IC/85/58

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

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INTERNATIONAL
ATOMIC ENERGY
AGENCY

UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION

1985 MIRAMARE-TRIESTE

IC/85/58

International Atomic Energy Agency

and

United Nations Educational Scientific and Cultural Organization

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International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

The Freund-Rubin mechanism is based on the equation $R_{ik}=\lambda g_{ik}$ (where $\lambda>0$), which, via Myers' Theorem, implies "spontaneous" compactification. The difficulties connected with the cosmological constant in this approach can be resolved if torsion is introduced and λ set equal to zero, but then compactification "by hand" is necessary, since the equation $R_{ik}=0$ can be satisfied both on compact and on non-compact manifolds. In this paper we discuss the global geometry of Ricci-flat manifolds with torsion, and suggest ways of restoring the "spontaneity" of the compactification.

MIRAMARE – TRIESTE

June 1985

* To be submitted for publication.

** Permanent address: Department of Mathematics, National University of Singapore, Kent Ridge 0511, Republic of Singapore.

1. WHAT IS "SPONTANEITY"?

The problem of understanding the relationship between gravitation and the other interactions is currently being addressed in a variety of ways – Kaluza-Klein, Supergravity, Superstrings, and so on. (See de Wit et al.(1984).) The majority of such models either suggest or require that the universe be represented by a manifold of more than four dimensions, and so are apparently faced with an immediate conflict with the observational evidence. This problem is usually circumvented by assuming that (i) the ground state of the universe is a product of some simple space-time with a spacelike manifold M, and (ii) that M is compact and has a very small diameter . (Non-compact models for M have also been proposed; for a general discussion and further references, see McInnes (1985 a,b). We shall not consider this possibility here.) This strategy is often loosely described as "spontaneous compactification", but, in fact, it is not at all evident that there is anything "spontaneous" in this procedure. Perhaps "compactification by hand" would be a more apt description.

In the particular case of eleven-dimensional supergravity, however, the Freund-Rubin (1980) mechanism does indeed inject a strong element of "spontaneity". If the generalized Maxwell field has non-zero vacuum expectation values $F_{\mu\nu\rho\sigma}$ proportional to the alternating tensor $e_{\mu\nu\rho\sigma}$ (Greek letters for space-time), then the Einstein equations imply that the internal space M satisfies $R_{ik}=\lambda g_{ik}$, with $\lambda>0$. Now if we further assume that M is connected and complete (in the sense that every Cauchy sequence in M converges to a point in M), then Myers' theorem (see below) implies that M <u>must</u> be compact. Thus, the compactness of M need not be presumed, but rather can be deduced from certain natural assumptions regarding the nature of the ground state. This compactification is therefore genuinely "spontaneous".

Taking the Freund-Rubin mechanism as our prototype, we can now give a much more general characterization of "spontaneous compactification" in the true sense. The key ingredients may be listed as follows.

(a) <u>Assumption concerning the vacuum expectation values of the matter fields.</u> Intuitively, we expect the ground state matter configuration to be as simple as possible. For example, the fact that the Freund-Rubin energy-momentum tensor (in space-time) is covariantly constant seems reasonable, since this means that the "matter" in the ground state is uniformly distributed. Similar considerations motivate the assumption that the off-diagonal components $F_{\mu\nu\rho k}$, $F_{\mu\nu ik}$, $F_{\mu ijk}$ vanish. There is, admittedly an element of vagueness in this procedure, but this is inevitable in the absence of a rigorous formulation of the concept of a "gravitational ground state".

(b) Assumption concerning the geometry of the ground state. It is essential to realize that the field equations alone cannot induce compactification, since they constrain the manifold only at a local level. Thus, for example, Freund and Rubin implicitly assume that the internal manifold M is a complete metric space, since otherwise Myers' theorem cannot be applied. Indeed, if M were not complete, it would necessarily be non-compact, independently of all other conditions. (This follows from the Hopf-Rinow theorem - see Cheeger and Ebin (1975)). Therefore, any compactification scheme must assume completeness. This is quite reasonable, since otherwise the internal manifold will have "holes" or "rips", and we do not expect the ground state to display such pathologies. (This is particularly true in those theories in which the higher dimensional geodesics have a direct physical interpretation . See McInnes (1985 b) for a discussion.) We conclude, then, that in setting up a compactification mechanism we will have no option but to make some assumptions as to the global structure of the ground state. Obviously these should be as general as possible, and should be physically motivated; but, above all, they must not themselves automatically imply compactness, since this would be tantamount to compactification "by hand". In the Freund-Rubin case, the assumptions that M is connected and complete do not, of course, imply that M is compact.

(c) Gravitational field equations. These allows us to translate the assumptions in (a) above into constraints on the geometry of M.

(d) A "compactification theorem". We must have a theorem which states that, under certain conditions compatible with (a), (b), (c) above, M is necessarily compact. This theorem should refer only to those aspects of

the geometry which are controlled by the gravitational field equations. For example, the Einstein equations give full information only about the Ricci tensor, not the curvature tensor: fortunately, however, the Myers "compactification theorem" only requires conditions on the Ricci tensor.

We have given this rather detailed and general formulation of spontaneous compactification in the hope that the technique can be extended beyond the specific model considered by Freund and Rubin. For, as is well known, that model encounters difficulties connected with the cosmological constant. Specifically, if the space-time cosmological constant is assumed to be very small, then the internal cosmological constant also becomes very small, and this is thought to be in conflict with the supposition that the internal manifold has a submicroscopic diameter. (Strictly speaking, this conclusion is unwarranted, because there does not seem to be any direct relationship between the "cosmological constant" of an Einstein manifold and its diameter, especially if it is multiply connected. Thus, for example, a sphere of given curvature has a totally different diameter to a real projective space of the same curvature. By means of a sufficiently large number of topological identifications, one might be able to reduce the diameter very substantially. But a topological structure of this level of complexity can probably only be justified in the context of quantum gravity.)

A solution of this problem, which is of course not peculiar to the Freund-Rubin mechanism, has been proposed by Orzalesi and collaborators (Destri et al. (1983) and references therein), who propose to consider manifolds with torsion as models of internal space. Various generalizations have been proposed, notably the "seven-sphere with torsion" and the "squashed seven- sphere with torsion" (Duff et al. (1984), Wu (1984)). In none of these cases is the compactification "spontaneous" in the sense in which we are using the term.

The principal purpose of this work is to consider the form which a genuine spontaneous compactification mechanism might take if the internal manifold is endowed with torsion. This will be done by relaxing some of the more restrictive assumptions made by Destri et al (1983) and then by proving a "compactification theorem" for the resulting class of manifolds. These manifolds have a vanishing (Riemann-Cartan) Ricci tensor. In the case of Riemannian manifolds, the vanishing of the Ricci tensor very severely restricts the isometry group, with serious consequences for Kaluza-Klein theories. We therefore explain in detail precisely why this does not, in general, occur if the torsion is non-zero. Finally, we consider the possibilities for generalizing or modifying our compactification scheme.

