IC/85/84

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

A SUPER SOLITON CONNECTION

M. Gürses

and

Ö. Oğuz

INTERNATIONAL
ATOMIC ENERGY
AGENCY

UNITED NATIONS
EDUCATIONAL
SCIENTIFIC
AND CULTURAL
ORGANIZATION

1985 MIRAMARE-TRIESTE

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

A SUPER SOLITON CONNECTION *

M. Gürses
TUBITAK, Research Institute for Basic Sciences,
P.O.Box 74 Gebze, Kocaeli, Turkey

and

Ö. Oğuz **
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

Integrable super non-linear classical partial differential equations are considered. A super sl(2,R) algebra valued connection 1-form is constructed. It is shown that curvature 2-form of this super connection vanishes by virtue of the integrable super equations of motion. A super extension of the AKNS scheme is presented and a class of super extension of the Lax hieararchy and super non-linear Schrödinger equation are found. O(N) extension and the Bäcklund transformations of the above super equations are also considered.

MIRAMARE - TRIESTE
July 1985

* To be submitted for publication.

** Permanent address: Department of Physics, University of Istanbul, Istanbul, Turkey.

I. INTRODUCTION

The inverse scattering method enables us to solve the initial value problem of non-linear evolution equations. Intensive development of this method has been devoted for solving non-linear partial differential equations (n.l.p.d.e.) such as Korteweg de Vries(KdV), modified KdV, Sine-Gordon and non-linear Schrödinger equations and the others [1]. Ablowitz, Kaub, Newell and Segur (AKNS) [2] have proposed a scheme which contains a large class of integrable n.l.p.d.e. including the above equations. For this reason there have been considerable efforts to put n.l.p.d.e. arising from mathematics and physics into AKNS scheme. A nice property of the AKNS equations is that they define a connection which has values in sl(2,R) algebra [3]. Its curvature vanishes by virtue of the n.l.p.d.e. under consideration. The Bäcklund transformations when supplemented by reduction problem [4], are nothing but the guage transformations in this formulation.

Recently there is an increasing interest in the supersymmetric model theories where anticommuting fields of the Grassmann algebra are equally treated with the commuting fields. The quantum and classical integrability of the super extension of the standard integrable systems have been studied by several authors [5,6,7,8,9]. Quite recently a new classically integrable super equation was presented by Kupershmidt [10,11]. This is a super extension of the KdV equation.

In order to have a compact formulation for the super integrable systems such as the AKNS scheme which handles certain classes of n.l.p.d.e. we have, very recently, proposed a super extension of the AKNS scheme [12]. A class of super integrable equations containing the super extension of the Lax hierarchy is found.

In present work we shall first give a short survey of the AKNS scheme and of its geometrical interpretation. In Section III we define a super sl(2,R) valued connection 1-form. Then the zero curvature condition leads to a new class of super integrable non-linear evolution equations associated with the super AKNS scheme. An integrable class containing the super non-linear Schrodinger equation is obtained. The super extension of the Lax hierarchy is given in section IV. The section V is devoted to O(N) extension of the super AKNS scheme. In particular, for an illustration, we examine O(2) extended super integrable non-linear evolution equations. In conclusion, the Bäcklund transformations of these super equations are considered.

II. THE AKNS SCHEME AND THE SOLITON CONNECTION

Ablowitz et.al. have proposed the Lax pair $\mathcal{L}$ and $\mathcal{A}$ as

$$\mathcal{L} = \begin{pmatrix} i\partial_x & -iq(t,x) \\ ir(t,x) & -i\partial_x \end{pmatrix} \tag{2.1}$$

$$\mathcal{A} = \begin{pmatrix} A(t,x,\lambda) & B(t,x,\lambda) \\ C(t,x,\lambda) & -A(t,x,\lambda) \end{pmatrix} \tag{2.2}$$

where functions A,B,C depend on t and x through their dependence on the potentials q and r, λ is the spectral parameter, and the wave function Ψ is a column vector with components ψ_1 and ψ_2. The linear eigenvalue equations, $\mathcal{L}\Psi = \lambda\Psi$, and associated evolution equations, $\Psi_t = \mathcal{A}\Psi$, are respectively given by

$$\psi_{1,x} = -i\lambda\psi_1 + q\psi_2 , \tag{2.3a}$$

$$\psi_{2,x} = i\lambda\psi_2 + r\psi_1 , \tag{2.3b}$$

$$\psi_{1,t} = A\psi_1 + B\psi_2 , \tag{2.4a}$$

$$\psi_{2,t} = C\psi_1 - A\psi_2 . \tag{2.4b}$$

The integrability of the above equations leads to the Lax equation, $\mathcal{L}_t = [\mathcal{L}, \mathcal{A}]$, or,

$$A_x + rB - qC = 0 , \tag{2.5a}$$

$$B_x + 2i\lambda B - q_t + 2Aq = 0 , \tag{2.5b}$$

$$C_x - 2i\lambda C - r_t - 2Ar = 0 \tag{2.5c}$$

which are called AKNS equations. The corresponding n.l.p.d.e. are obtained by expanding A,B and C as a power series of λ in eqs.(2.5) and by comparing the coefficients of λ^n.

