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Quantification of the Clustering Properties
of Nuclear States

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The amount of a particular type of clustering in a nuclear state is defined in this paper as the norm square of the projection of the wave function onto the particular cluster+model subspace. It is pointed out that, although the clusters cannot be localized in space by measurement, the amount of clustering characterizes the cluster formation in close analogy with a quantum mechanical probability. The cluster+model component of the wave function is proved to have a variational property. This facilitates the computation of the amount of clustering. The model dependence of the amounts of various clusterings and their relationship to the corresponding spectroscopic factors are studied via numerical examples for two models of ${}^6\text{Li}$. It is concluded that, in a relative sense, the spectroscopic factor, which is more directly related to experiment, is also characteristic of the clustering contents of different states of the same nucleus, but it cannot be used for comparisons between different nuclei or clusterings.

1. INTRODUCTION

Although the success of nuclear cluster models [1,2] is undisputed, the term "clustering" is still used in a variety of conflicting senses. This is just a reflection of the fact that antisymmetrization causes wave functions based on diametrically opposite philosophies to overlap and sometimes even makes them identical [1]. In this paper our objective is to outline a coherent theoretical framework for characterizing the clustering properties of nuclear states and thereby to give a quantitative definition for the amount of clustering. In doing so, we adopt the viewpoint of the fully developed cluster models as represented by the resonating-group method (RGM) or the generator-coordinate method (GCM). To explain our motivation, we now review some previous methods for characterizing the clustering properties of states.

In the harmonic-oscillator (h.o.) cluster model Neudatchin introduced the cluster separation parameter α , which is the ratio of the h.o. width of the cluster relative motion to that of the cluster internal motion [3]. For $\alpha=1$ the model state is identical to an h.o. shell-model state, thus α measures the departure, from a particular shell-model state, caused by clustering. Arima's terms "shell-model clustering" and "localized clustering" [4] distinguish the clustering due to $0\hbar\omega$ shell admixtures from that due to higher admixtures. Obviously, the use of such concepts is limited to particular models and states.

A most objective and general definition of N -nucleon clus-

tering may be formulated in terms of N -particle correlation functions [5]. Such a function takes on zero value when the wave function factorizes in single-particle (s.p.) coordinates and is large when the N -nucleon internal wave function factorizes from the rest. However, owing mainly to the Pauli principle, none of these pure cases does ever occur in reality. Antisymmetrization effects allow a substantial enhancement in the N -particle cluster correlation only for large cluster-core separations. This cluster correlation effect is still insignificant unless the wave function is large enough in this region, and that is so only for states lying above, or at least near, the cluster separation threshold [6]. These are very specific clustering effects, which are most aptly described by the cluster models. But there is no reason to restrict the term clustering to such effects since the scope of the cluster models is substantially wider. They are capable of describing not only surface but also bulk properties of nuclei [7], and may be applicable not only to excitations above the threshold but also to tightly bound ground states [7]. Indeed, many nuclear states are most aptly described as antisymmetrized products of cluster internal and relative wave functions. It is thus sensible to use the term clustering to signify the property of nuclear states that they contain such wave function components. With this definition adopted, a characterization of the clustering properties of nuclear states requires other quantities.

In this paper we confine our attention to clustering into

two clusters. We explicitly discuss only bound states. Since, however, resonant states can be described analogously [8], it is straightforward to generalize our considerations to resonant states as well. The type of a clustering is fully defined by specifying the cluster intrinsic states, the relative angular momentum and its coupling with the intrinsic angular momentum. For simplicity, we will mostly suppress the treatment of the angular momenta. To obtain a measure for a particular type of clustering, one has to define a pure two-cluster state or a subspace of such states to measure with. Then one can characterize the cluster content of any other state, whether or not it is given in terms of a cluster model, by projecting it onto this object. The familiar spectroscopic factor, which is most often used to quantify the clustering properties of states, is the integral of the modulus square of such projections. It is usual to identify the spectroscopic factor with the probability of clustering, and the related amplitude function with the probability amplitude of finding two clusters at a given relative distance [4]. This interpretation, however, has been disputed [9]. Fließbach has introduced a renormalized amplitude, and claims that this quantity carries the proper probability meaning [9].

The present paper attempts to clarify this matter. In Section 2 we shall see that none of these rival interpretations is entirely correct since, due to the Pauli principle, there is no way to localize clusters in a nucleus. For the intercluster relative motion an analogue of a s.p. wave function can only be defined in a pure two-cluster model. But for the interpretation of this function the analogy does not hold: this function cannot be identified with the probability amplitude of

finding the two clusters at a relative distance. It is still possible to define, through a projection, the weight of a cluster-model component in a wave function, and we shall call this quantity the amount of clustering. It will be shown to be equal to the norm square of Fliessbach's renormalized amplitude [9]. Its meaning will be elucidated by showing that the cluster-model component of a wave function satisfies a variational condition. This provides a convenient framework for the calculation of the amount of clustering, which is the subject of Section 3. To get a feeling for the relative behaviour of the amounts and spectroscopic factors of various clusterings, in Section 4 we give some numerical examples. We consider the $\alpha+d$, ${}^5\text{He}+p$ and $t+\pi$ clustering in two different cluster models of ${}^6\text{Li}$. These examples show that the spectroscopic factor can also be used to characterize any one particular clustering aspect of different states of the same nucleus. The significance of this finding will become clear in Section 5, where it is pointed out that the spectroscopic factor is more directly accessible to experimental investigation than the amount of clustering.

2. THE AMOUNT OF CLUSTERING

This section is devoted to the analysis of a nuclear wave function in terms of amplitudes belonging to a particular two-cluster component. As we shall see, all complications are caused

by the Pauli principle. To emphasize this fact, we present the antisymmetrized case after, in Subsection A, a non-antisymmetrized case is discussed. In Subsection B the amount of clustering in an antisymmetrized state is introduced as the expectation value of a projector. In Subsection C we give an equivalent definition in terms of a variational prescription and elaborate on the meaning of the notions introduced. Subsection D discusses the related question of intercluster relative wave functions.

A. Non-Antisymmetrized System

Let us consider two clusters of n_1, n_2 distinguishable constituents, whose (normalized) internal states are $\phi_1(\xi_1)$ and $\phi_2(\xi_2)$. Let the Hamiltonian of the system in the centre-of-mass (c.m.) frame be denoted by H . If the two clusters are fixed at a relative separation \underline{x} , the wave function is

$$\psi_{\underline{x}}^{\alpha}(\xi) = \phi_1(\xi_1) \phi_2(\xi_2) \delta(\underline{x} - \underline{x}_{12}) \equiv \Phi_{\alpha}(\xi) \delta(\underline{x} - \underline{x}_{12}), \quad (2.1)$$

where $\xi \equiv \{\xi, \underline{x}_{12}\} \equiv \{\xi_1, \xi_2, \underline{x}_{12}\}$ are the internal coordinates of the entire system; the wave function of the c.m. is assumed to be factored out and is dropped. The function $\psi_{\underline{x}}^{\alpha}$ is an eigenfunction of an observable \underline{x}_{12} , and therefore belongs to a set of orthogonal functions $\{\psi_{\underline{x}}^{\alpha}\}$:

$$\langle \psi_{\underline{x}}^{\alpha} | \psi_{\underline{x}'}^{\alpha'} \rangle = \delta(\underline{x} - \underline{x}'). \quad (2.2)$$

The probability of finding clusters $\underline{1}$ and $\underline{2}$ (in a unit volume) at a separation \underline{r} in any state Ψ ($\langle \Psi | \Psi \rangle = 1$) of the system is obviously

$$S_{\underline{r}}^{\alpha} = |\gamma_{\alpha}(\underline{r})|^2, \quad (2.3)$$

where the probability amplitude $\gamma_{\alpha}(\underline{r})$ is given by

$$\gamma_{\alpha}(\underline{r}) = \langle \psi_{\underline{r}}^{\alpha} | \Psi \rangle \equiv \langle \Phi_{\alpha}(\xi) \delta(\underline{r} - \underline{r}_{12}) | \Psi \rangle. \quad (2.4)$$

Because of (2.2), the total probability, or the (global) amount, of the clustering $\alpha = \underline{1} + \underline{2}$ in Ψ can be obtained as

$$S_{\alpha} = \int d\underline{r} S_{\underline{r}}^{\alpha} \equiv (\gamma_{\alpha} | \gamma_{\alpha}). \quad (2.5)$$

(By round bra-kets we denote matrix elements that involve integration over the "parameter coordinate" \underline{r} .)