Notation: In general, we adhere strictly to the conventions of Kobayashi and Nomizu (1963). A Riemann-Cartan manifold or RC manifold is a manifold endowed with a *positive-definite metric tensor* g and a *connection* ∇ with $\nabla g = 0$. On such a manifold there are always (if the torsion $T \neq 0$) two fundamental connections, namely ∇ and the Levi-Cività connection $\overset{\circ}{\nabla}$ generated by g. Hence we need a dual notation (R is the curvature tensor for ∇, and $\overset{\circ}{R}$ that for $\overset{\circ}{\nabla}$) and also a dual nomenclature. We shall use the letters RC to indicate that we are referring to quantities generated by ∇, and the ordinary term for $\overset{\circ}{\nabla}$. Thus, for *example*, a tensor A will be called constant if $\overset{\circ}{\nabla} A = 0$, and "RC-constant" if $\nabla A = 0$. Finally, note that we follow Kobayashi and Nomizu in using S to denote the Ricci tensor, but R_{ik} to denote its components. The symmetric part of S will be denoted (S).

2. COMPACTIFICATION WITH LIE GROUPS

A somewhat drastic solution of the cosmological constant problem in the Freund-Rubin framework would be to assume that all components of the vacuum expectation value of the tensor F vanish. The space-time cosmological constant is then precisely zero, and the internal space M satisfies $\overset{\circ}{R}_{ik} = 0$, *which can be satisfied on certain compact manifolds.*

There are two objections to this procedure, if we retain Riemannian geometry without torsion. In order to discuss the first, we need the following theorem (which unifies various results in Kobayashi and Nomizu (1963)).

THEOREM 1. Let M be a connected compact Ricci-flat Riemannian manifold. Then

(i) The connected component of the identity of the isometry group Isom (M) is Abelian.

(ii) If $\dim(\mathrm{Isom}(M)) \geq \dim M$, then M is a flat manifold (and is therefore $\mathbf{R}^n/D$ for some discrete group D).

(iii) If the universal covering manifold of M is compact, then $\mathrm{Isom}(M)$ is a finite group.

(The proof will be given later as a consequence of a more general result.)

From the point of view of Kaluza-Klein theories, this result is disastrous, since it means that only an Abelian gauge group (at best) can be obtained. (This comment does not apply to superstring theories, but, even there, the absence of symmetries is a major technical impediment to explicit calculation with the metric. Note that the spaces considered by Candelas, Horowitz et al. (1984) as vacuum configurations for superstrings are of the form (compact and simply connected)/(discrete) and therefore satisfy part (iii) of the above theorem.)

A second, and in our view, equally serious objection to internal manifolds with $\overset{\circ}{R}_{ik} = 0$ is that this equation can be satisfied both on compact and on complete non-compact manifolds. The compactification is therefore not spontaneous.

The introduction of torsion immediately resolves the first of these objections: for example, any semisimple compact Lie group can be endowed with a Riemann-Cartan connection which is RC Ricci-flat (in fact, RC flat), but obviously the symmetry group is not Abelian. Indeed, Destri et al. (1983) propose just such a space as a model of M. *In more detail*, these authors assume

(a) The Einstein-Cartan theory of gravitation. (See Hehl et al.(1976).) The field equations are (A, B, C, D being indices for the full multi-dimensional universe)

$$R_{AB} - \frac{1}{2} g_{AB} R = k E_{AB}$$

$$T^A{}_{BC} + \delta^A{}_B T^D{}_{CD} - \delta^A{}_C T^D{}_{BD} = k S^A{}_{BC} ,$$

where E_{AB} is the canonical (non-symmetric) energy-momentum tensor, T^A_{BC} is the torsion, and S^A_{BC} is the spin tensor.

(b) That the internal manifold M is a semisimple connected Lie group with a positive-definite Cartan-Killing metric.

(c) That the spin tensor S^A_{BC} has non-zero vacuum expectation values only in the internal space: $S^i_{jr} = f^i_{jr}/k$ (where f^i_{jk} are the structural constants in the appropriate basis), and that $E_{AB} = 0$.

The resulting ground state is a product of Minkowski space with a compact Lie group. However, it is clear that the compactification is not spontaneous. In particular, the formal condition (b) - which has no clear physical meaning - automatically entails the compactness of M: a well-known theorem of Weyl (see Kobayashi and Nomizu (1969) for a geometric proof) states that any connected semisimple Lie group with positive-definite Cartan-Killing metric is compact. Thus, the compactness of M is implicit in the geometric assumptions at the outset.

The difficulty with this mechanism, then, arises from the lack of a physical motivation for (b) and (c). The status of these assumptions can be greatly clarified with the aid of the following theorem, which simplifies and slightly modifies certain results of Hicks (1959) and Wolf (1972).

THEOREM 2. Let M be a complete, connected, simply connected Riemann-Cartan manifold such that

(i) The curvature tensor $R = 0$;

(ii) The "fully covariant" torsion tensor, defined by $T(X,Y,Z) = g(X,T(Y,Z))$, is totally antisymmetric.

Then M is a homogeneous (coset) space. If, in addition, we have

(iii) T is constant, i.e. , $\overset{\circ}{\nabla} T = 0$, then M is a connected, simply connected Lie group.

Proof (outline):

Using (i), let $\{e_j\}$ be a global basis of RC constant vector fields. From the definition of torsion we obtain, for each j, k, the equations $[e_j, e_k] = -T^r_{jk} e_r$. Now consider a vector field of the form $a^i e_i$, where

the a^i are fixed numbers. Then $X = a^i e_i$ is a Killing vector field, since (if L denotes the Lie derivative)

$$(L_X g)(e_j, e_k) = a^i e_i g_{jk} + g_{rk} T^r_{ij} a^i + g_{jr} T^r_{ik} a^i .$$

Here the second and third terms vanish by total antisymmetry, and the first term vanishes since $\nabla g = 0$ implies $e_i g_{jk} = 0$ in this basis. Furthermore, X is obviously of constant length, and its integral curves are therefore geodesics.

Now let x and y be arbitrary points in M. Then since M is complete, there exists (by the Hopf-Rinow theorem- see Cheeger and Ebin (1975)) a minimizing geodesic joining x to y. (This curve is both a ∇ geodesic and a $\overset{\circ}{\nabla}$ geodesic.) The tangent vector to this curve at x may be expressed as a linear combination $a^i e_i(x)$. Now define $X = a^i e_i$ as a vector field; then, as we have seen, the integral curves of X are geodesics. By the uniqueness of geodesics with given initial conditions, the geodesic joining x to y is an integral curve of X. But the isometry group G of M acts on M through motions along the integral curves of the Killing fields: hence G maps x into y. Since x and y are arbitrary, G acts transitively, and so M must be a homogeneous space G/H.