Since the operators $\mathcal{L}$ and $\mathcal{A}$ are traceless it is possible to define a connection 1-form which has the values in the sl(2,R) algebra. Namely the sl(2,R) valued connection 1-form, Ω, can be expressed by

$$\Omega = e_i\theta_i = \begin{pmatrix} \theta_0 & \theta_1 \\ \theta_2 & -\theta_0 \end{pmatrix} \tag{2.6}$$

where e_i (i = 0,1,2) are the generators of the sl(2,R) algebra and 1-forms θ_i are

$$\theta_0 = A\,dt - i\lambda\,dx , \tag{2.7a}$$

$$\theta_1 = C\,dt + r\,dx , \tag{2.7b}$$

$$\theta_2 = B\,dt + q\,dx . \tag{2.7c}$$

Then the linear eigenvalue and the time evolution equations (2,3) and (2,4) can be unified in the form

$$d\Psi = -\Omega\Psi \tag{2.8}$$

where d is the exterior derivative. The integrability of this equation that the curvature implies that the curvature 2-form associated to this connection vanishes [3],

$$d\Omega + \Omega\wedge\Omega = 0 \tag{2.9}$$

i.e. Ω is a flat connection and $\wedge$ denotes exterior product. Hence the basic problem is AKNS scheme turns out to be a search for a flat connection associated with n.l.p.d.e. in this formulation.

III. A SUPER-sl (2,R) VALUED SOLITON CONNECTION

In order to get a super soliton connection we first embed sl(2,R) algebra into a super algebra which we call it the super-sl(2,R) algebra. A 3x3 matrix representation of the super-sl(2,R) algebra is defined by three bosonic e_i (i=0,1,2) and two fermionic q_a (a = 1,2) generators.

They are given by

$$e_0=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\quad e_1=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\quad e_2=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},$$
$$q_1=\begin{pmatrix}0&0&1\\0&0&0\\0&-1&0\end{pmatrix},\quad q_2=\begin{pmatrix}0&0&0\\0&0&1\\1&0&0\end{pmatrix}. \tag{3.1}$$

The commutation, [,], and anticommutation, { , }, relations are

$$\begin{array}{lll}
[e_0,e_1]=2e_1 & [e_0,e_2]=-2e_2 & [e_1,e_2]=e_0\\
[e_0,q_1]=q_1 & [e_0,q_2]=-q_2 & [e_1,q_1]=0\\
[e_1,q_2]=q_1 & [e_2,q_1]=q_2 & [e_2,q_2]=0\\
\{q_1,q_1\}=-2e_1 & \{q_2,q_2\}=2e_2 & \{q_1,q_2\}=e_0
\end{array} \tag{3.2}$$

and the modified Jacobi identities are satisfied. A connection 1-form which has values in this algebra is defined by

$$\Omega = e_i\theta_i + q_a\Pi_a = \begin{pmatrix}\theta_0 & \theta_1 & \Pi_1\\ \theta_2 & -\theta_0 & \Pi_2\\ \Pi_2 & -\Pi_1 & 0\end{pmatrix} \tag{3.3}$$

We call Ω as the super soliton connection. Following the parametrization of the sl(2,R) soliton connection, θ_i and π_a are given by

$$\begin{array}{ll}
\theta_0 = A(t,x,\lambda)\,dt - i\lambda\,dx & \Pi_1 = \alpha(t,x,\lambda)\,dt + \beta(t,x)\,dx\\
\theta_1 = C(t,x,\lambda)\,dt + r(t,x)\,dx & \Pi_2 = \rho(t,x,\lambda)\,dt + \epsilon(t,x)\,dx\\
\theta_2 = B(t,x,\lambda)\,dt + q(t,x)\,dx &
\end{array} \tag{3.4}$$

Here A,B,C are assumed to be commuting, α and ρ are anticommuting super AKNS functions while q and r are commuting, ϵ and β are anticommuting super AKNS potentials.

The linear eigenvalue and the evolution equations for the super Jost function Ψ are contained in the following exterior differential equation.

$$d\Psi + \Omega\Psi = 0 \tag{3.5}$$

where $\Psi^T = (\psi_1, \psi_2, \phi)$. Here ψ_1 and ψ_2 are commuting functions ϕ is an anticommuting function. The integrability of eq. (3.5) implies zero curvature condition

$$d\Omega + \Omega\wedge\Omega = 0 \tag{3.6}$$

i.e. Ω is a flat connection 1-form.

A class of integrable super n.l.p.d.e. can be found by allowing only the positive power series

$$A = \sum_{m=0}^{M} a_m\lambda^m,\quad B = \sum_{m=0}^{M} b_m\lambda^m,\quad C = \sum_{m=0}^{M} c_m\lambda^m$$
$$\alpha = \sum_{m=0}^{M} \alpha_m\lambda^m,\quad \rho = \sum_{m=0}^{M} \rho_m\lambda^m \tag{3.7}$$

substituting these in eq. (3.6) we obtain the non-linear evolution equations,

$$q_t = b_{0,x} - 2\rho_0\epsilon + 2a_0 q, \tag{3.8a}$$
$$r_t = c_{0,x} + 2\alpha_0\beta - 2a_0 r, \tag{3.8b}$$
$$\beta_t = \alpha_{0,x} - a_0\beta + \rho_0 r - \epsilon c_0, \tag{3.8c}$$
$$\epsilon_t = \rho_{0,x} - \beta b_0 + q\alpha_0 + \epsilon a_0 \tag{3.8d}$$

and the recursion equations,

$$a_{m,x} - \alpha_m\epsilon + \rho_m\beta - c_m q + b_m r = 0, \quad m \geq 0, \tag{3.9a}$$
$$b_{m,x} + 2i\,b_{m-1} - 2\rho_m\epsilon + 2a_m q = 0, \quad m \geq 1, \tag{3.9b}$$
$$c_{m,x} - 2i\,c_{m-1} + 2\alpha_m\beta - 2a_m r = 0, \quad m \geq 1. \tag{3.9c}$$
$$\alpha_{m,x} - i\alpha_{m-1} + \rho_m r - \beta a_m - \epsilon c_m = 0, \quad m \geq 1, \tag{3.9d}$$
$$\rho_{m,x} + i\rho_{m-1} - \beta b_m + q\alpha_m + \epsilon a_m = 0, \quad m \geq 1. \tag{3.9e}$$