If Ψ is a pure cluster state, i.e. $\Psi = \Phi_{\alpha}(\xi) \varphi_{\alpha}(\underline{r}_{12})$, then $\gamma_{\alpha}(\underline{r}) = \varphi_{\alpha}(\underline{r})$, and $\varphi_{\alpha}(\underline{r})$ is just an eigenfunction of the hermitian s.c. Hamiltonian $\langle \Phi_{\alpha} | H | \Phi_{\alpha} \rangle$:

$$\langle \Phi_{\alpha} | H | \Phi_{\alpha} \rangle \varphi_{\alpha} = E \varphi_{\alpha}, \quad (2.6)$$

i.e. it is a relative wave function, and $S_{\alpha} = 1$. If Ψ is a superposition of such states, $\Psi = \sum_{\alpha} \Phi_{\alpha} \varphi_{\alpha}$ with $\langle \Phi_{\alpha} | \Phi_{\beta} \rangle = \delta_{\alpha\beta}$, then still $\gamma_{\alpha}(\underline{r}) = \varphi_{\alpha}(\underline{r})$, but φ_{α} is no longer an eigenfunction

of $\langle \Phi_\alpha | H | \Phi_\alpha \rangle$. Thus, strictly speaking, the probability amplitude γ_α is no longer a s.p. wave function. One can, however, define a matrix Hamiltonian $H_{\alpha\beta} = \langle \Phi_\alpha | H | \Phi_\beta \rangle$, and then the column vector of the functions φ_α satisfies a Schrödinger-like equation:

$$\sum_{\beta} H_{\alpha\beta} \varphi_{\beta} = E \varphi_{\alpha} . \quad (2.7)$$

Thus the notion of a relative wave function survives only as a set of functions φ_α defined simultaneously for all (orthogonal) clusterizations. It shows the soundness of this generalization that the probabilities $S_\alpha = (\gamma_\alpha | \gamma_\alpha)$ add up to unity: $\sum_{\alpha} S_{\alpha} = 1$.

It should be noted, however, that the orthogonality of the clusterizations, $\langle \Phi_\alpha | \Phi_\beta \rangle = \delta_{\alpha\beta}$, is automatically satisfied only if all Φ_α contain states of the same pair of clusters. For Ψ being a superposition of non-orthogonal clusterizations, the probability amplitude, γ_α of (2.4), still exists, but is not related so simply to φ_α . As we shall see, in the antisymmetrized case the non-orthogonality hampers already the definition of a probability amplitude.

B. Aspects of Clustering in the Antisymmetrized Case

For a two-cluster system of indistinguishable particles (2.1) is not a legitimate state. To satisfy the Pauli principle, one should instead define

$$\Psi_T^{\alpha}(\zeta) = \mathcal{A}_{12} \{ \Phi_{\alpha}(\xi) \delta(\tau - \tau_{12}) \} = \mathcal{A}_{12} \{ \phi_1(\xi_1) \phi_2(\xi_2) \delta(\tau - \tau_{12}) \} , \quad (2.8)$$

where the intercluster antisymmetrizer \mathcal{A}_{12} contains a statistical factor:

$$\mathcal{A}_{12} = [n_1! n_2! / (n_1 + n_2)!]^{1/2} \sum_{\mathcal{P}} (-)^{\mathcal{P}} \mathcal{P}, \quad (2.9)$$

and ϕ_i are now assumed internally antisymmetrized. By not distinguishing protons from neutrons, we implicitly adopted the isospin formalism.

While each term of Ψ_{\pm}^{α} is an eigenfunction of a particular observable $\mathcal{P}_{\pm 12}$, belonging to the same eigenvalue \pm , there is no observable whose eigenfunction is Ψ_{\pm}^{α} with eigenvalue \pm . (Otherwise $\langle \Psi_{\pm}^{\alpha} | \Psi_{\pm'}^{\alpha} \rangle$ would be equal to $\delta(\pm - \pm')$, which is not the case; see (2.13) below.) Nevertheless, the overlap $\langle \Psi_{\pm}^{\alpha} | \Psi \rangle$, where Ψ is an antisymmetrized state, is proportional to a quantity $\langle \psi_{\pm}^{\alpha} | \Psi \rangle$, which looks like the probability amplitude of finding the system in an eigenstate ψ_{\pm}^{α} , with eigenvalue \pm , of the observable \mathcal{P}_{12} :

$$g_{\alpha}(\pm) \equiv \langle \Psi_{\pm}^{\alpha} | \Psi \rangle \quad (2.10a)$$

$$= \binom{n_1 + n_2}{n_1}^{1/2} \langle \psi_{\pm}^{\alpha} | \Psi \rangle, \quad (2.10b)$$

where in ψ_{\pm}^{α} of (2.1) ϕ_i are now antisymmetrized. The function $g_{\alpha}(\pm)$ is nothing but the reduced-width amplitude related to the conventional spectroscopic factor,

$$S_{\alpha} \equiv (g_{\alpha} | g_{\alpha}). \quad (2.11)$$

These considerations apparently confirm the conventional view [4] that $g_\alpha(\underline{r})$ is (proportional to) the probability amplitude of finding the two clusters displaced by \underline{r} relative to each other. Such a formulation, however, is always to be put between quotation marks since \underline{r}_{12} is not a physical [10] observable because it is not permutation invariant. Therefore it does not represent a measurable physical quantity [10].

The weight of the wave function component that lies in the cluster-model subspace can still be characterized quantitatively. The proper quantity for this purpose is the length square of the cluster-model component, which is equal to the expectation value of the projector P_α that projects onto the subspace spanned by $\{\Psi_{\underline{r}}^\alpha\}$. We thus define the amount of clustering α in the state Ψ as

$$S_\alpha = \langle \Psi | P_\alpha | \Psi \rangle. \quad (2.12)$$

The merit of S_α , in contrast to λ_α , is that it has the form of a quantum mechanical probability, so it has all the formal properties of a probability. In constructing P_α , due care is to be taken of the fact that the subscript \underline{r} of $\Psi_{\underline{r}}^\alpha$ is not an eigenvalue label, hence the overlap

$$A_{\alpha\alpha}(\underline{r}, \underline{r}') = \langle \Psi_{\underline{r}}^\alpha | \Psi_{\underline{r}'}^\alpha \rangle \quad (2.13)$$

is, in general, not equal to $\delta(\underline{r} - \underline{r}')$. In fact, $A_{\alpha\alpha}(\underline{r}, \underline{r}')$ is

nothing but the RGM overlap kernel of clusterization α , which is well-known to tend to $\delta(\underline{x}-\underline{x}')$ only with $|\underline{x}|, |\underline{x}'| \rightarrow \infty$. Because of this non-orthogonality, $P_\alpha \neq \int |\Psi_{\underline{x}}^\alpha\rangle d\underline{x} \langle \Psi_{\underline{x}'}^\alpha|$. (It is for this reason that s_α of (2.11) is not unity even in a pure two-cluster model.)