Now suppose that T is constant. Choosing the e_j to be orthonormal, one finds that the components of the Levi-Civita connection in this basis are $\overset{\circ}{\Gamma}{}^i_{jk} = -\frac{1}{2} T^i_{jk}$, and so $\overset{\circ}{\nabla} T = 0$ yields

$$e^i T^j_{km} - \frac{1}{2} T^r_{km} T^j_{ir} + \frac{1}{2} T^j_{rm} T^r_{ik} + \frac{1}{2} T^j_{kr} T^r_{im} = 0 ,$$

or

$$e_i T^j_{km} + \frac{1}{2} T^j_{ir} T^r_{mk} + \frac{1}{2} T^j_{mr} T^r_{ki} + \frac{1}{2} T^j_{kr} T^r_{im} = 0 .$$

Permuting twice on ikm and adding, we find

$$e_i T^j_{km} + e_k T^j_{mi} + e_m T^j_{ik} + \frac{3}{2}(T^j_{ir} T^r_{mk} + T^j_{mr} T^r_{ki} + T^j_{kr} T^r_{im}) = 0 .$$

But the Jacobi identities corresponding to the basis $\{e_i\}$ are

$$e_i T^j{}_{km} + e_k T^j{}_{mi} + e_m T^j{}_{ik} + (T^j{}_{ir} T^r{}_{mk} + T^j{}_{mr} T^r{}_{ki} + T^j{}_{kr} T^r{}_{im}) = 0.$$

Subtracting these two equations, we find

$$T^j{}_{ir} T^r{}_{mk} + T^j{}_{mr} T^r{}_{ki} + T^j{}_{kr} T^r{}_{im} = 0$$

and so $e_i T^j{}_{km} = 0$, that is, the $T^j{}_{km}$ are constants. In view of $[e_i e_j] = -T^r{}_{ij} e_r$, it is evident that the $\{e_i\}$ generate a Lie algebra. Let G be the corresponding simply connected, connected Lie group. Then, G acts on M through motions along the integral curves of the vector fields $a^i e_i$. Arguing as before, we find M = G/H. But now we have (since the Lie algebra of G is generated by the $\{e_i\}$) the relations dim G = dim M = dim G - dim H, whence H must be discrete. Since M is simply connected, H must be trivial, and so M = G. This concludes the proof.

REMARK 1: If M is not assumed to be simply connected, then one finds (by applying the above argument to the universal covering space of M) that M has the form G/D, where G is a connected, simply connected Lie group, and D is a discrete subgroup.

REMARK 2: It is easy to see that the given connection on M coincides with the standard (-) connection on G. But the given metric on M may not coincide with the Cartan-Killing metric or G - indeed G may not be semisimple. See the Remarks after the proof of Proposition 1 in the next section.

The great virtue of this theorem from our present point of view is that we can now do away with the assumption that M is a Lie group. Instead we can make assumptions regarding the torsion and RC curvature of M, and it may be possible to motivate such assumptions in a physical way through the field equations. Thus, assumtpion (b) of the mechanism of Destri et al. can be reformulated as follows:

(b') The internal manifold M is a complete, connected, simply connected

Riemann-Cartan manifold which satisfies

(i) $R^i{}_{jk\ell} = 0$;

(ii) T_{ijk} is totally antisymmetric;

(iii) T_{ijk} is constant. (Thus M is a Lie group);

(iv) The metric on M is the Cartan-Killing metric.

(Note that the fourth assumption is independent of the others - the Cartan-Killing metric is not the only possible metric on a Lie group.) Now let us consider the justification for these assumptions. Since the second Einstein-Cartan field equation relates the torsion to the spin tensor by an invertible algebraic equation, (iii) means that the vacuum expectation value of the spin tensor S_{ijk} is constant: in other words, the spin source is uniformly distributed throughout M. This appears to be quite reasonable for a ground state - indeed, as we have already mentioned, one reason for accepting anti-de Sitter space as a space-time ground state is that the "matter" is uniformly distributed throughout the space (that is, the cosmological term is constant). Similarly, (ii) means that the spin tensor is taken to be totally antisymmetric. Destri et al. (1983) justify this on the grounds that the simplest source of spin - the Dirac field - does indeed generate a totally antisymmetric spin tensor. Thus a spin tensor of this type is naturally associated with spin - ½ condensates. While this is plausible, in practice one might also be interested in Rarita-Schwinger condensates, for which the spin tensor need not be totally antisymmetric (Wu (1984)). However, while (ii) is less well motivated than (iii), it does at least have a physical interpretation.

The same cannot, unfortunately, be said of (i), and still less of (iv). Assumption (i) can be partially justified on the grounds that it implies the vanishing of the internal energy - momentum tensor - again, this is a reasonable condition for the ground state to satisfy - but is obviously unnecessarily strong, since the vanishing of the RC Ricci tensor would suffice for this. Finally, (iv) appears to have no physical meaning whatever; again, since M is spacelike, this assumption amounts (via the theorem of Weyl mentioned earlier) to compactification by hand.

This analysis clearly indicates that some of these assumptions must be relaxed or replaced if we are to obtain a genuinely spontaneous compactification mechanism. Such modifications will lead us to spaces which are not necessarily Lie groups, but this is desirable: in particular, one would hope that (at least some) homogeneous (coset) manifolds can be included. A specific procedure will be proposed in the next section.

We conclude this section with a brief discussion of the "seven-sphere with torsion" solutions. As is well known, the round seven-sphere with torsion (Englert 1982) satisfies the first two assumptions listed under (b') above, but, since it is not a Lie group, it cannot satisfy the third. The squashed seven-sphere with torsion violates (i) and (iii) but satisfies (ii) (see Duff et al. (1984)). Finally, Wu (1984) proposes to retain (i) but neither (ii) nor, in general, (iii). We shall not consider these models further, since in each case the topology is assumed at the outset to be that of the seven-sphere.

3. A FRAMEWORK FOR SPONTANEOUS COMPACTIFICATION WITH TORSION

We now propose to extend the concept of spontaneous compactification to spaces with torsion by generalizing the mechanism of Destri et al. (1983). In this paper we shall not propose a specific model, but rather a general framework to guide the construction of such models. We proceed according to the principles laid down in the first section, and list our assumptions in the same way.

(a) Assumptions concerning the vacuum expectation values.

(i) We shall retain the following assumptions of Destri et al. First we take the canonical energy - momentum tensor E_{AB} to be zero. (Thus R_{AB} is zero and the problem of the cosmological constant does not arise.) Second, we assume that the spin tensor has non-vanishing components S_{ijk} only in the internal direction. (Thus space-time may be taken as Minkowski space.) Third, we assume that the "spin condensate" is uniformly distributed throughout the internal ground state manifold M - that is, we take S_{ijk} to be a covariant constant tensor. We do not assume that the RC curvature

tensor vanishes.

(ii) We shall not assume that the tensor S_{ijk} is totally antisymmetric; however, we will be forced to assume that (in a certain technical sense to be clarified later) the totally antisymmetric part gives the dominant contribution to S_{ijk}.