Letting a_M = constant and $b_M = c_M = \alpha_M = \rho_M = 0$ we obtain a subclass of super integrable n.l.p.d.e. As an example, let us take M = 3, then the evolution eqs.(3.8) take the following form

$$q_t = a_2\left[-\frac{1}{2}q_{xx} + q^2 r + 2\epsilon_x\epsilon + 2q\beta\epsilon\right] + ia_3\left[-\frac{1}{4}q_{xxx} + \right.$$
$$\left. + \frac{3}{2}rqq_x + 3(\epsilon_x\epsilon)_x - 3q\beta_x + 3q\beta\epsilon_x\right], \quad (3.10a)$$

$$r_t = a_2\left[\frac{1}{2}r_{xx} - qr^2 + 2\beta_x\beta - 2r\beta\epsilon\right] + ia_3\left[-\frac{1}{4}r_{xxx} + \right.$$
$$\left. + \frac{3}{2}qrr_x - 3(\beta_x\beta)_x + 3r\beta_x\epsilon - 3r\beta\epsilon_x\right], \quad (3.10b)$$

$$\epsilon_t = a_2\left[-\epsilon_{xx} + q\beta_x + \frac{1}{2}\beta q_x + \frac{1}{2}qr\epsilon\right] + ia_3\left[-\epsilon_{xxx} + \right.$$
$$\left. + \frac{3}{4}rq_x\epsilon + \frac{3}{4}qr_x\epsilon + \frac{3}{2}qr\epsilon_x + \frac{3}{2}q_x\beta_x + \frac{3}{4}\beta q_{xx}\right], \quad (3.10c)$$

$$\beta_t = a_2\left[\beta_{xx} - r\epsilon_x - \frac{1}{2}\epsilon r_x - \frac{1}{2}qr\beta\right] + ia_3\left[-\beta_{xxx} + \right.$$
$$\left. + \frac{3}{4}rq_x\beta + \frac{3}{4}qr_x\beta + \frac{3}{2}qr\beta_x + \frac{3}{2}r_x\epsilon_x + \frac{3}{4}\epsilon r_{xx}\right] \quad (3.10d)$$

where a_2 and a_3 are commuting constants. As special cases we list

i) The AKNS coupled n.l.p.d.e. [2] : $\epsilon = \beta = 0$

ii) The super KdV : $r = r_0$(constant), $\beta = 0$, $a_2 = 0$, $ia_3 = 4$

$$q_t = -q_{xxx} + 6r_0 qq_x - 12(\epsilon\epsilon_x)_x , \quad (3.11a)$$

$$\epsilon_t = -4\epsilon_{xxx} + 6r_0 q\epsilon_x + 3r_0 q_x\epsilon . \quad (3.11b)$$

This system was given by Kupershmidt [10].

iii) The super modified KdV : $r = k_1\bar{q}$, $\beta = k_2\bar{\epsilon}$, $a_2 = 0$, $ia_3 = 4$
k_1 = real constant ($k_1 = \bar{k}_1$).

$$q_t = -q_{xxx} + 6k_1|q|^2 q_x + 12(\epsilon_x\epsilon)_x - 12k_2 q\bar{\epsilon}_x\epsilon + 12k_2 q\bar{\epsilon}\epsilon_x , \quad (3.12a)$$

$$\epsilon_t = -4\epsilon_{xxx} + 3k_1|q|^2_x\epsilon + 6k_1|q|^2\epsilon_x + 6k_2 q_x\bar{\epsilon}_x + 3k_2\bar{\epsilon}q_{xx} . \quad (3.12b)$$

This extension of the modified KdV is different from the one given by Kupershmidt [10].

iv) The super non-linear Schrödinger equation : $r = k_1\bar{q}$, $\beta = k_2\bar{\epsilon}$, $a_3 = 0$,
$a_2 = -2i$, k_1 = real constant ($k_1 = \bar{k}_1$).

$$q_t = -2i\left[-\frac{1}{2}q_{xx} + k_1|q|^2 q + 2\epsilon_x\epsilon + 2k_2 q\bar{\epsilon}\epsilon\right] , \quad (3.13a)$$

$$\epsilon_t = -2i\left[-\epsilon_{xx} + \frac{1}{2}k_1|q|^2\epsilon + k_2 q\bar{\epsilon}_x + \frac{1}{2}k_2 q_x\bar{\epsilon}\right] . \quad (3.13b)$$

A bar over a quantity denotes the Berezin adjoint operation in the Grassmann algebra [13]. For a usual complex valued functions this operation reduces to complex conjugation. In the cases (iii) and (iv) one finds that $k_1 = k_2^2$. Hence the signs of the self interacting terms in the bosonic parts are fixed by the super extension.

