RADIATIVE PROCESSES IN GAUGE THEORIES *

CALKUL Collaboration

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ABSTRACT

It is shown how the introduction of explicit polarization vectors of the radiated gauge particles leads to great simplifications in the calculation of bremsstrahlung processes at high energies. (author)

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I. INTRODUCTION

Bremsstrahlung processes play an important role in present-day high energy physics. Reactions like $e^+e^- \to e^+e^-\gamma$ and $e^+e^- \to \mu^+\mu^-\gamma$ are vital to precision tests of QED and to the determination of electroweak interference effects. Also, QCD processes like $e^+e^- \to$ 3 jets, 4 jets,... get contributions from subprocesses in which one or more gluons are radiated.

In the framework of perturbative quantum field theory, there is no fundamental problem associated with the evaluation of bremsstrahlung cross sections. In practice, however, the calculations turn out to be very lengthy when one uses the standard methods. Generally, very cumbersome and untransparent expressions result, unless one spends an enormous effort on simplifying the formulae.

In Sec.II, we will show, however, that single bremsstrahlung cross sections do have a remarkably simple structure, provided fermion masses can be neglected. This is usually the case for high energy reactions involving electrons, muons, and/or light quarks. We find that all these cross sections factorize into two factors, one of which is associated with the well-known "infrared factor" [1], while the second one is a suitable generalization of the non-radiative cross section.

Simple results should however be obtained in a simple way, and, in Sec. III, we describe the formalism, based on a covariant description of helicity amplitudes, which achieves this goal.

In Sec.IV, we give some applications of this formalism to multiple bremsstrahlung processes like $e^+e^- \to \mu^+\mu^- + n\gamma$ and $e^+e^- \to$ 4 jets. Finally, in Sec. V, we summarize our results.

II. SINGLE BREMSSTRAHLUNG[2)]

Consider first the process

$$e^+(p_+) + e^-(p_-) \rightarrow \mu^+(q_+) + \mu^-(q_-) + \gamma(k) \; , \tag{1}$$

where the momenta of the particles are given between parentheses, and let us introduce the following notation:

$$\begin{aligned} s &= (p_+ + p_-)^2 \; , \; t = (p_+ - q_+)^2 \; , \; u = (p_+ - q_-)^2 \; , \\ s' &= (q_+ + q_-)^2 \; , \; t' = (p_- - q_-)^2 \; , \; u' = (p_- - q_+)^2 . \end{aligned} \tag{2}$$

All particles being massless, we have furthermore

$$s + s' + t + t' + u + u' = 0. \tag{3}$$

To simplify the discussion, we shall assume first that the photon is emitted by the muons only. We then have to consider the two top Feynman diagrams of Fig.1.

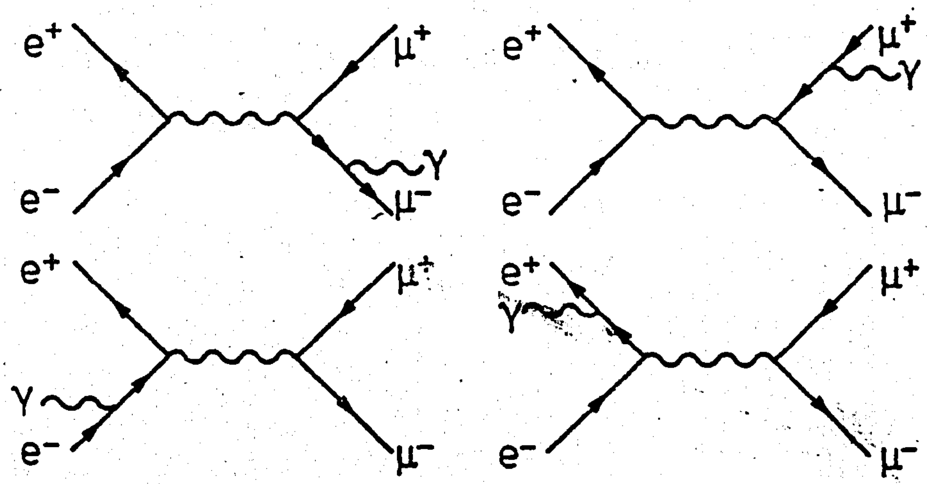

Fig.1: Feynman diagrams for $e^+e^- \rightarrow \mu^+\mu^-\gamma$.