Expressed in terms of a non-orthogonal basis, the length square of a vector Ψ is the sum of the products of the contravariant and covariant components: $\|\Psi\|^2 = \sum_i \Gamma_i q_i$. The covariant components of Ψ in the basis $\{\Psi_{\underline{x}}^\alpha\}$ of the subspace of our interest are $q_\alpha(\underline{x})$, whereas the contravariant components are defined as

$$\Gamma_\alpha(\underline{x}) = \int d\underline{x}' A_{\alpha\alpha}^{-1}(\underline{x}, \underline{x}') q_\alpha(\underline{x}'), \quad (2.14)$$

where $A_{\alpha\alpha}^{-1}(\underline{x}, \underline{x}')$ is the kernel of the inverse of the integral operator $\hat{A}_{\alpha\alpha} = \int d\underline{x}' A_{\alpha\alpha}(\underline{x}, \underline{x}')$. (The circumflex is the distinctive sign on operators that act in the space of the parameter coordinate \underline{x} .) The inverse of $\hat{A}_{\alpha\alpha}$ is defined as $\hat{A}_{\alpha\alpha}^{-1} \hat{A}_{\alpha\alpha} = \hat{\Lambda}$, where $\hat{\Lambda}$ is the projector onto the subspace spanned by the eigenfunctions, of $\hat{A}_{\alpha\alpha}$, that belong to non-zero eigenvalues. The eigenstates with eigenvalue zero, excluded from $\hat{\Lambda}$, are called the redundant states. To satisfy the formula for the length square, one has to define P_α as

$$P_\alpha = \int d\underline{x} \int d\underline{x}' |\Psi_{\underline{x}}^\alpha\rangle A_{\alpha\alpha}^{-1}(\underline{x}, \underline{x}') \langle \Psi_{\underline{x}'}^\alpha|. \quad (2.15)$$

This agrees with Feshbach's definition of an antisymmetrized projector [11]. Substituting (2.15) into (2.12) and using

(2.10a) and (2.14), we indeed get

$$S_{\alpha} = (\Gamma_{\alpha} | g_{\alpha}), \quad (2.16)$$

which is just the length-square formula written for an innumerably infinite dimensional case.

One may find it convenient to introduce Fliessbach's re-normalized reduced-width amplitude [9]

$$G_{\alpha} \equiv \hat{A}_{\alpha\alpha}^{-\frac{1}{2}} g_{\alpha} = \hat{A}_{\alpha\alpha}^{\frac{1}{2}} \Gamma_{\alpha}. \quad (2.17)$$

Using this, we can rewrite S_{α} as

$$S_{\alpha} = (G_{\alpha} | G_{\alpha}), \quad (2.18)$$

which shows that the amount of clustering is equal to Fliessbach's "new spectroscopic factor" [9].

To conclude this subsection, it is due to note the following. The definition (2.12) of the amount of clustering is just the formula for the quantum mechanical probability of finding the system in the subspace defined by the projector P_{α} . This formula, however, carries this meaning only if the projection is onto a space spanned by a subset of the eigenvectors of an observable [10]. Since $\{\Psi_{\alpha}^{\alpha}\}$ is defined in a different way, a probability interpretation of S_{α} seems to require an extension of the notion of the quantum mechanical probability. We shall

confirm this conjecture in Subsection D.

C. Variational Property

The $\alpha = \underline{1+2}$ clustering content of a state Ψ may also be characterized via constructing the function $\Gamma_\alpha^1(\underline{x})$ that, inserted in the pure two-cluster state

$$\Psi_\alpha = \int d\underline{x} \Gamma_\alpha^1(\underline{x}) \Psi_\alpha^\alpha, \quad (2.19)$$

produces the best approximation to Ψ . The best approximant Ψ_α may be defined as that which minimizes the norm of $\Psi - \Psi_\alpha$:

$$\delta \langle \Psi - \Psi_\alpha | \Psi - \Psi_\alpha \rangle = 0, \quad (2.20a)$$

$$\delta^2 \langle \Psi - \Psi_\alpha | \Psi - \Psi_\alpha \rangle > 0, \quad (2.20b)$$

and the clustering may be characterized by

$$S_\alpha^1 = 1 - \langle \Psi - \Psi_\alpha | \Psi - \Psi_\alpha \rangle, \quad (2.21)$$

with Ψ_α being the optimal Ψ_α . The quantity S_α^1 is appropriate for this purpose since $0 \leq S_\alpha^1 \leq 1$, and the equalities hold for Ψ orthogonal and equal (almost everywhere) to Ψ_α , respectively. (To verify $0 \leq S_\alpha^1$ is non-trivial at this stage, but it is a trivial consequence of Eq. (2.28) to be derived below.)

From (2.20a) it follows that

$$\langle \delta \Psi_\alpha | \Psi - \Psi_\alpha \rangle \equiv \int d\tau \delta \Gamma_\alpha'(\tau) \langle \Psi_\tau^\alpha | \Psi - \Psi_\alpha \rangle = 0, \quad (2.22)$$

and, whenever this is satisfied, (2.20b) also holds:

$$\delta^2 \langle \Psi - \Psi_\alpha | \Psi - \Psi_\alpha \rangle = 2 \langle \delta \Psi_\alpha | \delta \Psi_\alpha \rangle > 0. \quad (2.23)$$

Thus all stationary points (2.20a) are minima. Eq. (2.22) is fulfilled for any $\delta \Gamma_\alpha'$ if and only if

$$\langle \Psi_\tau^\alpha | \Psi - \Psi_\alpha \rangle = 0. \quad (2.24)$$

With Ψ_α of (2.19) inserted and the definitions in (2.13) and (2.10a) used, this may be written as

$$\int d\tau' A_{\alpha\alpha}(\tau, \tau') \Gamma_\alpha'(\tau') = g_\alpha(\tau). \quad (2.25)$$

By inverting the integral operator and comparing the result with (2.14), we immediately see that

$$\Gamma_\alpha'(\tau) = \Gamma_\alpha(\tau). \quad (2.26)$$

(One can verify by direct substitution that, if Γ_α' is a solution of (2.25), then so is $\Gamma_\alpha' + \sum_i c_i \varphi_i^R$, where φ_i^R are the redundant states. Since the inverse operator $\hat{A}_{\alpha\alpha}^{-1}$ is defined on the $\hat{\Lambda}$ subspace, in (2.26) we excluded such solutions.

This is convenient since such solutions would yield the same

many-body state Ψ_α , and, consequently, the same value for S'_α .)

The quantity S'_α can be reduced as follows. Eq. (2.24) implies that

$$\langle \Psi - \Psi_\alpha | \Psi_\alpha \rangle = 0, \quad (2.27)$$

and, consequently,

$$S'_\alpha = 1 - \langle \Psi | \Psi \rangle + \langle \Psi_\alpha | \Psi \rangle = \langle \Psi_\alpha | \Psi \rangle. \quad (2.28)$$

Using (2.19) and (2.26) and then the definition of g_α in (2.10a), and comparing the result with (2.16), we obtain

$$S'_\alpha = \int d\mathbf{r} \Gamma_\alpha^*(\mathbf{r}) \langle \Psi_\alpha^a | \Psi \rangle = (\Gamma_\alpha | g_\alpha) = S_\alpha. \quad (2.29)$$

Thus the "variational principle" (2.20) results in a definition, of the amount of clustering, that is equivalent to the first definition, Eq. (2.12). The second definition, formulated in momentum space, was introduced in a special model in our earlier work [12].

The variational property of the amount of clustering sheds some light on its physical meaning. From (2.27) we have learned that the cluster component Ψ_α is defined so that the rest of Ψ be orthogonal to it. Eq. (2.28) shows that S_α is just the projection of the (non-normalized) best cluster-model state Ψ_α onto Ψ . To visualize these formulae, a two-dimensional model is shown in Fig. 1. The states are represented by vectors. The

vector Ψ is well-defined, and (2.19) specifies the angle of Ψ_α with respect to Ψ . The length of Ψ_α is set by condition (2.20), which requires that the distance between the end-points of Ψ and Ψ_α be minimum. In accord with (2.27), this is equivalent to requiring that $\Psi - \Psi_\alpha \perp \Psi_\alpha$. This shows that the best two-cluster state Ψ_α is just the projection of Ψ onto $\{\Psi_\alpha^\alpha\}$, which enters into the definition of S_α in (2.12). According to (2.12), the amount of clustering S_α is the length square of this basis vector. In two dimensions the equivalence of (2.12) and (2.21) follows from the Pythagorean theorem. (Note that the length of Ψ is unity.) It is also easy to see that the length square of $\Psi_\alpha = P_\alpha \Psi$, Eq. (2.12), is equal to the length of its projection onto Ψ , Eq. (2.28). Indeed, the length of Ψ_α is $\cos \epsilon$, and its square is just as well $\cos^2 \epsilon$ as the double projection $\langle \Psi_\alpha | \Psi \rangle$.

The variational principle offers a convenient starting point for the calculation of S_α . Such an application will be shown in Section 3.