IV. A SUPER LAX HIERARCHY

We also obtain a super extension of the Lax hierarchy [14] by allowing $r = r_0$ (a real constant) and $\beta = 0$ and having M as an arbitrary nonnegative integer. Then the evolution equations are given by

$$q_t = 2K_{M,x} , \qquad \epsilon_t = \Sigma_M \quad (4.1)$$

where K_M and Σ_M satisfy the following recursion relations

$$K_M = r_0 L_q K_{M-1,x} - L_\epsilon \Sigma_{M-1} , \quad (4.2a)$$

$$\Sigma_M = H_q \Sigma_{M-1} + r_0 H_\epsilon K_{M-1} \quad (4.2b)$$

and the operators L_q, L_ϵ, H_q and H_ϵ are defined by

$$L_q = -\frac{1}{4r_0}\partial_{xx} + q + \frac{1}{2}q_x\int^x \cdot\, dx , \quad (4.3a)$$

$$L_\epsilon = \frac{3}{2}\epsilon\partial_x + \frac{1}{2}\epsilon_x , \quad (4.3b)$$

$$H_q = -\partial_{xx} + qr_0 , \qquad H_\epsilon = \epsilon_x + \frac{3}{2}\epsilon\partial_x . \quad (4.3c)$$

Assuming $K_{-1} = \frac{4}{r_0}$ and $\Sigma_{-1} = 0$ as initial data, the first three members for $M = 0,1,2$ are respectively given by

$$K_0 = 4q \ , \qquad \Sigma_0 = 4\epsilon_x \ , \tag{4.4a}$$

$$K_1 = -\frac{1}{2} q_{xx} + \frac{3}{2} r_0 q^2 - 6\epsilon\epsilon_x \ , \tag{4.4b}$$

$$\Sigma_1 = -4\epsilon_{xxx} + 6 r_0 q \epsilon_x + 3 r_0 \epsilon q_x \ , \tag{4.4c}$$

$$K_2 = \frac{1}{8} q_{4x} - \frac{5}{4} r_0 q q_{xx} - \frac{5}{8} r_0 q_x^2 - 15 r_0 q \epsilon\epsilon_x + \frac{5}{4} r_0^2 q^3 + \frac{15}{2} r_0 \epsilon\epsilon_{xxx} - \frac{5}{2}\epsilon_x\epsilon_{xx} \ , \tag{4.4d}$$

$$\Sigma_2 = 4\epsilon_{5x} - \frac{25}{2} r_0 q_{xx}\epsilon_x - 15 r_0 q_x \epsilon_{xx} - 10 r_0 q \epsilon_{xxx} - \frac{15}{4} r_0 \epsilon q_{xxx} + \frac{15}{2} r_0^2 q^2 \epsilon_x + \frac{15}{2} r_0^2 q q_x \epsilon \ . \tag{4.4e}$$

As can be seen from the eqs. (4.4b) and (4.4c), the case $M = 1$ corresponds to the super KdV equations.

V. O(N) EXTENDED SUPER AKNS SCHEME

Another nice property of the super AKNS scheme is that, by embedding the super-sl(2,R) algebra into a higher dimensional super algebra, the number of fermionic potentials can be increased as much as desired. For this purpose one can introduce an internal index on the fermionic part of the algebra as q_1^i, q_2^i ($i = 1,2,\ldots,N$). In order to close this larger algebra one is forced to include the generators of the algebra of O(N), J^{ij}. Then the extended super connection having values in this larger super algebra may be expressed as

$$\Omega = E_A \theta_A + q_a^i \Pi_a^i + J^{ij} \tau^{ij} = \begin{pmatrix} \theta & \Pi \\ -\Pi^t\sigma & \tau \end{pmatrix} \tag{5.1}$$

$$(A = 0,1,2 \ , \ a = 1,2 \ , \ i,j = 1,2,\ldots,N)$$

where θ is the usual sl(2,R) part of the connection

$$\theta = \begin{pmatrix} \theta_0 & \theta_1 \\ \theta_2 & -\theta_0 \end{pmatrix}$$

and the fermionic part, π, is now a 2xN matrix and π^t is the transposed of the π and $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The fermi-fermi part, τ, of the connection is an antisymmetric NxN matrix 1-form. The connection 1-form given in eq.(3.3) corresponds to the case $N = 1$ ($\tau = 0$).

For any integer N we can parametrize the elements of the connection (5.1) as follows

$$\theta_0 = A(t,x,\lambda)\,dt - i\lambda\,dx \ , \qquad \Pi_1^i = \alpha^i(t,x,\lambda)\,dt + \beta^i(t,x)\,dx \ ,$$
$$\theta_1 = C(t,x,\lambda)\,dt + r(t,x)\,dx \ , \qquad \Pi_2^i = \rho^i(t,x,\lambda)\,dt + \epsilon^i(t,x)\,dx \ ,$$
$$\theta_2 = B(t,x,\lambda)\,dt + q(t,x)\,dx \ , \qquad \tau^{ij} = \eta^{ij}(t,x,\lambda)\,dt + \phi^{ij}(t,x)\,dx$$

where the super functions A,B,C,η^{ij} and the super potentials q,r,ϕ^{ij} are bosonic, and the super functions α^i, ρ^i and the super potentials ϵ^i, β^i are fermionic fields. As in the previous cases the zero curvature condition yields the following O(N) extended super AKNS equations,

$$A_x + rB - qC + \beta^i\rho^i - \alpha^i\epsilon^i = 0 \ , \tag{5.2a}$$

$$B_x - q_t + 2i\lambda B + 2Aq + 2\epsilon^i\rho^i = 0 \ , \tag{5.2b}$$

$$C_x - r_t - 2i\lambda C - 2Ar - 2\beta^i\alpha^i = 0 \ , \tag{5.2c}$$

$$\alpha_x^i - \beta_t^i - i\lambda\alpha^i - A\beta^i - C\epsilon^i + r\rho^i + \beta^k\eta^{ki} - \alpha^k\phi^{ki} = 0 \ , \tag{5.2d}$$

$$\rho_x^i - \epsilon_t^i + i\lambda\rho^i + q\alpha^i - B\beta^i + A\epsilon^i + \epsilon^k\eta^{ki} - \rho^k\phi^{ki} = 0 \ , \tag{5.2e}$$

$$\eta_x^{ij} - \phi_t^{ij} - \beta^i\rho^j + \alpha^i\epsilon^j - \rho^i\beta^j + \epsilon^i\alpha^j + \phi^{ik}\eta^{kj} - \eta^{ik}\phi^{kj} = 0 \ . \tag{5.2f}$$