(b) Assumption concerning the geometry of the ground state. Beyond the standard assumptions that the universe splits into a product of space-time with a connected, complete, spacelike manifold M, we wish to impose the following additional condition: the universe splits only once. That is, M itself must not split into a product. Let us be more precise about this. Let N be any Riemannian manifold, and x be an arbitrary point in N. Suppose that there exist two submanifolds N' and N'' of N and an open neighbourhood U of x such that U is the Riemannian product of U' with U", where U' is an open neighbourhood of x in N', and U" is an open neighbourhood of x in N". Then we shall say that N is "locally decomposable". More simply, a locally decomposable manifold is one in which, at each point, it is possible to find a coordinate system such that the metric has block-diagonal form throughout a neighbourhood around that point - in short, a locally decomposable manifold is what is usually (but not quite correctly) described in the physics literature as a "product manifold". We propose to forbid such manifolds as candidates for M.

There are two major reasons for imposing this condition. The first is that, in general, a locally decomposable manifold has an isometry group which is not simple. In the Kaluza-Klein context, this would lead to a non-simple gauge group with (possibly) different gauge couplings, and so unification would be lost even among the non-gravitational interactions (unless the Candelas-Weinberg (1984) method can be applied, which seems somewhat problematic). It is true that the presence of torsion complicates this picture; in principle, one could use the torsion to break completely all of the factors in the gauge group save one. But this is clearly very unnatural.

The second reason for requiring local indecomposability is more vague but also more deep. Throughout this paper, we have assumed as usual that the ground state of the universe is a (pseudo-) Riemannian product $W = L \times M$, where L is space-time. Why should W split in this way? The answer is of course unknown, but it is significant that W should split into a compact part and a part which cannot be compact. As is well known, compact space-times violate causality (Geroch (1967)). Thus we may speculate that space-time splits off from W in order that causality be preserved. (Freund and Rubin (1980) have attempted to explain why precisely four dimensions split off.) The fact that the Myers compactification theorem is not valid for pseudo-Riemannian manifolds (Beem and Ehrlich (1981)) reinforces this idea. The point is that if the existence of time is indeed responsible for the splitting of W, then M, being entirely spacelike, should not split.

To summarize, then, we assume that the internal manifold M is a spacelike, connected, complete, locally indecomposable manifold. These conditions in themselves are quite general and certainly do not imply compactness.

(c) Gravitational field equations

We use the Einstein-Cartan theory.

(d) A "compactification theorem"

Before starting the appropriate result, we must use (c) to express the assumptions under (a) in geometrical form. First, $E_{AB} = 0$ implies that the RC Ricci tensor is zero. Second, the assumption that S_{ijk} is constant means that the torsion T is constant. The idea that the "spin-1/2-like" contribution to S_{ijk} is dominant can be expressed as follows. (As we shall see, our torsion has to be traceless, and so T_{ijk} and S_{ijk} essentially coincide; therefore we refer directly to T_{ijk}.) The tensor T_{ijk} corresponds to a reducible representation of the appropriate rotation group, but can be split into two "irreducible" parts (not three, because T is traceless) according to

$$T_{ijk} = T_{[ijk]} + U_{ijk} , \qquad (3.1)$$

where $T_{[ijk]}$ is the totally antisymmetric part and

$$U_{ijk} = \frac{2}{3}\left(T_{ijk} - T_{[jk]i}\right) . \qquad (3.2)$$

The deviation of T from total antisymmetry can be measured by the quantity

$$\theta = U_{ijk}\, U^{ijk} / T_{[ijk]}\, T^{[ijk]} . \qquad (3.3)$$

Since T is constant, θ is a fixed number which is zero if and only if T is totally antisymmetric. The statement that the totally antisymmetric part of T is dominant simply means that θ is small.

We are now in a position to state and prove our main result. (We denote the symmetric part of the RC Ricci tensor by (S).)

PROPOSITION 1. Let M be a complete, connected, locally indecomposable Riemann-Cartan manifold with constant non-zero torsion. If $(S) = 0$ and $\theta < \frac{1}{2}$, M must be compact.

COROLLARY. Let M be a complete, connected, locally indecomposable Riemann-Cartan manifold with constant, non-zero, totally antisymmetric torsion. If $(S) = 0$, then M must be compact.

This proposition has the following partial converse.

PROPOSITION 2. Let M be a complete, connected, locally indecomposable Riemann-Cartan manifold with constant non-zero torsion, and with a continuous non-Abelian isometry group. If $(S) = 0$, then M is compact if and only if $\theta < \frac{1}{2}$.

These results will be proved with the aid of the following definitions and theorems. (See Kobayashi and Nomizu (1963), (1969).) Let N be any Riemannian manifold (assumed to be connected, but not necessarily complete). The holonomy group acts (via parallel transport around closed loops) as a group of linear transformations on the tangent space T_x at any point x. Thus, T_x yields a representation of the holonomy group. If this representation is reducible, then N is said to be a reducible manifold. Similarly, if the restricted holonomy group (obtained by parallel transport around null-homotopic loops) acts reducibly, then we shall say that N is locally reducible. We now have

THEOREM 3. (Local de Rham decomposition theorem). Every connected locally reducible Riemannian manifold is locally decomposable.

THEOREM 4. (Global de Rham decomposition theorem). Every connected, simply connected, complete, reducible Riemannian manifold is globally decomposable, that is, globally isometric to a Riemannian product.

We shall also use the following results.

THEOREM 5. (Myers). Let N be a connected, complete Riemannian manifold. If all eigenvalues of the Ricci tensor are bounded from below by a strictly positive constant, then N must be compact.

THEOREM 6. (Bochner). Let N be a compact, connected Riemannian manifold, with a negative-definite Ricci tensor. Then the isometry group of N is finite.

THEOREM 7. (Schur). Let G be a subgroup of the orthogonal group $O(n)$ which acts irreducibly on $\mathbb{R}^n$. Then every symmetric bilinear form on $\mathbb{R}^n$ which is invariant by G is a multiple of the standard Euclidean inner product.

We may now prove Propositions 1 and 2.