D. Intercluster Relative Wave Function and Probability Interpretation

For a hypothetical system of distinguishable particles, in (2.4) we constructed the probability amplitude of finding the clusters at displacement \underline{r} . For identical particles, in (2.12) we defined the global amount of clustering without reference to such a quantity. However, the clustering is often quantified

locally as well, in terms of an intercluster "relative wave function". It is therefore important and pertinent to our subject to clarify in what sense this phrasing is allowable. The present subsection is concerned with this problem.

Let us first consider a pure two-cluster model. The model space of this problem is spanned by a basis $\{\Psi_{\alpha}^{\sigma}\}$ with elements defined in (2.8). In this space the Schrödinger equation $H\Psi_{\alpha} = E\Psi_{\alpha}$ takes the form

$$\int d\tau' (\tau | \hat{H}_{\alpha\alpha} | \tau') \Gamma_{\alpha}(\tau') = E g_{\alpha}(\tau), \quad (2.30)$$

with $(\tau | \hat{H}_{\alpha\alpha} | \tau') \equiv \langle \Psi_{\alpha}^{\sigma} | H | \Psi_{\alpha}^{\sigma} \rangle$. In keeping with (2.14), we assume that the Γ_{α} in (2.30) contains no redundant component. After multiplication by $\hat{A}_{\alpha\alpha}^{-\frac{1}{2}}$ and insertion of $1 = \hat{A}_{\alpha\alpha}^{-\frac{1}{2}} \hat{A}_{\alpha\alpha}^{\frac{1}{2}}$ + $\sum_i |\varphi_i^{\alpha}\rangle \langle \varphi_i^{\alpha}|$, Eq. (2.30) may be written as

$$\int d\tau' (\tau | \hat{\mathcal{H}}_{\alpha\alpha} | \tau') G_{\alpha}(\tau') = E G_{\alpha}(\tau) \quad (2.31a)$$

or

$$\hat{\mathcal{H}}_{\alpha\alpha} G_{\alpha} = E G_{\alpha}, \quad (2.31b)$$

where

$$\hat{\mathcal{H}}_{\alpha\alpha} = \hat{A}_{\alpha\alpha}^{-\frac{1}{2}} \hat{H}_{\alpha\alpha} \hat{A}_{\alpha\alpha}^{-\frac{1}{2}}, \quad (2.32)$$

and G_{α} is defined in (2.17). Eq. (2.31) is reminiscent of a s.p. Schrödinger equation in the space of the parameter coordinate τ for two reasons. First, $\hat{\mathcal{H}}_{\alpha\alpha}$ is hermitean [11] and can often be approximated very well by the sum of the relative

kinetic energy and a local potential [13]. Second, the orthonormality of two states of the same intrinsic cluster structure, $\langle \Psi_\alpha^i | \Psi_\alpha^j \rangle = \delta_{ij}$, implies $(G_\alpha^i | G_\alpha^j) = \delta_{ij}$. This analogy is important because it makes it possible to introduce s.p. cluster models [14,2a]. To emphasize the analogy, it is natural to call $G_\alpha(t)$ a relative wave function.

As we saw in Section A, the notion of the relative wave function can in some sense be extended to a superposition of orthogonal clusterizations. However, when the wave function $\Psi = \sum_\alpha \Psi_\alpha$ is antisymmetrical, the various clusterizations $\Psi_\alpha = \int d\tau \varphi_\alpha(\tau) \Psi_\tau^\alpha$ cease to be orthogonal. None the less, one may still recover a formal analogy with (2.31)-(2.32) by defining a matrix \hat{A} of the integral operators $\hat{A}_{\alpha\beta}$ whose kernels are $A_{\alpha\beta}(t, t') = \langle \Psi_\tau^\alpha | \Psi_\tau^\beta \rangle$. It is then straightforward to verify the following equation:

$$\hat{\mathcal{H}} \underline{G} = E \underline{G} \quad (2.33a)$$

or

$$\sum_\beta \hat{\mathcal{H}}_{\alpha\beta} G_\beta = E G_\alpha, \quad (2.33b)$$

where $\hat{\mathcal{H}}$ and \underline{G} stand, respectively, for the square matrix of $\hat{\mathcal{H}}_{\alpha\beta}$ and the column vector of G_α , defined as

$$\hat{\mathcal{H}}_{\alpha\beta} = \sum_{\tau\tau'} \hat{A}_{\alpha\tau}^{-\frac{1}{2}} \hat{H}_{\tau\tau'} \hat{A}_{\tau'\beta}^{-\frac{1}{2}}, \quad (2.34)$$

$$G_\alpha = \sum_\beta \hat{A}_{\alpha\beta}^{-\frac{1}{2}} g_\beta = \sum_\beta \hat{A}_{\alpha\beta}^{\frac{1}{2}} \varphi_\beta, \quad (2.35)$$

with

$$(\underline{t} | \hat{H}_{\gamma\delta} | \underline{t}') = \langle \Psi_{\underline{t}}^{\gamma} | H | \Psi_{\underline{t}'}^{\delta} \rangle, \quad (2.36)$$

and the matrix square root is defined as $\hat{A}_{\alpha\beta} = \sum_{\gamma} \hat{A}_{\alpha\gamma}^{\frac{1}{2}} \hat{A}_{\gamma\beta}^{\frac{1}{2}}$. The matrix Hamiltonian $\hat{\mathcal{H}}$ is hermitean, and hence is amenable to an approximation with a matrix of local potentials [15]; moreover, $(\underline{G} | \underline{G}) = \langle \Psi | \Psi \rangle (=1)$. Nevertheless, in spite of this analogy, it is not meaningful to interpret any $G_{\alpha}(\underline{t})$ as an intercluster relative wave function since each of them consists of a superposition of all clusterizations $\{ \Psi_{\underline{t}}^{\beta} \}$. Therefore, we prefer to use the term relative wave function only in conjunction with a pure two-cluster model, and now we return to this case.

Based on the formal similarity of $G_{\alpha}(\underline{t})$ to a wave function, Fliessbach and others [9,16] interpret it as the probability amplitude of finding two clusters separated by \underline{t} . (In fact, although $G_{\alpha}(\underline{t})$ has such properties only in a pure two-cluster model, Fliessbach does not seem to restrict this interpretation to this case [9].) Let us now examine whether this interpretation can be substantiated.

In subsection A we did not exclude the existence of a proper intercluster observable. We thus have to attempt to construct the observable that corresponds to the probability amplitude $G_{\alpha}(\underline{t})$. To be a probability amplitude of an observable in a state Ψ_{α} , $G_{\alpha}(\underline{t})$ should be expressible in the form $G_{\alpha}(\underline{t}) = \langle E_{\underline{t}}^{\alpha} | \Psi_{\alpha} \rangle$, where $E_{\underline{t}}^{\alpha}$ is the eigenfunction of the observable, with eigenvalue \underline{t} . Using the definitions (2.17) and

(2.10a), one can obtain this form with

$$|\Xi_{\underline{x}}^{\alpha}\rangle = \hat{A}_{\alpha\alpha}^{-\frac{1}{2}} |\Psi_{\underline{x}}^{\alpha}\rangle \quad (2.37)$$

Since $\langle \Xi_{\underline{x}}^{\alpha} | \Xi_{\underline{x}'}^{\alpha} \rangle = \delta(\underline{x} - \underline{x}')$, one can construct the operator

$$\hat{X}_{\alpha} = \int d\underline{x} |\Xi_{\underline{x}}^{\alpha}\rangle \langle \Xi_{\underline{x}}^{\alpha}|, \quad (2.38)$$

which has the property $\hat{X}_{\alpha} |\Xi_{\underline{x}}^{\alpha}\rangle = \underline{x} |\Xi_{\underline{x}}^{\alpha}\rangle$. This is an operator that acts in the genuine physical state space and not in the artificial parameter-coordinate space. It is permutation invariant and hermitean, but, alas, its eigenfunctions span only the subspace $\{|\Psi_{\underline{x}}^{\alpha}\rangle\}$, i.e. they do not form a complete set in the space of the physical states. The operator \hat{X}_{α} is thus not an observable. On the other hand, its meaning is implausible, and there seems to be no compelling reason to identify it with an operator for the intercluster separation. The definition of \hat{X}_{α} may be extended to the whole space by arbitrarily adding further orthogonal terms to it, but this would further impede its interpretation. On these accounts, $G_{\alpha}(\underline{x})$ cannot be viewed as a probability amplitude of finding the two clusters at \underline{x} , not even in some restricted sense. No support for this interpretation can be gained from a reformulation fashioned after Eq. (2.12) either; the projection operator belonging to $|G_{\alpha}(\underline{x})|^2$ is $P_{\underline{x}}^{\alpha} = |\Xi_{\underline{x}}^{\alpha}\rangle \langle \Xi_{\underline{x}}^{\alpha}|$, and this operator, of course, has no more to do with the intercluster separation than \hat{X}_{α} itself.