Expanding super functions A,B,C,α^i, ρ^i, η^{ij} as positive power series of the spectral parameter λ one obtains a class of non-linear evolution equations

$$r_t = c_{0,x} - 2a_0 r - 2\beta^i\alpha_0^i \ , \tag{5.3a}$$

$$q_t = b_{0,x} + 2a_0 q + 2\epsilon^i\rho_0^i \ , \tag{5.3b}$$

$$\beta^i_t = \alpha^i_{0,x} - a_0\beta^i - c_0\epsilon^i + r\rho^i_0 + \beta^k\eta^{ki}_0 - \alpha^k_0\phi^{ki} \quad , \tag{5.3c}$$

$$\epsilon^i_t = \rho^i_{0,x} + q\alpha^i_0 - b_0\beta^i + a_0\epsilon^i + \epsilon^k\eta^{ki} - \rho^k_0\phi^{ki} \quad , \tag{5.3d}$$

$$\phi^{ij}_t = \eta^{ij}_{0,x} + \alpha^i_0\epsilon^j - \beta^i\rho^j_0 + \epsilon^i\alpha^j_0 - \rho^i_0\beta^j + \phi^{ik}\eta^{kj}_0 - \eta^{ik}_0\phi^{kj} \tag{5.3e}$$

and the recursion equations

$$a_{m,x} + rb_m - c_m q + \beta^i\rho^i_m - \alpha^i_m\epsilon^i = 0 \, , \qquad m \geq 0 \, , \tag{5.4a}$$

$$b_{m,x} + 2ib_{m-1} + 2a_m q + 2\epsilon^i\rho^i_m = 0 \, , \qquad m \geq 1 \, , \tag{5.4b}$$

$$c_{m,x} - 2ic_{m-1} - 2a_m r - 2\beta^i\alpha^i_m = 0 \, , \qquad m \geq 1 \, , \tag{5.4c}$$

$$\alpha^i_{m,x} - i\alpha^i_{m-1} - a_m\beta^i - c_m\epsilon^i + r\rho^i_m + \beta^k\eta^{ki}_m - \alpha^k_m\phi^{ki} = 0, \; m \geq 1, \tag{5.4d}$$

$$\rho^i_{m,x} + i\rho^i_{m-1} + q\alpha^i_m - b_m\beta^i + a_m\epsilon^i + \epsilon^k\eta^{ki}_m - \rho^k_m\phi^{ki} = 0, \; m \geq 1, \tag{5.4e}$$

$$\eta^{ij}_{m,x} - \beta^i\rho^j_m + \alpha^i_m\epsilon^j - \rho^i_m\beta^j + \epsilon^i\alpha^j_m + \phi^{ik}\eta^{kj}_m - \eta^{ik}_m\phi^{kj} = 0, \; m \geq 1. \tag{5.4f}$$

As an illustration, let us consider the $N = 2$ case, i.e., $O(2)$ extended super AKNS scheme. In this case the following class of the integrable non-linear evolution equations are found. (We assume $M = 3$ and a_3 = constant, $b_3 = c_3 = \alpha^i_3 = \rho^i_3 = \eta^{ij}_3 \equiv \eta^{ij}_0 = 0$)

$$\begin{aligned} r_t = {} & a_2\left[\tfrac{1}{2}r_{xx} - r^2q - 2r\beta^k\epsilon^k - 2\beta^k\beta^k_x + \right. \\ & \left. + 2\beta^\ell\beta^k\phi^{k\ell}\right] + ia_3\left[-\tfrac{1}{4}r_{xxx} + \tfrac{3}{2}qrr_x + 3(\beta^k\beta^k_x)_x + \right. \\ & + 3r\beta^k_x\epsilon^k - 3r\beta^k\epsilon^k_x - 6\beta^k_x\beta^\ell\phi^{\ell k} + 6r\beta^\ell\epsilon^k\phi^{k\ell} - \\ & \left. - 3\beta^\ell\beta^k\phi^{k\ell}_x\right] + \left[4\beta^\ell_x\beta^k - 4q\beta^\ell\epsilon^k\right]\eta^{k\ell}_2 \quad , \end{aligned} \tag{5.5a}$$

$$\begin{aligned} q_t = {} & a_2\left[-\tfrac{1}{2}q_{xx} + rq^2 + 2q\beta^k\epsilon^k - 2\epsilon^k\epsilon^k_x + 2\epsilon^\ell\epsilon^k\phi^{k\ell}\right] + \\ & + ia_3\left[-\tfrac{1}{4}q_{xxx} + \tfrac{3}{2}rqq_x - 3(\epsilon^k\epsilon^k_x)_x - 3q\epsilon^k_x\beta^k + \right. \\ & \left. + 3q\epsilon^k\beta^k_x + 6\epsilon^\ell_x\epsilon^k\phi^{k\ell} - 6q\epsilon^\ell\beta^k\phi^{k\ell} + 3\epsilon^\ell\epsilon^k\phi^{k\ell}_x\right] + \\ & + \left[-4\epsilon^\ell_x\epsilon^k + 4q\epsilon^\ell\beta^k\right]\eta^{k\ell}_2 \quad , \end{aligned} \tag{5.5b}$$