PROOF OF PROPOSITION 1: Define a (1,2) tensor on M by $K(X,Y)=\nabla_X Y - \overset{\circ}{\nabla}_X Y$ where X and Y are vector fields. In components with respect to an arbitrary basis, K is

$$K^i{}_{j\ell} = -\frac{1}{2} g^{ir}(T_{j\ell r} + T_{\ell jr}) + \frac{1}{2} T^i{}_{j\ell} \quad . \tag{3.4}$$

Note that $K_{ij\ell} = g_{ir}K^r{}_{j\ell}$ is antisymmetric in its first and third indices. A straightforward computation yields

$$R_{ij} = \overset{\circ}{R}_{ij} + K^n{}_{nm} K^m{}_{ji} - K^n{}_{jm} K^m{}_{ni} + \overset{\circ}{\nabla}_n K^n{}_{ji} - \overset{\circ}{\nabla}_j K^n{}_{ni} \tag{3.5}$$

as the general relation between R_{ij} and $\overset{\circ}{R}_{ij}$. In the present case, the last two terms on the right-hand side vanish, as also does the second - for the following reason. Since M is locally indecomposable, it must also (by Theorem 3) be locally irreducible. Now T and (therefore) $T^m{}_{mi}$ are constant with respect to $\overset{\circ}{\nabla}$ and so $T^m{}_{mi}$ is an invariant vector under the action of the restricted holonomy group. Hence $T^m{}_{mi}$ is zero, that is, T is traceless. But $K^m{}_{mi} = T^m{}_{mi}$, and so only the first and third terms in (3.5) are non-zero. Thus, setting the symmetric part of R_{ij} equal to zero, we find

$$R_{ij} = \frac{1}{2} K^m{}_{ni} K^n{}_{jm} + \frac{1}{2} K^m{}_{nj} K^n{}_{im} \quad . \tag{3.6}$$

Clearly $\overset{\circ}{\nabla}_p \overset{\circ}{R}_{ij} = 0$, and so $\overset{\circ}{R}_{ij}$ is a symmetric bilinear form invariant under the action of the restricted holonomy group. As we have seen, this action is irreducible, and so by Theorem 7 we have

$$R_{ij} = c\, g_{ij} \quad . \tag{3.7}$$

Substituting this into (3.6), contracting and setting $n = \dim M$, we obtain,

$$cn = K_{mni}K^{nim} \quad .$$

Using the antisymmetry of K and the easily derived relations

$$K^{nim} = \frac{1}{2} T^{inm} + \frac{1}{2}(T^{nim} - T^{min}), \qquad \frac{1}{2}(K_{mni} - K_{nmi}) = \frac{1}{2} T_{imn} \quad ,$$

one obtains

$$cn = \frac{1}{4} T_{imn} T^{inm} + \frac{1}{2} T_{imn} T^{nim} \quad .$$

Setting $T_{imn} = T_{[imn]} + U_{imn}$ and using the relations $T_{imn} U^{imn} = 0$ and $\frac{1}{2}(U_{imn} - U_{min}) = \frac{1}{2} U_{nmi}$, one finds

$$cn = \frac{1}{4} T_{[imn]} T^{[imn]} - \frac{1}{2} U_{imn} U^{imn} \quad . \tag{3.8}$$

From the definition of θ, assuming $T_{[imn]}$ is not zero, we have finally

$$cn = \frac{1}{2}\left(\frac{1}{2} - \theta\right) T_{[imn]} T^{[imn]} \quad . \qquad (3.9)$$

Thus $c > 0$ if $\theta < \frac{1}{2}$. Since M is connected and complete, it now follows from (3.7) and Theorem 5 that M is compact. This completes the proof.

REMARK 3: *In our general discussion, we assumed that the full Ricci tensor vanishes, but clearly this assumption can be weakened to (S)=0. However, this has no clear physical meaning.*

REMARK 4: Combining Theorem 2 with Proposition 1, we can show that any complete, connected, simply connected; locally indecomposable RC manifold which satisfies the three conditions of Theorem 2 must be a simply connected compact Lie group. Furthermore, the given connection coincides with the (–) connection, and the given metric coincides with the Cartan-Killing metric. (This follows from (3.6), which becomes $cg_{ij} = 1/4 T^m_{ni} T^n_{jm}$. Here the T^m_{ni} are essentially the structural constants.) Thus, the group must also be semisimple in this case.

We now prove Proposition 2: The proof is precisely as above, except that we now must also show that M is non-compact if $\theta > \frac{1}{2}$. In this case $c < 0$ and so if M were compact we could use either Theorem 6 (if $\theta > \frac{1}{2}$) or Theorem 1 (if $\theta = \frac{1}{2}$) to produce a contradiction. This completes the proof.

REMARK 5: Proposition 2 fails without the assumption that the isometry group is continuous and non-Abelian. In Kaluza-Klein theories we must in practice have such an isometry group, and so in this context the condition $\theta < \frac{1}{2}$ is not only sufficient but also necessary for compactification.

REMARK 6: If $T_{[ijk]} = 0$, then θ is undefined, but it is clear from (3.8) that $c < 0$ in this case; therefore, we can include this case in the statements of Propositions 1 and 2 by formally allowing θ to be infinite.

With Proposition 1 we conclude our general description of spontaneous compactification for spaces with torsion. The problem of constructing non-trivial particular examples will be considered in the next section.

4. SYMMETRIES OF MANIFOLDS WITH TORSION

As Theorem 1 clearly shows, the vanishing of the Ricci tensor of a compact Riemannian manifold strongly restricts the symmetry group. The example of Lie groupsshows that the restrictions are less severe in the Riemann-Cartan case. But this tells us nothing about the more interesting case in which the RC Ricci tensor vanishes but the RC curvature does not. Part (ii) of Theorem 1 means that, in the Riemannian case, such a manifold is less symmetric than its flat counterparts. It is important to determine whether this is so for RC manifolds.

In this section we shall show that, unless it is supplemented by some quite unnatural technical conditions, RC Ricci - flatness imposes only a very weak condition on the symmetry group of a compact RC manifold. It is even conceivable that some Riemannian coset manifolds can be "Ricci - flattened" by torsion without losing any symmetry, though we have no examples as yet.

Let Aff(M) be the identity component of the group of affine automorphisms of a RC manifold M: that is, Aff(M) consists of mapping of M into itself which preserve the affine connection. In the case of compact Riemannian manifolds, Aff(M) coincides with the identity component of the isometry group, denoted Isom(M), but for a general RC manifold this is not so. A vector field X on M which generates local isometries will be called a metric Killing vector, while a field which generates local affine automorphisms will be called an affine Killing vector. The corresponding algebras can, under certain natural conditions (Kobayashi and Nomizu (1963)), be identified with the Lie algebras of Isom(M) and Aff(M). The following simple result specifies the relationships between these various objects.

PROPOSITION 3. A vector field X on a compact Riemann-Cartan manifold is an affine Killing vector if and only if it is a metric Killing vector and also the Lie derivative $L_X T$ is zero.