Although these considerations do not lead to a sound interpretation of $G_\alpha(t)$, they further elucidate its formal properties. With the use of \tilde{E}_2^α , Eq. (2.31a) may be rewritten as

$$\int d\tau' \langle \tilde{E}_2^\alpha | H | \tilde{E}_2^\alpha \rangle \langle \tilde{E}_2^\alpha | \Psi_\alpha \rangle = E \langle \tilde{E}_2^\alpha | \Psi_\alpha \rangle. \quad (2.39)$$

From this form it is clear that similar equations derive from $\langle \Psi_\alpha | H | \Psi_\alpha \rangle$ for any arbitrary ortho-normal system $\{\tilde{E}_2^\alpha\}$ that spans $\{\Psi_2^\alpha\}$, and both the hermiticity of the hamiltonian matrix and the ortho-normality $(\tilde{G}_\alpha^i | \tilde{G}_\alpha^j) = \delta_{ij}$ of $\tilde{G}_\alpha^k(t) = \langle \tilde{E}_2^\alpha | \Psi_\alpha \rangle$ ($k = i, j$) hold for any such set $\{\tilde{E}_2^\alpha\}$. The different \tilde{G}_α are related to each other (and, of course to G_α) as different representations of a s.p. state vector.

Having failed to find a genuine intercluster observable, we have failed to find support for the probability interpretation of S_α , too. We are thus compelled to accept that both $|G_\alpha(t)|^2$ and S_α are "probabilities" of "finding" the system in subspaces (in E_2^α and in $\{\Psi_2^\alpha\} = \{\tilde{E}_2^\alpha\}$, respectively) that are unidentifiable by measurements in the strict quantum mechanical sense. Nevertheless, the difference between the two applications of the term "probability" is still considerable. While the individual states \tilde{E}_2^α have no distinguished role or plausible meaning, the subspace $\{\Psi_2^\alpha\}$ does have some. This subspace is the entity whose choice specifies the type of a particular cluster model and the object that is probed by cluster reactions with the mediation of model assumptions for the reac-

tion mechanism. For this reason, we deem the use of the term "probability" excusable for the amount of clustering, but misleading for the modulus square of the relative wave function.

If the probability interpretation of G_α were to be taken seriously, it should be expected that G_α enters into the evaluation of matrix elements. The relevant matrix elements are those which contain operators Q that can be decomposed into c.m., intercluster and intracluster terms

$$Q = Q_{c.m.} + Q_{i2} + (Q_1 + Q_2). \quad (2.40)$$

As was shown by Lomnitz-Adler [17], the matrix elements of such an operator between pure two-cluster states Ψ_α and Ψ'_α can be uniquely reduced to terms that correspond to those of (2.40). A probability interpretation of G_α would suggest that the intercluster term is just $\langle G_\alpha | Q_{i2} | G'_\alpha \rangle$, i.e.

$$\langle \Psi_\alpha | Q | \Psi'_\alpha \rangle = \langle G_\alpha | Q_{i2} | G'_\alpha \rangle, \quad (2.41)$$

as is indeed assumed in Ref. [16]. But the exact result does not meet this expectation [17,19]. The intercluster term is reduced to (2.41) only if G_α and G'_α are eigenfunctions, of Q_{i2} , that belong to the same eigenvalue.

The matrix elements of asymmetrical structure are also of interest. One can obtain such matrix elements by decomposing the matrix element of $Q - Q_{c.m.}$ between a pure two-cluster

state $\Psi_\alpha = \mathcal{A}_{12} \{ \Gamma_\alpha \Phi_\alpha \}$ and any other state Ψ' :

$$\langle \Psi_\alpha | \underline{Q} - \underline{Q}_{c.m.} | \Psi' \rangle = \langle \Phi_\alpha \Gamma_\alpha | \mathcal{A}_{12} (\underline{Q} - \underline{Q}_{c.m.}) | \Psi' \rangle \quad (2.42a)$$

$$= \binom{n_1 + n_2}{n_1}^{\frac{1}{2}} \langle \Phi_\alpha \Gamma_\alpha | \underline{Q} - \underline{Q}_{c.m.} | \Psi' \rangle \quad (2.42b)$$

$$= \binom{n_1 + n_2}{n_1}^{\frac{1}{2}} \langle \Phi_\alpha \Gamma_\alpha | \underline{Q}_1 + \underline{Q}_2 | \Psi' \rangle + \binom{n_1 + n_2}{n_1}^{\frac{1}{2}} \langle \Phi_\alpha \Gamma_\alpha | \underline{Q}_{12} | \Psi' \rangle. \quad (2.42c)$$

Here we used $\mathcal{A}_{12} (\underline{Q} - \underline{Q}_{c.m.}) = (\underline{Q} - \underline{Q}_{c.m.}) \mathcal{A}_{12}$ and $\mathcal{A}_{12}^2 \sim \mathcal{A}_{12}$. The matrix element of our concern is the second term of (2.42c) since such asymmetrical matrix elements often enter into reaction calculations [19]. Using (2.10a), we can cast it into the form

$$\begin{aligned} \binom{n_1 + n_2}{n_1}^{\frac{1}{2}} \langle \Phi_\alpha \Gamma_\alpha | \underline{Q}_{12} | \Psi' \rangle &= \binom{n_1 + n_2}{n_1}^{\frac{1}{2}} \int d\pm \Gamma_\alpha^*(\pm) \hat{Q}_{12} \langle \Phi_\alpha \delta(\pm - \pm_{12}) | \Psi' \rangle \\ &= (\Gamma_\alpha | \hat{Q}_{12} | g'_\alpha). \end{aligned} \quad (2.43)$$

where \hat{Q}_{12} , unlike \underline{Q}_{12} , acts in the parameter-coordinate space. Eq. (2.43) is another example for a reduction that does not involve the relative wave function. The covariant-contravariant pair can only be replaced by a pair of G functions at the expense of introducing the non-hermitean operator

$$\hat{Q}_{12} = \hat{A}_{\alpha\alpha}^{-\frac{1}{2}} \underline{Q}_{12} \hat{A}_{\alpha\alpha}^{\frac{1}{2}}, \quad (2.44)$$

which yields

$$\binom{n_1+n_2}{n_1}^{\frac{1}{2}} \langle \Phi_\alpha \Gamma_\alpha | Q_{12} | \Psi' \rangle = (G_\alpha | \hat{O}_{12} | G'_\alpha). \quad (2.45)$$

Recalling the discussion above, one should not be surprised to find that the reductions of many-body matrix elements are not consistent with the probability interpretation of $G_\alpha(\pm)$. It is another matter that Eq. (2.41) may be a good approximation to the exact formulae. For a state Ψ_α whose relative-motion part is dominated by one eigenfunction of $\hat{A}_{\alpha\alpha}$, the functions g_α , G_α and Γ_α are nearly proportional to each other. If $\Psi' \approx \Psi'_\alpha$ is also dominated by one eigenfunction of $\hat{A}_{\alpha\alpha}$, and this belongs to the same eigenvalue, then $\Gamma_\alpha/G_\alpha \approx G'_\alpha/g'_\alpha$, which renders the approximation (2.41) good. Such is the case for two-cluster states that belong to the same "cluster band", i.e. to the same value of $N=2n+\ell$, where n is the node number and ℓ is the orbital momentum of the intercluster relative motion. For some matrix elements between members of the ground-state cluster band of $^{20}\text{Ne} = ^{16}\text{O} + \alpha$, the approximation (2.41) agrees with the exact result within a few per cent [17].

3. CALCULATION OF THE AMOUNT OF CLUSTERING

A. General Considerations

The formulae (2.16) and (2.18) for the amount of clustering contain the various amplitudes g_α , G_α and Γ_α . Once any

of them is known and the eigenvalue problem of the single-channel RGM overlap operator $\hat{A}_{\alpha\alpha}$ has been solved [2b], the other functions may be determined by using (2.17).