$$\begin{aligned} \beta^i_t = {} & a_2\left[\beta^i_{xx} - \tfrac{1}{2}rq\beta^i - \beta^k\epsilon^k\beta^i - \tfrac{1}{2}r_x\epsilon^i - r\epsilon^i_x - 2\beta^k_x\phi^{ki} + r\epsilon^k\phi^{ki} - \right. \\ & \left. - \beta^k\phi^{ki}_x\right] + ia_3\left[-\beta^i_{xxx} + \tfrac{3}{4}qr_x\beta^i + \tfrac{3}{2}rq\beta^i_x + \tfrac{3}{4}r_{xx}\epsilon^i + \right. \\ & + \tfrac{3}{2}r_x\epsilon^i_x + \tfrac{3}{4}rq_x\beta^i + 3\beta^k_x\epsilon^k\beta^i + \beta^k\epsilon^k_x\beta^i + 2\beta^k\epsilon^k\beta^i_x + \\ & + r\beta^k\epsilon^k\epsilon^i - \beta^k\beta^k_x\epsilon^i + 3\beta^k_{xx}\phi^{ki} - \tfrac{3}{2}r_x\epsilon^k\phi^{ki} - \tfrac{3}{2}rq\beta^k\phi^{ki} + \\ & + \beta^\ell\beta^k\epsilon^i\phi^{k\ell} - 2\beta^\ell\epsilon^\ell\beta^k\phi^{ki} + 2\beta^\ell\epsilon^k\beta^i\phi^{k\ell} + 3\beta^k_x\phi^{ki}_x + \\ & \left. + \beta^k\phi^{ki}_{xx}\right] + \left[-\beta^k_{xx} + r_x\epsilon^k + rq\beta^k\right]\eta^{ki}_2 + \\ & + \left[2\beta^k_x\phi^{\ell i} + \beta^k\phi^{\ell i}_x - \beta^\ell\epsilon^k\beta^i - \beta^\ell\beta^k\epsilon^i\right]\eta^{k\ell}_2 \quad , \end{aligned} \tag{5.5c}$$

$$\begin{aligned} \epsilon^i_t = {} & a_2\left[-\epsilon^i_{xx} + \tfrac{1}{2}rq\epsilon^i + \beta^k\epsilon^k\epsilon^i + \tfrac{1}{2}q_x\beta^i + q\beta^i_x + 2\epsilon^k_x\phi^{ki} - q\beta^k\phi^{ki} - \right. \\ & \left. + \epsilon^k\phi^{ki}_x\right] + ia_3\left[-\epsilon^i_{xxx} + \tfrac{3}{4}rq_x\epsilon^i + \tfrac{3}{2}rq\epsilon^i_x + \tfrac{3}{4}q_{xx}\beta^i + \right. \\ & + \tfrac{3}{2}q_x\beta^i_x + \tfrac{3}{4}qr_x\epsilon^i + 3\beta^k\epsilon^k\epsilon^i + \beta^k_x\epsilon^k\epsilon^i + 2\beta^k\epsilon^k\epsilon^i_x + \\ & + q\beta^k\epsilon^k\beta^i + \epsilon^k\epsilon^k_x\beta^i + 3\epsilon^k_{xx}\phi^{ki} - \tfrac{3}{2}q_x\beta^k\phi^{ki} - \tfrac{3}{2}rq\epsilon^k\phi^{ki} - \\ & - \epsilon^\ell\epsilon^k\beta^i\phi^{k\ell} - 2\beta^\ell\epsilon^k\epsilon^i\phi^{k\ell} - 2\beta^\ell\epsilon^\ell\epsilon^k\phi^{ki} + 3\epsilon^k_x\phi^{ki}_x + \\ & \left. + \epsilon^k\phi^{ki}_{xx}\right] + \left[-\epsilon^k_{xx} + q_x\beta^k + rq\epsilon^k\right]\eta^{ki}_2 + \\ & + \left[2\epsilon^k_x\phi^{\ell i} + \epsilon^k\phi^{\ell i}_x + \epsilon^\ell\epsilon^k\beta^i + \beta^\ell\epsilon^k\epsilon^i\right]\eta^{k\ell}_2 \quad , \end{aligned} \tag{5.5d}$$

$$\phi_t^{ij} = \Big\{ a_2 (\beta^i \epsilon^j)_x + i a_3 \big[(\beta^i \epsilon_x^j - \beta_x^i \epsilon^j)_x + 2(\epsilon^i \beta_x^k - \beta^i \epsilon_x^k)\phi^{kj} + (\epsilon^i \beta^k - \beta^i \epsilon^k)\phi_x^{kj} \big] + (\beta^i \epsilon_x^k - \epsilon^i \beta_x^k)\eta_2^{kj} \Big\} - \Big\{ (i \leftrightarrow j) \Big\} + i a_3 \big[(r \epsilon^i \epsilon^j)_x - (q \beta^i \beta^j)_x \big] \tag{5.5e}$$

where a_2, a_3 and η_2^{ij} are commuting constants.

