0$ and the accuracy of (21) is $O(\Delta)$. One can define $R(\lambda)$ via the classical r-matrix which can be written in the form

$$r(\lambda)=\frac{\alpha}{2\lambda}\left(\mathcal{P}-K\right)$$

where $\mathcal{P}$ is the graded permutation operator in $\mathbb{C}^{2,1}\otimes\mathbb{C}^{2,1}$ and K is the rank one projector: $K^2=K$. The quantum R-matrix has the same structure

$$R(\lambda)=\sigma_1(\lambda)I+\sigma_2(\lambda)\mathcal{P}-\sigma_3(\lambda)K \tag{22}$$

The coefficients $\sigma_j(\lambda)$ follow from the graded Yang-Baxter equation for $R(\lambda)$ [3]

$$\sigma_1(\lambda)=\lambda(\ell i\lambda+3\alpha/2),$$

$$\sigma_2(\lambda)=\alpha(\lambda-i3\alpha/4),$$

$$\sigma_3(\lambda)=\alpha\lambda$$

The quantum L-operator satisfying (21) differs from the classical one (13) by the quantum correction

$$L_q(x,\lambda)=L_{cl}(x,\lambda)+i\frac{\alpha}{4}(h^2-1/4) \tag{23}$$

Let us underline that the expressions (22), (23) are valid and the equation (21) is satisfied in the fundamental representation only. The quantum R-matrices and the L-operators depend upon the representation.

According to QISM we transform the local variables $q(x)$, $q^+(x)$, $\psi(x)$, $\psi^+(x)$ which are defined in the tensor product of the Fock spaces $\mathcal{H}=\mathcal{H}_q\otimes\mathcal{H}_\psi$ to the new one which are the entries of the monodromy matrix $T_\ell(\lambda)$:

$$T_\ell(\lambda)=T(\ell,\lambda)$$

$$-i\frac{d}{dx}T(x,\lambda)=:L_q(x,\lambda)T(x,\lambda):,\quad T(-\ell,\lambda)=I. \tag{24}$$

The monodromy matrix entries

$$T_\ell(\lambda)=\begin{pmatrix} A_1^{(\ell)}(\lambda) & B_1^{(\ell)}(\lambda) & B_3^{(\ell)}(\lambda)\\ C_1^{(\ell)}(\lambda) & A_2^{(\ell)}(\lambda) & B_2^{(\ell)}(\lambda)\\ C_3^{(\ell)}(\lambda) & C_2^{(\ell)}(\lambda) & A_3^{(\ell)}(\lambda)\end{pmatrix} \tag{25}$$

are operators in $\mathcal{H}$ with the commutation relations being defined by $R(\lambda)$ (22)

$$R(\lambda-\nu)\left(T_\ell(\lambda)\otimes T_\ell(\nu)\right)=\left(I\otimes T_\ell(\nu)\right)\left(T_\ell(\lambda)\otimes I\right)R(\lambda-\nu) \tag{26}$$

As a result of (26) the transfer matrix $t_\ell(\lambda)$ is a commuting operator family

$$t_\ell(\lambda)=\mathrm{str}\,T_\ell(\lambda),\qquad [t_\ell(\lambda),t_\ell(\nu)]=0. \tag{27}$$

The structure of the R-matrix (22) and the commutation relations (26) are similar to that of the supersymmetric sine-Gordon and anticommuting massive Thirring models [3]. Hence we do not use the algebraic Bethe ansatz but the analytic one [11]. The eigenstates of $t_\ell(\lambda)$ are parametrized by a set of complex numbers $\{\nu_j\}$ satisfying the equations

$$e^{i\nu_j \ell} = \prod_{k=1}^{n} S_2(\nu_j-\nu_k)\, S_{-1}(\nu_j-\nu_k) \tag{28}$$

where $S_m(\lambda) = (\lambda + i\alpha m/4)/(\lambda - i\alpha m/4)$. The corresponding eigenvalues are given by $(\gamma = i\alpha/\ell)$

$$\begin{aligned}\Lambda(\lambda) = e^{i\lambda\ell} \prod_k S_{-1}(\lambda-\nu_k - i3\alpha/4) - \\ - e^{i\gamma\ell}\prod_k S_1(\lambda-\nu_k - i\alpha/4)\, S_{-1}(\lambda-\nu_k - i\alpha/2) + \\ + e^{-i\lambda\ell}\prod_k S_1(\lambda-\nu_k).\end{aligned} \tag{29}$$

The eigenvalues of the charge (17), the momentum (18) and the Hamiltonian (19) are the following

$$n \quad , P = \tfrac{1}{2}\sum_{k=1}^{n} \nu_k \quad , \quad E = \tfrac{1}{2}\sum_{k=1}^{n} \nu_k^2$$

In the limit $\ell \to \infty$ we have to consider two different cases: 1. the finite number of excitations, when the ground state is the Fock vacuum $|0\rangle$: $\varphi(x)|0\rangle = \psi(x)|0\rangle = 0$; 2. the system with finite density $\rho = n/\ell$. $\ell \to \infty$, $n \to \infty$ when the ground state is described by the distribution of excitation momenta $\sigma(\lambda)$ which is a solution of an integral equation.

We summarize the characteristics of the system in the first case of a finite number of excitations. There are massive bosons with scattering matrix

$$S^{(\ell)}(\lambda) = S_3(\lambda)\, S_2(\lambda)\, S_{-1}(\lambda), \tag{30}$$

fermions with half of the boson mass and S-matrix

$$S^{(f)}(\lambda) = S_2(\lambda)\, S_{-1}(\lambda), \tag{31}$$

The S-matrix of the boson-fermion scattering is given by

$$S^{(\ell f)}(\lambda) = S_{5/2}(\lambda)\, S_{-1/2}(\lambda). \tag{32}$$

To conclude with let us point out that the generalizations of NS (1.2) associated with symmetric superspaces Osp(2m/2n)/U(m/n) can be obtained using the abovementioned approach and the classical and quantum R-matrices [10, 12].

It is worthwhile to mention that the results obtained can also be used to solve the quantum supersymmetric σ-models because in the pure bosonic case the integrable spin systems and the Bethe ansatz were used extensively [13,14]. The R-matrix (22) and the Bethe equations (28) are the starting point in the construction of such integrable graded spin systems for the SUSY σ models.

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