Except for the pure two-cluster models, however, in which S_{α} is trivially unity, none of the major microscopic structure models provides any of the functions g_{α} , G_{α} , Γ_{α} directly. In particular, Γ_{α} is equal to the RGM relative wave function (apart from redundant components in the latter) only in a pure two-cluster model: $\Psi_{\alpha} = \int d\tau \Gamma_{\alpha}(\tau) \Psi_{\tau}^{\alpha}$. (This formula can be verified by substituting (2.10a), applied to $\Psi = \Psi_{\alpha}$, into (2.14) and using (2.13).) The orthogonality-condition model (OCM) [2a] does not give a direct access to any of these functions either. Albeit in the pure two-cluster case the basic equation of the OCM is an approximation to (2.31) thus its solution is G_{α} , the general OCM equation is an approximation to (2.33) thus its solution is the set of \mathcal{G}_{α} and not G_{α} . Therefore, to calculate any of the functions G_{α} , one needs to know, in addition to all \mathcal{G}_{α} , the spectral decomposition of the full RGM overlap operator \hat{A} . The solution of the eigenvalue problem of the RGM overlap operators requires highly specialized techniques [2b,20], which we will not discuss here.

In general, from among the three functions g_{α} , G_{α} , Γ_{α} , the covariant component $g_{\alpha}(\tau)$ is the most amenable to computation since it is given by an explicit formula, Eq. (2.10a). The construction of $g_{\alpha}(\tau)$ in the shell model is given e.g.

in [21], that in the RGM and GCM is discussed e.g. in [2b,22, 23,12]. Its determination in the OCM is dealt with in [24].

Once $g_\alpha(t)$ is known, Eq. (2.14) or (2.25) may be used to compute $\Gamma_\alpha(t)$. The form given in (2.14) is useful only if $\hat{A}_{\alpha\alpha}^{-1}$ is known, i.e. if the eigenvalue problem of $\hat{A}_{\alpha\alpha}$ has been solved. Otherwise the inversion of the integral operator just amounts to solving the integral equation (2.25). It may be more straightforward to solve (2.25) directly than to solve the eigenvalue problem and then to use (2.14). If the GCM formalism is used, the GCM overlap kernel has to be transformed into the RGM kernel. This may be done through a two-step Fourier transformation [2b]. The first step of this transformation yields the RGM kernel in momentum representation. One can avoid performing the second step by writing (2.25) in momentum representation, which is as straightforward to work with as the coordinate representation. This is the method used in [12]; since it is adequately discussed there, we do not enlarge upon it any more.

Rather, here we discuss, with some detail, an entirely different alternative method, which avoids calculating any of g_α , G_α and Γ_α , and reduces the computation of S_α to matrix algebraic manipulations.

B. Generator-Coordinate Formalism

In this subsection we implement the variational method of Section 2.C for a nucleus described as a superposition of vari-

ous two-cluster states in the GCM. The specific feature of this method compared with our earlier approach [12] is that now we express not only the nuclear wave function, but also the pure two-cluster basis state $\Psi_{\underline{z}}^{\alpha}$, involved in (2.19), in the GCM framework.

The model wave function is assumed to consist of a superposition of various clusterizations α :

$$\Psi = \sum_{\alpha} \sum_{\underline{z}} F_{\alpha}(\underline{z}_i) \Phi_{\alpha}(\underline{z}_i), \quad (3.1)$$

with \underline{z}_i standing for the discretized values of the generator coordinates (GC's). As is usual, the GC's \underline{z} are the distances between the centres of the h.o. wells generating the cluster states; the discrete values \underline{z}_i are chosen, for simplicity, the same for all clusterizations. The generating function $\Phi_{\alpha}(\underline{z})$ for any clusterization $\alpha = \underline{1} + \underline{2}$ has the form

$$\bar{\Phi}_{\alpha}(\underline{z}) = \mathcal{N}_{\underline{1}\underline{2}} \{ \phi_{\underline{1}} \phi_{\underline{2}} \varphi_{\underline{3}}^{\alpha}(\underline{z}_{\underline{1}\underline{2}}) \}, \quad (3.2)$$

where

$$\varphi_{\underline{3}}^{\alpha}(\underline{z}_{\underline{1}\underline{2}}) = \mathcal{N} \int d\hat{\Omega} Y_{\underline{l}\underline{2}}(\hat{\Omega}) \exp\left\{-\frac{1}{2} \frac{n_{\underline{1}}\beta_{\underline{1}}n_{\underline{2}}\beta_{\underline{2}}}{n_{\underline{1}}\beta_{\underline{1}}+n_{\underline{2}}\beta_{\underline{2}}} (\underline{z} - \underline{z}_{\underline{1}\underline{2}})^2\right\}, \quad (3.3)$$

with \underline{l} being the relative orbital momentum of $\underline{1}$ and $\underline{2}$, $\beta_i = m\omega_i/\hbar$ the h.o. parameter for a nucleon in cluster i and \mathcal{N} a normalization factor. (Here the circumflex denotes a solid angle.) For simplicity, we have not written the angular momentum coupling between the cluster internal and interclus-

ter angular momenta explicitly. The overlaps between the various generating functions $\Phi_\alpha(s)$ constitute the GCM overlap matrix

$$N_{\alpha\beta}(s, s') = \langle \Phi_\alpha(s) | \Phi_\beta(s') \rangle. \quad (3.4)$$

The wave function (3.1) may be considered an approximation to the RGM wave function

$$\Psi = \sum_\alpha \int d\tau u_\alpha(\tau) Y_{\ell\ell}(\hat{\tau}) \Psi_\tau^\alpha. \quad (3.5)$$

By inserting (3.2) into (3.1) and (2.8) into (3.5), we see that the approximation consists in the discretization

$$u_\alpha(\tau) = \sum_i F_\alpha(s_i) \int d\hat{\tau} Y_{\ell\ell}^*(\hat{\tau}) \varphi_{s_i}^\alpha(\boldsymbol{x}). \quad (3.6)$$

The alternative definition, Eqs. (2.21), (2.28) for the amount of clustering requires an explicit formula for the pure two-cluster function Ψ_α , too. Eq. (2.19) shows that this function has the form of an RGM function (3.5) specialized to a single clusterization α . It is therefore plausible to assume that a good approximation to S_α may be obtained by applying the same discretization to function Ψ_α as that used for Ψ in (3.1):

$$\Psi_\alpha = \sum_i f_\alpha(s_i) \Phi_\alpha(s_i), \quad (3.7)$$

with the same values of the GC's as in (3.1).

With the use of the trial function (3.7), the variational procedure of Section 2.C can be repeated, to yield, in accord with (2.24), $\langle \Phi_\alpha(\alpha_i) | \Psi - \Psi_\alpha \rangle = 0$. Using (3.1), (3.7) and (3.4), we may write this equation for each value of i as

$$\sum_j N_{\alpha\alpha}(\alpha_i, \alpha_j) f_\alpha(\alpha_j) = \sum_p \sum_j N_{\alpha p}(\alpha_i, \alpha_j) F_p(\alpha_j). \quad (3.8)$$

In keeping with (2.28), for the amount of clustering we obtain $S_\alpha = \langle \Psi_\alpha | \Psi \rangle$. Substituting (3.7) and (3.1) into this formula, and using (3.4), we obtain

$$S_\alpha = \sum_f \sum_{k,l} f_\alpha^*(\alpha_k) N_{\alpha f}(\alpha_k, \alpha_l) F_f(\alpha_l). \quad (3.9)$$

The computation of S_α thus requires the solution of the set of linear equations (3.8) to obtain $f_\alpha(\alpha_j)$, which is then to be substituted into (3.9). Having carried through this procedure formally, we can write

$$S_\alpha = \sum_p \sum_{ijkl} F_p^*(\alpha_i) N_{p\alpha}(\alpha_i, \alpha_j) N_{\alpha\alpha}^{-1}(\alpha_j, \alpha_k) N_{\alpha p}(\alpha_k, \alpha_l) F_p(\alpha_l), \quad (3.10)$$

where we used the fact that N is a hermitean matrix.