We now list the possible extended integrable super n.l.p.d.e. arising from eqs. (5.5)

i) The AKNS coupled n.l.p.d.e. : $\beta^i = \epsilon^i = \phi^{ij} = 0$

ii) O(2) extended super KdV : $\beta = 0, r = r_0$(constant), $a_2 = 0, ia_3 = 4$

$$q_t = -q_{xxx} + 6 r_0 q q_x - 12(\epsilon^k \epsilon_x^k)_x + 24 \epsilon_x^l \epsilon^k \phi^{kl} + 12 \epsilon^l \epsilon^k \phi^{kl} - 4 \epsilon_x^l \epsilon^k \phi^{kl} , \tag{5.6a}$$

$$\epsilon_t^i = -4\epsilon_{xxx}^i + 3 r_0 q_x \epsilon^i + 6 r_0 q \epsilon_x^i + 12 \epsilon_{xx}^k \phi^{ki} - 6 r_0 q \epsilon^k \phi^{ki} + 12 \epsilon_x^k \phi_x^{ki} + 4 \epsilon^k \phi_{xx}^{ki} + \big[-\epsilon_{xx}^k + r_0 q \epsilon^k \big] \eta_2^{ki} + \big[2\epsilon^k \phi^{li} + \epsilon^k \phi_x^{li} \big] \eta_2^{kl} \tag{5.6b}$$

$$\phi_t^{ij} = 4 r_0 (\epsilon^i \epsilon^j)_x \tag{5.6c}$$

iii) O(2) extended super modified KdV ; $r = k_1 \bar{q}$, $\beta^i = k_2 \bar{\epsilon}^i$, $a_2 = 0$, $ia_3 = 4$, k_1 = real constant $(k_1 = \bar{k}_1)$

$$q_t = -q_{xxx} + 6 k_1 |q|^2 q_x - 12(\epsilon^k \epsilon_x^k)_x - 12 k_2 \epsilon_x^k \bar{\epsilon}^k + 12 k_2 q \epsilon^k \bar{\epsilon}_x^k + 24 \epsilon_x^l \epsilon^k \phi^{kl} - 24 k_2 q \epsilon^l \bar{\epsilon}^k \phi^{kl} + 12 \epsilon^l \epsilon^k \phi_x^{kl} + 4(-\epsilon_x^l \epsilon^k + k_2 q \epsilon^l \bar{\epsilon}^k)\eta_2^{kl} \tag{5.7a}$$

$$\epsilon_t^i = -4\epsilon_{xxx}^i + 3k_1 \bar{q} q_x \epsilon^i + 6 k_1 q \bar{q} \epsilon_x^i + 6 k_2 q_x \bar{\epsilon}_x^i + 6 k_2 q_{xx} \bar{\epsilon}^i + 3 k_1 q \bar{q}_x \epsilon^i + 12 k_2 \bar{\epsilon}^k \epsilon_x^k \epsilon^i + 8 k_2 \bar{\epsilon}^k \epsilon^k \epsilon_x^i + 4 k_2 \epsilon^k \epsilon_x^k \bar{\epsilon}^i + 4 k_2 \bar{\epsilon}_x^k \epsilon^k \epsilon^i + 4 k_2^2 q \bar{\epsilon}^k \epsilon^k \bar{\epsilon}^i + 12 \epsilon_{xx}^k \phi^{ki} - 6 k_2 q_x \bar{\epsilon}^k \phi^{ki} - 6 k_1 q \bar{q} \epsilon^k \phi^{ki} - 4 k_2 \epsilon^l \epsilon^k \bar{\epsilon}^i \phi^{kl} - 8 k_2 \bar{\epsilon}^l \epsilon^k \epsilon^i \phi^{kl} - 8 k_2 \bar{\epsilon}^l \epsilon^l \epsilon^k \phi^{ki} + 12 \epsilon_x^k \phi_x^{ki} + 4 \epsilon^k \phi_{xx}^{ki} + (-\epsilon_{xx}^k + k_2 q_x \bar{\epsilon}^k + k_1 |q|^2 \epsilon^k)\eta_2^{ki} + (2\epsilon_x^k \phi^{li} + \epsilon^k \phi_x^{li} + k_2 \epsilon^l \epsilon^k \bar{\epsilon}^i + k_2 \bar{\epsilon}^l \epsilon^k \epsilon^i)\eta_2^{kl} , \tag{5.7b}$$

$$\phi_t^{ij} = k_2 \big[4(\bar{\epsilon}^i \epsilon_x^j - \bar{\epsilon}_x^i \epsilon^j)_x + 8(e \bar{\epsilon}_x^k - \bar{\epsilon}^i \epsilon_x^k)\phi^{ki} + 4(\epsilon^i \bar{\epsilon}^k - \bar{\epsilon}^i \epsilon^k)\phi_x^{kj} + (\bar{\epsilon}^i \epsilon_x^k - \epsilon^i \bar{\epsilon}_x^k)\eta^{kj} \big] - k_2 \big[(i \leftrightarrow j) \big] + 4\big[k_1 (\bar{q} \epsilon^i \epsilon^j)_x - k_2^2 (q \bar{\epsilon}^i \bar{\epsilon}^j)_x \big] \tag{5.7c}$$

with $k_1 = k_2^2$ and $\phi^{ij} = \bar{\phi}^{ij}$.

iv) O(2) extended super non-linear Schrödinger equation :