PROOF: Let K be the tensor defined in the proof of Proposition 1. Clearly $L_X T = 0$ if and only if $L_X K = 0$. Now it may be shown that X is an affine Killing vector if and only if, for every vector field Y,

$L_x \nabla_y - \nabla_y L_x = \nabla_{[xy]}$. Let X, Y, Z be arbitrary vector fields. Then using $\nabla_y Z = \overset{\circ}{\nabla}_y Z + K(Y,Z)$ one finds

$$L_x \nabla_y Z - \nabla_y L_x Z - \nabla_{[xy]} Z = L_x \overset{\circ}{\nabla}_y Z - \overset{\circ}{\nabla}_y L_x Z - \overset{\circ}{\nabla}_{[xy]} Z + (L_x K)(Y, Z) \tag{4.1}$$

Thus it is clear that if X is a metric Killing field and $L_x T = 0$, then X is also an affine Killing field. Conversely, if X is an affine Killing field, the left-hand side vanishes. Exchanging Y and Z and subtracting, we find

$$L_x(\overset{\circ}{\nabla}_y Z - \overset{\circ}{\nabla}_z Y) + (\overset{\circ}{\nabla}_z L_x Y - \overset{\circ}{\nabla}_{[xy]} Z) + (\overset{\circ}{\nabla}_{[xz]} Y - \overset{\circ}{\nabla}_y L_x Z) + (L_x T)(Y,Z) = 0$$

Since $\overset{\circ}{\nabla}$ is torsionless, we have

$$[X,[Y,Z]] + [Z,[X,Y]] + [[X,Z],Y] + (L_x T)(Y,Z) = 0 \quad ,$$

and so, by the Jacobi identities, $L_x T = 0$. Substituting this into (4.1) we find that X is a metric Killing field. This completes the proof.

<u>COROLLARY</u>. Aff(M) is a subgroup of Isom(M).

In physical language, one would say that, unless the torsion is invariant by the isometry group, it breaks the symmetry from Isom(M) down to Aff(M). Thus, the "symmetry group" is Aff(M), not Isom(M).

In order to proceed, we introduce the following notation. For any vector field X, let $A_x = L_x - \nabla_x$. Since A_x annihilates any function, it may be treated as a (1,1) tensor. If X is a metric Killing vector, then A_x, regarded as a (0,2) tensor, is antisymmetric. For any vector fields X, Y, one has $A_x Y = - T_x Y - \nabla_y X$, where T_x is the (1,1) tensor defined by $T_x(Z) = T(X,Z)$. It is also possible to show that if X is an affine Killing vector and Y is arbitrary, then $\nabla_y(A_x) = R(X,Y)$,

where R is the RC curvature tensor. (See Kobayashi and Nomizu (1963)).

The following result is a direct generalization of Theorem 1.

<u>PROPOSITION 4.</u> Let M be a compact connected RC manifold with traceless torsion and RC Ricci tensor equal to zero. Suppose either that every affine Killing vector satisfies

(i) Trace $A_x T_x \leqslant 0$ everywhere on M,

or that every affine Killing vector satisfies

(ii) $T_x = b A_x$ everywhere on M, where b is a constant. Then

(a) $[X,Y] = \pm T(X,Y)$ for every pair X, Y, of affine Killing fields, where the (-) sign occurs only in case (ii) and only if b = -1.

(b) If dim Aff(M) $\geqslant$ dim M and T is totally antisymmetric, then M has the form G/D, where G is a simply connected Lie group and D is a discrete subgroup.

<u>PROOF</u> (Outline): Let X be an affine Killing vector field , and let Y, Z be arbitrary vector fields. Then

$$(\nabla_y A_x)(Z) = \nabla_y(A_x(Z)) - A_x(-A_z Y - T_z Y)$$
$$= R(X,Y)Z = - R(Y,X)Z \quad .$$

By definition, the Ricci tensor S is given as the trace S(X,Z) of the map $Y \to R(Y,X)Z$, and so

$$- S(X,Z) = \operatorname{div}(A_x Z) + \operatorname{Trace} A_x A_z + \operatorname{Trace} A_x T_z \quad . \tag{4.2}$$

Now Stokes' theorem for a compact orientable RC manifold takes the form

$$\int \operatorname{div} W \, dv = - \int \operatorname{Trace} T_W \, dv \quad ,$$

where W is any vector field and dv is the volume element. In our case T_w is traceless, and so, setting S = 0 in (4.2) and integrating, we obtain

$$\int (\operatorname{Trace} A_x A_z + \operatorname{Trace} A_x T_z) dv = 0 \quad . \tag{4.3}$$

(If M is not orientable, we can take the appropriate twofold covering, without altering our conclusions.)

Now assume condition (i). Set $Z = X$ in (4.3). Since A_X is anti-symmetric, Trace $A_X A_X \leq 0$ at each point, and since the same is true of Trace $A_X T_X$, we must have Trace $A_X A_X = 0$ and therefore $A_X = 0$ at each point. Thus if X and Y are affine Killing fields, we have $T_X Y + \nabla_Y X = 0$. Exchanging X and Y and subtracting, one finds (from the definition of T) that $[X,Y] = T(X,Y)$.

Now if $\dim \mathrm{Aff}(M) \geq \dim M$, then at any point one can set up a basis of affine Killing vectors $\{X_i\}$. We have just seen that $A_X = 0$ for an affine Killing field, and so $R(X_i, X_j) = \nabla_{X_j} A_{X_i} = 0$. Thus M is RC flat. Now from Proposition 3 and the relation $[X_i, X_j] = T(X_i, X_j)$, one finds

$$[X_i, T(X_j, X_k)] = T(T^r{}_{ij} X_r, X_k) + T(X_j, T^r{}_{ik} X_r)$$

which after simplification, becomes

$$X_i T^m{}_{jk} = T^m{}_{ir} T^r{}_{kj} + T^m{}_{kr} T^r{}_{ji} + T^m{}_{jr} T^r{}_{ik} \quad . \tag{4.4}$$

Permuting twice on ijk and adding, one obtains an equation which can be compared with the Jacobi identities for the basis $\{X_i\}$. The result is that both sides of (4.4) vanish. Now using $L_X g = 0$ and the assumed total anti-symmetry of T, one shows that $X_i g_{jk} = 0$ in this basis and so, since the commutator coefficients are $T^i{}_{jk}$, $\mathring{\Gamma}^i{}_{jk} = \frac{1}{2} T^i{}_{jk}$. Hence

$$\mathring{\nabla}_i T^j{}_{k\ell} = X_i T^j{}_{k\ell} + \frac{1}{2} T^r{}_{k\ell} T^j{}_{ir} - \frac{1}{2} T^j{}_{r\ell} T^r{}_{ik} - \frac{1}{2} T^j{}_{kr} T^r{}_{i\ell},$$

which is zero since both sides of (4.4) vanish. Finally, M must be complete because it is compact. Thus, by Theorem 2, M is essentially a Lie group.

The proof is similar in case (ii). Putting $Z = X$ in (4.3), as well as $T_X = b A_X$, one has,

$$(1 + b) \int \mathrm{Trace}\, A_X A_X \, d\mathscr{V} = 0 \quad ,$$

whence $A_X = 0$ as above, unless $b = -1$. But in that case $\nabla_Y X = -(A_X + T_X)Y = 0$ for every affine Killing vector X and arbitrary Y. Thus every affine

Killing vector is RC constant, and so, from the definition of torsion, every pair X, Y of affine Killing fields satisfies $[X,Y] = -T(X,Y)$. Except for unimportant details, the proof that M is RC flat and is essentially a Lie group now proceeds as before. This concludes the proof.