It is seen that the amount of clustering α draws contributions from the other clusterization components as well. For the special case of a pure two-cluster model, the equality $S_\alpha = 1$ should hold, which offers a possibility to check the approximation in (3.7). Writing (3.10) for this special case,

and using (3.4) and $\langle \Psi | \Psi \rangle = 1$, we obtain

$$\begin{aligned}
 S_{\alpha} &= \sum_{i,j} F_{\alpha}^{*}(s_i) N_{\alpha\alpha}(s_i, s_j) F_{\alpha}(s_j) \\
 &= \langle \sum_i F_{\alpha}(s_i) \Phi_{\alpha}(s_i) | \sum_j F_{\alpha}(s_j) \Phi_{\alpha}(s_j) \rangle = \langle \Psi | \Psi \rangle = 1.
 \end{aligned}
 \tag{3.11}$$

Thus, for a pure two-cluster model, (3.7) does not imply any approximation.

4. APPLICATION TO ${}^6\text{Li}$

To get a feeling for the behaviour of the amount of clustering S , we computed it for the clusterings into the simplest (4+2)-, (5+1)- and (3+3)-nucleon clusters in the ${}^6\text{Li}$ nucleus described by two GCM models. For the sake of comparison, we also computed the corresponding spectroscopic factors λ . In Subsection A we briefly outline these models, and in Subsection B we present the results.

A. The Models

The GCM models underlying the calculations to be presented are described in detail in Ref. [12], and here we recapitulate only the main physical ingredients.

One of the models is a pure $\alpha+d$ cluster model, whereas the other includes, in addition, a 5+1 nucleon clusterization. Since the Coulomb force is neglected, this may as well be considered

a ${}^5\text{He}+p$ as a ${}^5\text{Li}+n$ component. Similarly, the values of S and λ for the $5+1$ decomposition bear these alternative meanings. We shall always refer to these components and quantities as " ${}^5\text{He}+p$ ".

The wave function Ψ has the form of (3.1)-(3.3). In each clusterization the nucleons are assumed to populate the lowest level(s) of cluster h.o. wells of the same parameter $\beta=0.48 \text{ fm}^{-2}$. We consider the two lowest-lying states of ${}^6\text{Li}$. The nucleon spins are assumed to add up to 1 in both states. In the ground state (g.s.) of $J^\pi=1^+$ the orbital angular momenta are assumed to couple to 0, while in the excited state (e.s), which is a triplet of $J^\pi=1^+, 2^+, 3^+$ in reality, they are assumed to add up to 2. In the $\alpha+d$ clusterization this total orbital momentum comes, of course, entirely from the intercluster motion, whereas in the ${}^5\text{He}+p$ clusterization it results from the p-wave neutron in ${}^5\text{He}$ coupled with the relative orbital momentum, restricted to be $l=1$.

This form of the model wave function is consistent with a central nucleon-nucleon interaction. We performed the calculations with two of the Volkov forces [25] (V1 and V2) as well as with Brink's force [26] (B).

The resulting binding, excitation and separation energies are tabulated in Table I. The models have been shown to reproduce the experimental spectroscopic factors and reduced-width amplitudes reasonably [12]. The predictions of the models for other g.s. and transition properties also appear

realistic [27]. The inclusion of the ${}^5\text{He}+p$ clusterization gives rise to a definite though not too large improvement in the reproduction of the data.

B. Results for the Clustering Properties

We calculated the amounts of clustering and spectroscopic factors for clusters that have the same h.o. parameters, the same orbital momenta and carry the same total nucleon spins as is assumed for the model states of ${}^6\text{Li}$ itself. The amounts of clustering and the spectroscopic factors have been calculated with the use of Eqs. (3.10) and (2.11), respectively. Eq. (3.10) has been checked against the exact formula (2.16) for the case of the $\alpha+d$ clustering. They were found to agree up to six digits, which is in the range of the numerical accuracy. The results for the $\alpha+d$, ${}^5\text{He}+p$ and $t+\pi$ clustering are presented in Tables II, III and IV, respectively. In discussing these results we shall make observations about the effects of the function spaces and forces included in the models and about the behaviour of the amount of clustering and the spectroscopic factor relative to each other.

In Table II we should first notice that in the pure $\alpha+d$ model the amount of $\alpha+d$ clustering is unity as it should be, and the $\alpha+d$ spectroscopic factor is also rather close to unity. The ${}^5\text{He}+p$ admixture hardly reduces S and s , which indicates that it absorbs a very little part of the wave function from the original function space. This tendency is quite uniform

for both states and all the three forces.

Table III shows that the pure $\alpha+d$ model already contains about 50 % of ${}^5\text{He}+p$ clustering, and, for the g.s. at least, the explicit inclusion of the ${}^5\text{He}+p$ clusterization increases the presence of this component to a surprisingly small extent. It is, however, interesting that the admixture has a more pronounced effect on the e.s., which, in the pure $\alpha+d$ model, contains less of the ${}^5\text{He}+p$ component. In the ${}^5\text{He}+p$ clustering property there is a marked difference between the Volkov forces and the Brink force. Table I shows that this is not just a separation-energy effect.

Although off-hand one would tend to think that the $t+\pi$ clustering has a much larger overlap with the $\alpha+d$ clustering than ${}^5\text{He}+p$, Table IV shows that this is not so. In fact, in the pure $\alpha+d$ model there is about as much of the $t+\pi$ clustering as ${}^5\text{He}+p$ clustering. What is conspicuous is that there is hardly any change in the $t+\pi$ clustering due to the ${}^5\text{He}+p$ admixture, especially in the g.s. This shows that there is very little overlap between those components of the ${}^5\text{He}+p$ and $t+\pi$ clusterizations which are not contained in the $\alpha+d$ subspace. The Brink force, again, gives substantially lower values than the other two although in the separation energies its predictions are not very different.

Coming to more general observations, we should first remark that the ${}^5\text{He}+p$ component causes little changes in the clustering properties of the g.s., but somewhat more in the

e.s. Another tendency common to the ${}^5\text{He}+p$ and $t+r$ clusterings is that the difference between the cluster contents of the two states is reduced by the inclusion of the ${}^5\text{He}+p$ clusterization.

As to the behaviour of S and s relative to each other, we should make three observations. First, the relationship between the two quantities for the two states is very much the same. From this we conclude that it is meaningful to compare any particular clustering contents of different states of the same nucleus in terms of the spectroscopic factor as well. Second, the changes in S and s caused by the ${}^5\text{He}+p$ admixture are strictly parallel. Thus the clustering contents of different models can also be compared in terms of spectroscopic factors as well. Third, for different clusterings of the same state, the amount of clustering and the spectroscopic factor behave in an uncorrelated manner relative to each other. E.g., for the $u+d$ and ${}^5\text{He}+p$ clusterings $s > S$, whereas for the $t+r$ clustering $s < S$. Thus, the spectroscopic factors are not suitable for comparing different clustering contents of a state. As a fourth remark, we should also add that the s value for a pure two-cluster state tends to decrease drastically for heavier nuclei [2]. It thus follows that the spectroscopic factor is not appropriate for comparing the clustering contents of states of different nuclei either.

To see the relationship of S and s more explicitly, let us rewrite (2.11) and (2.16) as

$$s_\alpha = \sum_n |(g_\alpha | \psi_n)|^2, \quad (4.1)$$

$$S_\alpha = (g_\alpha | \hat{A}_{\alpha\alpha}^{-1} | g_\alpha) = \sum_n a_n^{-1} |(g_\alpha | \psi_n)|^2, \quad (4.2)$$

where ψ_n and a_n are the eigenfunctions and eigenvalues of $\hat{A}_{\alpha\alpha}$, respectively, and in (4.2) the relationship $\Gamma_\alpha = \hat{A}_{\alpha\alpha}^{-1} g_\alpha$ (cf. (2.14)) is also used. In the examples we are aware of, for eigenvalues a_n that belong to the same ℓ either all $a_n < 1$ or all $a_n > 1$ (see e.g. [28]), and we assume that this is a general rule. Eqs. (4.1) and (4.2) show that in the former case $S_\alpha > s_\alpha$, whereas in the latter case $S_\alpha < s_\alpha$. Thus whether $s_\alpha > S_\alpha$ or vice versa depends solely on $\hat{A}_{\alpha\alpha}$, i.e. on the type and relative quantum numbers of the clustering but not on the state considered. The eigenvalues a_n are known for $\alpha+d$ and $t+\tau$ [29], and are consistent with the values of S/s found in our calculations. We note, however, that, for a pair of two light clusters, the eigenvalues of $\hat{A}_{\alpha\alpha}$ depend considerably on the parity of the relative motion [29]. If the h.o. parameters of the two clusters are different, the eigenvalues also depend (slightly [18]) on the ℓ value. These facts may limit the applicability of the spectroscopic factors for comparing clusterings of the same type but of different relative quantum numbers.