$r = k_1 \bar{q}$, $\beta^i = k_2 \bar{\epsilon}^i$, $a_3 = 0$, $a_2 = -2i$, k_1 = real constant $(k_1 = \bar{k}_1)$

$$q_t = -2i \big[-\tfrac{1}{2} q_{xx} + k_1 (q\bar{q}) q + 2 k_2 q \bar{\epsilon}^k \epsilon^k - 2\epsilon^l \epsilon_x^l + 2 \epsilon^l \epsilon^k \phi^{kl} \big] + 4\big[-\epsilon_x^l \epsilon^k + k_2 q \epsilon^l \bar{\epsilon}^k \big] \eta_2^{kl} \tag{5.8a}$$

$$
\begin{aligned}
\epsilon^i_t = -2i\Big[& -\epsilon^i_{xx} + \frac{1}{2}k_1|q|^2\epsilon^i + k_2\bar{\epsilon}^k\epsilon^k\epsilon^i + \frac{1}{2}k_2 q_x\bar{\epsilon}^i + \\
& + k_2 q\bar{\epsilon}^i_x + 2\epsilon^k_x\phi^{ki} - k_2 q\bar{\epsilon}^k\phi^{ki} + \epsilon^k\phi^{ki}_x\Big] + \\
& + \Big[-\epsilon^k_{xx} + k_2 q_x\bar{\epsilon}^k + k_1|q|^2\epsilon^k\Big]\eta^{ki}_2 + \Big[2\epsilon^k_x\phi^{\ell i} + \\
& + \epsilon^k\phi^{\ell i}_x + k_2\epsilon^\ell\epsilon^k\bar{\epsilon}^i + k_2\bar{\epsilon}^\ell\epsilon^k\epsilon^i\Big]\eta^{k\ell}_2
\end{aligned} \tag{5.8b}
$$

$$
\begin{aligned}
\phi^{ij}_t = \; & k_2\Big[-2i(\bar{\epsilon}^i\epsilon^j)_x + (\bar{\epsilon}^i\epsilon^k_x - \epsilon^i\bar{\epsilon}^k_x)\eta^{kj}_2\Big] - \\
& - k_2\Big[(i \longleftrightarrow j)\Big]
\end{aligned} \tag{5.8c}
$$

with $k_1 = k_2^2$ and $\phi^{ij} = \bar{\phi}^{ij}$. Here the bar over quantity also donates the Berezin adjoint operation. Furthermore for higher N values one may obtain a richer extended class of super integrable n.l.p.d.e. in this framework.

VI. CONCLUSION

The Bäcklund transformations in this formulation correspond to the guage transformations [15]. If we denote ψ' as a transformed super field connected to the old one Ψ by

$$\Psi' = S\,\Psi \tag{6.1}$$

where S is a 3x3 super matrix depending on λ as well then the super connections Ω' and Ω are related by

$$dS = S\Omega - \Omega' S \tag{6.2}$$

When the guage transformation (6.2) are supplemented by the Zakharov-Shabat [4] reduction procedure they can be considered as Bäcklund transformations for the super n.l.p.d.e. under consideration and provide n-soliton solutions.

As a conclusion, a super AKNS scheme is proposed by generalizing the sl(2,R) valued soliton connection to the super - sl(2,R) valued soliton connection. Furthermore introducing an internal symmetry we constructed the O(N) extended super AKNS scheme. Classes of super integrable n.l.p.d.e.'s are found for the cases N = 1 and N = 2 super AKNS scheme including the non-linear Schrödinger equation.

ACKNOWLEDGEMENTS

We thank R. Güven and Y. Nutku for discussions. One of the authors (Ö.O) thanks the Scientific and Technical Research Council of Turkey for partial financial support. He would also like to thank Prof. Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

1. For instance, see, R.K. Dodd, J.C. Eilbeck, J.D. Gibbon and H.C. Morris; "Solitons and Non-Linear Wave Equations" Academic Press, London, (1982)
2. M.J. Ablowitz, J.D. Kaub, A.C. Newell and H. Segur; Phys. Rev. Lett. 30, 1262, (1973); ibid 31, 125, (1973); Stud. App. Math 53, 249, (1974).
3. M. Crampin, F.A.E. Pirani and D.C. Robinson; Lett. Math. Phys. 2, 15, (1977). See also, M. Gürses and Y. Nutku; J. Math Phys. 22, 1393, (1981); R. Sasaki; Phys. Lett. A71, 390, (1979); ibid A73, 77, (1979).
4. V.E. Zakharov and S.B. Shabat; Funct. Anal. Appl. 13, 166, (1979).
5. S. Ferrara, L. Girardello and S. Sciuto; Phys. Lett. B76, 303, (1978). L. Girardello and S. Sciuto; Phys. Lett. B77, 267, (1978). M. Chaichian and P.P. Kulish; Phys. Lett. B78, 413, (1978). E.D. Hoker and R.Jackiw; Phys. Rev. D26, 3517, (1982).
6. R.D. Auria and S. Sciuto; Nucl. Phys. B171, 189, (1980). M.A. Olshanetsky; Commun. Math. Phys. 88, 63, (1983).
7. M. Omete and K. Inoue; Phys. Lett. B147, 317, (1984); Prog. Theor. Phys. 72, 641, (1984).
8. B.A. Kupershmidt; Prog. Natl. Acad. Sci. 81, 6562 (1984)
9. Yu I. Manin and A. Radul; Commun. Math. Phys. 98, 65, (1985)
10. B.A. Kupershmidt; Phys. Lett. A102, 213, (1984).
11. B.A. Kupershmidt; J. Phys. A: Math. Gen. 17, L863, (1984).
12. M. Gürses and Ö. Oğuz; Phys. Lett. A108, 437, (1985).
13. F.A. Berezin; "The Method of Second Quantization", Academic Press, New York, (1966).
14. P.D. Lax; Commun. Pure Appl. Math 24, 407, (1968).
15. M, Gürses; in "Solutions of Einstein's Equations: Techniques and Results" Eds. C. Hoenselaers and W. Dietz, Lecture Notes in Physics No.205 , Springer-Verlag, Berlin, Heidelberg, (1984).