REMARK 7: Any compact connected Riemannian manifold obviously satisfies the conditions of this proposition if its Ricci tensor is zero, and so one obtains the first two parts of Theorem 1 by setting $T = 0$ in Proposition 4. The third part of Theorem 1 is obtained as follows. It is clear from the above proof that for such a manifold, every Killing vector satisfies $A_X = 0$, and so $\nabla_Y X = \mathring{\nabla}_Y X = 0$. Now let M be the universal covering manifold of M, which inherits its local geometry from M. Then the algebra of Killing vectors is invariant under the action of the holonomy group, and so M is reducible and consequently splits, according to Theorem 4. The Killing vectors generate a flat simply connected manifold. But such a manifold is non-compact , which is impossible if M is compact. Hence, there can in fact be no Killing vectors, and so the Lie algebra of the isometry group is trivial and the isometry group is discrete. As M is compact, so also is its isometry group, which must therefore be not only discrete but actually finite.

REMARK 8: Any compact semisimple Lie group satisfies the conditions of Proposition 4. The (+) connection corresponds to case (i), and the (-) connection to case (ii) (with $b = -1$).

Apart from its general interest, Proposition 4 is mainly of interest to us because it shows that, in general, the vanishing of the Ricci tensor imposes a remarkably weak condition on the symmetry group of a Riemann Cartan manifold. The restrictions of Theorem 1 are so strong in the Riemannian case simply because these manifolds "accidentally" satisfy condition (i) of Proposition 4. But for a general *Riemann-Cartan manifold with zero* Ricci tensor, there is no reason whatever to expect that either (i) or (ii) will hold . In this case the only restriction is equation (4.3), which, being an integral equation, is a weak constraint. We conclude, therefore, that the Ricci flatness condition is unlikely to restrict the symmetry of a

Riemann-Cartan manifold to any significant extent.

The existence of symmetry is of great value in constructing explicit examples of manifolds. In the present context, homogeneous (coset) manifolds are of particular interest. The vast majority of such manifolds do not admit RC flat connections, but it is certainly possible that many may admit RC Ricci-flat connections. In view of our assumption (in Propositions 1 and 2) that $\overset{\circ}{\nabla} T = 0$, the following result suggests one approach. (Here we use the term "symmetric space" in the technical sense; see Kobayashi and Nomizu (1969).)

PROPOSITION 5. Let $M = G/H$ be a Riemannian symmetric space with a G-invariant metric g. Then if T is the torsion of any G-invariant Riemann-Cartan connection on M, we have $\overset{\circ}{\nabla} T = 0$, where $\overset{\circ}{\nabla}$ is the Levi-Civita connection for g.

PROOF: Since the RC connection is G-invariant, it follows from Proposition 3 that T is G-invariant. But standard results on symmetric spaces state (i) that the Levi-Civitá connection induced by a G-invariant metric coincides with the canonical connection, and (ii) that any G-invariant tensor is constant with respect to the canonical connection. Hence $\overset{\circ}{\nabla} T = 0$, which completes the proof.

REMARK 9: This result could be regarded as further motivation for the assumption in Proposition 1 that T is constant.

Proposition 5 suggests that examples of compact manifolds compatible with Proposition 1 may possibly be found by substituting the metric of a symmetric space (in particular, a space of constant curvature) into the left-hand side of (3.6) and solving for K^i_{jk} subject to the constraint $\theta < \frac{1}{2}$. One hopes that non-trivial solutions (with non-zero RC curvature) can be found in this way.

5. MODIFICATIONS OF THE COMPACTIFICATION THEOREM

As several of the known examples of internal manifolds with torsion do not satisfy all conditions of Proposition 1, it is of some interest to ask whether these conditions can be modified or dropped. Here we list

briefly some relevant remarks.

First, note that it is not possible to remove either completeness or local indecomposability. Without the first, the manifold would necessarily be non-compact. Without the second, we would be including manifolds of the type $M \times \mathbb{R}^n$, where M satisfies all conditions of Proposition 1. This manifold satisfies all conditions of Proposition 1 except local indecomposability, and fails to be compact .

Second, note that both the round and the squashed seven-spheres with torsion have $\overset{\circ}{\nabla} T$ totally antisymmetric but non zero. Although the physical motivation is not clear, one may ask whether the condition $\overset{\circ}{\nabla} T = 0$ in Proposition 1 can be relaxed to total antisymmetry (on all four indices) for $\overset{\circ}{\nabla} T$. Interestingly, the answer depends on the solution to a problem in pure mathematics which, to the author's knowledge, remains unresolved. Taking both T and $\overset{\circ}{\nabla} T$ to be totally antisymmetric in equation (3.5), and setting $R_{ij} = 0$, one obtains

$$\overset{\circ}{R}_{ij} = \frac{1}{4} T^m{}_{ni} T^n{}_{jm} \quad .$$

Clearly $\overset{\circ}{R}_{ij}$ is non-negative, but this certainly does not imply compactification. (Myers' theorem requires that the eigenvalues be positive and bounded away from zero.) However, if one calculates the gradient of the scalar curvature, it is found (by judicious use of the antisymmetry properties) that $\overset{\circ}{\nabla}_i \overset{\circ}{R} = -2 \overset{\circ}{\nabla}{}^j \overset{\circ}{R}_{ji}$. The Bianchi identities then imply that $\overset{\circ}{R}$ is a positive constant. It is apparently unknown at present whether it is possible for a non-compact complete manifold with non-negative Ricci tensor to have a constant positive scalar curvature, assuming local indecomposability of course. This is related to an extension of the well-known Yamabe conjecture.

Finally, one may wish to consider replacing $\overset{\circ}{\nabla} T = 0$ by the equation $\nabla T = 0$. The consequences of this can be explored as follows. Let M be a Riemann-Cartan manifold which is RC locally reducible - that is, the restricted holonomy group of ∇ (not $\overset{\circ}{\nabla}$) acts reducibly. Let $x \in M$ and let U be an open neighbourhood of x. Let H'_x be an invariant subspace of the tangent space at x, and let H''_x be the orthogonal complement of H'_x. Then H''_x is also invariant, and in fact this splitting of the tangent spaces can be extended throughout U in a consistent and continuous way. We shall say that the torsion T splits holonomically if, for each x, $T(H'_x, H'_x) \subseteq H'_x$, $T(H''_x, H''_x) \subseteq H''_x$, and $T(H'_x, H''_x) = 0$. The following result now generalizes the local de Rham decomposition theorem to Riemann-Cartan manifolds, and should be compared with Theorem 3. (The proof is given in the Appendix).

PROPOSITION 6. (Local de Rham for RC manifolds). Let M be a connected, RC locally reducible Riemann-Cartan manifold such that the torsion splits holonomically. Then M is locally decomposable, regarded as a Riemannian manifold. If in addition $\nabla T = 0$, then M is locally decomposable, regarded as an RC manifold.

REMARK 10: If ∇T is not zero, then the torsion may not decompose, and so one has only a Riemannian decomposition.