5. SUMMARY AND CONCLUSION

We have defined the amount S_α of a particular clustering α

in a state Ψ as the length square of the projection of Ψ onto the subspace of the cluster states concerned. We have expressed S_α in terms of the covariant and contravariant components g_α and Γ_α as well as in terms of a function G_α , which, in a pure two-cluster model, plays the role of a relative wave function: $S_\alpha = (\Gamma_\alpha | g_\alpha) = (G_\alpha | G_\alpha)$. The amount of clustering is truly characteristic of the clustering content of Ψ , and it is reasonable to liken it to a probability. Notwithstanding that no similar statement can be made about the spectroscopic factor $s_\alpha = (g_\alpha | g_\alpha)$, still this quantity is also useful as a relative measure of the clustering contents of states of the same nucleus. An interpretation of $g_\alpha(\mathbf{r})$ (or $\Gamma_\alpha(\mathbf{r})$) as a local probability amplitude of finding two clusters displaced by \mathbf{r} is ruled out by formal reasons. The relative wave function $G_\alpha(\mathbf{r})$, on the contrary, does have the formal properties of a probability amplitude. We showed, however, that the operator that $G_\alpha(\mathbf{r})$, as a probability amplitude, should belong to is not an observable, and it would be arbitrary to consider it an intercluster separation operator.

When a many-body matrix element can be separated into terms that depend only on the internal and relative motion, respectively, the relative-motion term usually does not reduce to a matrix element of the relative wave functions G . For a type of matrix elements that is often involved in direct reaction theories, the relative-motion term contains the covariant-contravariant pair rather than G . In the case of disintegra-

tion processes, like α decay, or of capture reactions it may be reasonable, depending on kinematical conditions, to approximate this matrix element by one involving G functions [19,30].

However, for reactions that involve nucleons other than those of the decomposed system, such an approximation is not reasonable. E.g., the transition amplitude of a stripping reaction $A(a,b)B$ ($a=b+x$, $B=A+x$) in the distorted wave Born approximation may be reduced as follows [30]:

$$T = \langle \mathcal{A}_{Bb} \{ \chi_{Bb}^{(-)} \phi_B \phi_b (H - H_{Bb}) \} | \mathcal{A}_{Aa} \{ \phi_A \phi_a \chi_{Aa}^{(+)} \} \rangle \quad (5.1)$$

$$\approx \langle \chi_{Bb}^{(-)} g_{B(Ax)} | U_{bx} | g_{a(bx)} \chi_{Aa}^{(+)} \rangle, \quad (5.2)$$

where H_{Bb} is an effective Hamiltonian that describes the $B+b$ elastic scattering and $U_{bx} = U_{bx}(r_{bx})$ is an effective interaction. Eq. (5.2) contains two covariant components, g , and two contravariant ones, χ . The "distorted waves" χ belong to high energies, where the corresponding operators \hat{A} are expected to have much smaller effect than in the bound-state problems of a and B , which are represented by the two g functions. Therefore, a replacement of both χ and g with the respective relative wave functions cannot compensate for each other. It looks more likely that the usual approximation of the contravariant components χ with solutions of local-potential problems, which is tantamount to replacing them by relative wave functions, makes little difference though this

is yet to be checked, too [23]. But the g functions are very different from the corresponding relative wave functions, and thus the former cannot be substituted by the latter [30].

From these considerations we conclude that it is not the amount of clustering, but the conventional spectroscopic factor that enters into direct reaction formulae [12,30]. It is justifiable to approximate these formulae by ones that contain the amount of clustering only in exceptional cases. We have thus to acquiesce in that the amount of clustering cannot be "measured" directly. But the spectroscopic factor is also a uniquely defined quantity, and, to facilitate comparisons with experiment, it is this quantity that is to be calculated. We just have to keep in mind that it does not have the precise meaning that is usually attributed to it. It is for this reason that, for comparing various models, the amount of clustering is a much more useful tool.

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Table I

Binding, Excitation and Separation Energies in MeV

Force Model	V1		V2		B		Experiment
	$\alpha+d$	mixed	$\alpha+d$	mixed	$\alpha+d$	mixed	
E_{Li} (binding)	25.91	26.54	27.46	28.08	25.71	26.26	31.99
E_{Li}^* (exc.)	3.54	3.25	3.60	3.27	3.55	3.35	3.6
$E_{Li} - E_{\alpha} - E_d$	1.16	1.78	1.68	2.29	0.55	1.10	1.48
$E_{Li} - E_{He}$	1.60	2.23	2.07	2.68	2.32	2.87	4.59
$E_{Li} - E_t - E_r$	13.22	13.85	13.89	14.51	12.68	13.23	15.80

Table II

Amount S and Spectroscopic Factor \mathcal{S} of the $\alpha+d$ Clustering in the Pure $\alpha+d$ and in the Mixed Model for the Ground State (g.s.) and First Excited State (e.s.) of ${}^6\text{Li}$ with the V1, V2 and B Force

State	Model	V1		V2		B	
		S	\mathcal{S}	S	\mathcal{S}	S	\mathcal{S}
g.s.	$\alpha+d$	1	1.071	1	1.073	1	1.059
	mixed	0.967	1.038	0.969	1.041	0.972	1.033
e.s.	$\alpha+d$	1	1.065	1	1.069	1	1.046
	mixed	0.940	1.010	0.942	1.013	0.952	1.006

Table III

Amount S and Spectroscopic Factor s of the ${}^5\text{He}+p$ Clustering in the Pure $\alpha+d$ and in the Mixed Model for the Ground State (g.s.) and First Excited State (e.s.) of ${}^6\text{Li}$ with the V1, V2 and B Force

State	Model	V1		V2		B	
		S	s	S	s	S	s
g.s.	$\alpha+d$	0.517	0.610	0.534	0.631	0.420	0.494
	mixed	0.601	0.694	0.612	0.708	0.509	0.584
e.s.	$\alpha+d$	0.467	0.563	0.495	0.597	0.310	0.376
	mixed	0.614	0.715	0.631	0.736	0.474	0.551

Table IV

Amount S and Spectroscopic Factor λ of the $t+t$ Clustering in the Pure $\alpha+t$ and in the Mixed Model for the Ground State (g.s.) and First Excited State (e.s.) of ${}^6\text{Li}$ with the V1, V2 and B Force

State	Model	V1		V2		B	
		S	λ	S	λ	S	λ
g.s.	$\alpha+t$	0.521	0.469	0.539	0.485	0.425	0.384
	mixed	0.502	0.470	0.535	0.482	0.437	0.395
e.s.	$\alpha+t$	0.484	0.435	0.513	0.461	0.328	0.296
	mixed	0.518	0.466	0.537	0.482	0.394	0.356

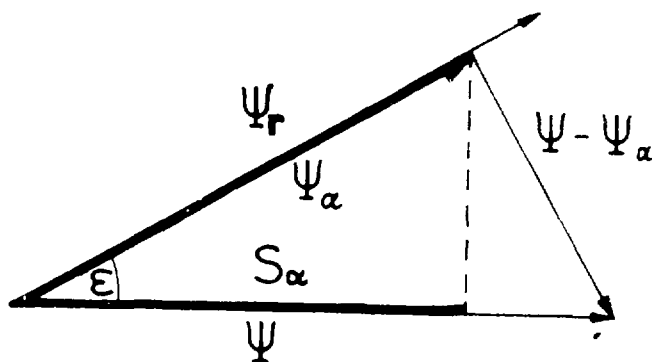


Fig. 1. Geometrical model for the visualization of a state Ψ , its best cluster approximant Ψ_α and the amount S_α of clustering α .

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