It is now clear that $\overset{\circ}{\nabla} T = 0$ cannot be replaced by $\nabla T = 0$, because Proposition 6 means that local indecomposability does not imply RC local irreducibility, and this is what one needs in order for Schur's lemma to apply and for the proof to go through. In fact, all other parts of the proof of Proposition 1 can be suitably modified, and the result remains valid if $\nabla T = 0$, but only if T splits holonomically. But there is no physical motivation for this last assumption.

In conclusion, then, we see that Proposition 1 is very sensitive to modifications of the hypotheses.

6. CONCLUSION

In this work we have examined the foundations of the Freund-Rubin spontaneous compactification technique, and have indicated the form which an extension of these ideas to Riemann-Cartan manifolds could take. Our purpose has been to provide a framework which not only guides the construction of particular models (in the sense that these should be compatible with the hypotheses of Proposition 1 or some similar result),

but which also sheds some light on the whole question of compactification. Although our treatment has been primarily concerned with the Kaluza-Klein approach, many of the results apply also to the "field-theoretic limit" of superstring theories and possibly to other multidimensional theories. (This is why we have avoided, as far as possible, any assumptions as to the symmetry group of the internal manifold: in particular, Proposition 1 is independent of any such assumption.)

The specific technical problems which afflict Kaluza-Klein theories (chiral fermions, zero-mass modes - see Witten (1983) and Muzinich (1984)) need not be rehearsed here. We shall conclude instead by pointing out some more general problems which deserve greater attention.

It is often claimed as a virtue of Kaluza-Klein theories that they reduce gauge theories to gravitation, and that they explain the origin of gauge symmetries. This is somewhat dubious, however, because these theories postulate at the outset that the internal space has a non-trivial isometry group. Most manifolds, of course, do not have this property. In general relativity theory, non-trivial isometry groups are imposed only as a useful approximation. The electromagnetic gauge symmetry, however, is practically exact. From this point of view, the Kaluza Klein manifolds are thus highly "non-generic". The virtue of the Kaluza-Klein formulation of gauge theory is not so much that it explains the symmetry as that it may provide a route to an explanation. One could imagine high degrees of symmetry arising, for example, from quantum gravitational effects of the type which tend to reduce anisotropies in cosmology (Parker (1984)). Another line of approach is suggested by the work of Isenberg and Moncrief (1985), who show that, under certain conditions, a space-time must inevitably develop non-trivial isometries.

At the other extreme, the claim is often made that gravitation can be reduced to gauge theory or to the field theory of spin-2 particles. The analyses of Trautman (1980) and Penrose (1980), respectively, make it quite clear that these viewpoints are considerably, and perhaps grossly, oversimplified.

However, the most basic problem confronting all higher- dimensional theories is that of understanding the reason for the "factorization" of the universe into "internal " and "external" parts. Spontaneous compactification has the relatively modest aim of explaining the topological differences of the two factors in terms of their geometric differences - the point being that Myers' theorem (and related results such as Proposition 1) is not valid for pseudo- Riemannian manifolds. Thus, the internal space differs topologically from space-time because the latter has a time dimension and the former does not (and cannot, lest causality be violated). But in all this, the "factorization" is presumed to be given - it is certainly not explained.

We have already remarked, in Sec. 3, that the existence of a time dimension may be partly responsible for the factorization of the universe, since it implies that the full multidimensional space cannot be compact. But this is obviously a very incomplete explanation. No theory which makes use of multidimensional spaces can be considered complete <u>unless</u> it gives a detailed account of the origin of the internal/external dichotomy. It is sometimes stated that this problem can be resolved by considering non-trivial fibre bundles instead of product spaces, but this is in fact not correct. Not every manifold can be regarded as the bundle space of a fibre bundle - the structure must be imposed by means of a postulate which is hardly less arbitrary than the assumption that the space factorizes. What is required is a physically motivated scheme which splits the universe "spontaneously". The de Rham decomposition theorems and their generalizations may be of value here.

ACKNOWLEDGMENTS

The author is grateful to Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for the hospitality at the International Centre for Theoretical Physics, Trieste, where this work was completed. He also wishes to thank Dr. J.A. Hëlayel - Neto and Dr. O. Foda for several very helpful discussions.

APPENDIX

PROOF OF PROPOSITION 6

Here we shall only indicate those parts of the proof which differ from the corresponding parts in the Riemannian case; the remainder will be asserted without proof. (See Kobayashi and Nomizu (1963).)

Let H' and H'' be the distributions defined throughout U in the way described in the text. It may be shown that these distributions are differentiable and that, if X is any vector field, $\nabla_X(H') \subseteq H'$ and $\nabla_X(H'') \subseteq H''$. Thus if X and Y are vector fields belonging to H', $[X,Y]$ also belongs to H', since $[X,Y] = \nabla_X Y - \nabla_Y X - T(X,Y)$ and T splits holonomically. Thus H', and similarly H'', is involutive. Given the point x, one can therefore (by the Frobenius integrability theorem) find a submanifold M' generated by H' and containing x, and similarly for M'' generated by H''. It can now be shown that there exists a neighbourhood V around x in U which is of the form $V' \times V''$, where V' is an open neighbourhood of x in M', and similarly for V'' in M''. Let $k = \dim H'$, and let $\{x^1 \dots x^k, x^{k+1} \dots x^n\}$ be a coordinate system adapted to the splitting, **that** is, $\partial_1 \dots \partial_k$ belong to H', and $\bar{\partial}_{k+1} \dots \bar{\partial}_n$ belong to H''. We now show that the metric in V' is independent of the coordinates in V'', and vice versa.

The "off-diagonal" components of g, $g(\partial_i, \bar{\partial}_j)$ are obviously zero. To see that $\partial_i(g(\bar{\partial}_j, \bar{\partial}_m))$ is zero, we use $\nabla g = 0$ to write

$$\partial_i(g(\bar{\partial}_j, \bar{\partial}_m)) = g(\nabla_{\partial i}\, \bar{\partial}_j, \bar{\partial}_m) + g(\bar{\partial}_j, \nabla_{\partial i}\, \bar{\partial}_m) \quad . \qquad (A.1)$$

Now by definition of T we have

$$\nabla_{\partial i}\, \bar{\partial}_i - \nabla_{\bar{\partial} j}\, \partial_i - [\partial_i, \bar{\partial}_j] = T(\partial_i, \bar{\partial}_j) \quad .$$

The third term on the left vanishes since we are dealing with a coordinate basis, and the right-hand side vanishes since T splits holonomically. Thus, the first two terms are equal. But the first belongs to H'', and the second to H': hence, both must be zero. Substituting into (A.1), we find that the V''

part of the metric is independent of the V' coordinates. Similarly, the V' part of the metric is independent of the V" coordinates, and so V is the Riemannian product of V' and V".

Notice that the proof depends only on $\nabla g = 0$. Thus T will decompose in the same way if $\nabla T = 0$, and so, in this case V is also the Riemann-Cartan product of V' and V". This completes the proof.

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