Their corresponding Feynman amplitudes are given by

$$M_1 = \frac{ie^3}{2s(q_-k)} \bar{v}(p_+)\gamma^\mu u(p_-) \bar{u}(q_-) \not{\epsilon} (\not{q}_- + \not{k})\gamma_\mu v(q_+),$$
$$M_2 = \frac{ie^3}{2s(q_+k)} \bar{v}(p_+)\gamma^\mu u(p_-) \bar{u}(q_-)\gamma_\mu(-\not{q}_+ - \not{k}) \not{\epsilon} v(q_+). \qquad (4)$$

With the standard methods of covariant summation over the polarization degrees of freedom, one finds that

$$\sum_{pol} |M_1|^2 = - \frac{8e^6}{s^2(q_-k)} (tt' + uu' + st + su' + 2tu'). \qquad (5)$$

Note that one power of (q_-k) has been cancelled in the denominator. Similarly,

$$\sum_{pol} |M_2|^2 = - \frac{8e^6}{s^2(q_+k)} (tt' + uu' + su + st' + 2t'u),$$
$$2 \,\mathcal{R}e \sum_{pol} M_1 M_2^* = \frac{4e^6}{s^2(q_-k)(q_+k)} [(s+t+u')(ss'+tt'+uu'-2tu') + (s+t'+u)(ss'+tt'+uu'-2t'u) + 2s' (ss'-tu'-t'u)] . \qquad (6)$$

However, if one takes the trouble of combining these three expressions [Eqs. (5) and (6)] over a common denominator, a much simpler expression results:

$$\sum_{pol} |M_1 + M_2|^2 = 4e^6 \frac{s'}{(q_+k)(q_-k)} \frac{t^2+t'^2+u^2+u'^2}{ss'} . \qquad (7)$$

Note that this time the double pole in s has also been replaced by a single pole $(s^{-2} \rightarrow s^{-1})$.

Let us now compare Eq. (7) with the infrared limit formula. For $k \to 0$, we have

$$M_1 + M_2 = \frac{ie^3}{s}\left[\frac{(q_-\varepsilon)}{(q_-k)} - \frac{(q_+\varepsilon)}{(q_+k)}\right] \bar{v}(p_+)\gamma^\mu u(p_-)\, \bar{u}(q_-)\gamma_\mu v(q_+)\,, \tag{8}$$

and

$$\sum_{\rm pol} |M_1 + M_2|^2 = 8e^6 \frac{s'}{(q_+k)(q_-k)} \frac{t^2+u^2}{s^2} \,. \tag{9}$$

It is obvious that the formulae for hard and soft bremsstrahlung are very similar.

Hard bremsstrahlung cross sections can be written as a product of two factors:

1) an "infrared factor"[1), in this case

$$e^2 \frac{s'}{(q_+k)(q_-k)}\,, \tag{10}$$

which, of course, must be evaluated for $k \neq 0$, and

2) a suitable extension of the non-radiative cross section, which in this case is obtained by substituting

$$t^2 \to \frac{1}{2}(t^2 + t'^2)\,,\ u^2 \to \frac{1}{2}(u^2 + u'^2)\,,\ \ s^2 \to ss'. \tag{11}$$

For the complete process, which includes radiation from the electron lines, we have the four Feynman diagrams of Fig.1. We find analogously

$$\sum_{\rm pol} |M_1+M_2+M_3+M_4|^2 = S(k)\ 4e^4 \frac{t^2+t'^2+u^2+u'^2}{ss'}\,, \tag{12}$$

but now S(k) stands for the complete "infrared factor":

$$S(k) = -e^2\left[\frac{p_+}{(p_+k)} - \frac{p_-}{(p_-k)} - \frac{q_+}{(q_+k)} + \frac{q_-}{(q_-k)}\right]^2 \tag{13}$$

$$= e^2 \left[\frac{s}{(p_+k)(p_-k)} + \frac{s'}{(q_+k)(q_-k)} - \frac{t}{(p_+k)(q_+k)} - \frac{t'}{(p_-k)(q_-k)} + \frac{u}{(p_+k)(q_-k)} + \frac{u'}{(p_-k)(q_+k)} \right].$$

In Ref.2, other single bremsstrahlung processes were examined as well. They included $e^+e^- \to e^+e^-\gamma$, $e^+e^- \to \gamma\gamma\gamma$, $q\bar{q} \to q\bar{q}g$, $q\bar{q} \to q'\bar{q}'g$, $q\bar{q} \to ggg$, and $gg \to ggg$. Every time, the above mentioned factorization was found to hold. However, the amount of work which was neeeded required extensive use of algebraic manipulation programs on a computer. This was mainly a consequence of the large number of Feynman diagrams which were involved. The process $gg \to ggg$ has 25 of them! Clearly, a more efficient method is needed, as one can easily imagine that one day even higher order bremsstrahlung processes might be investigated.

III. HELICITY FORMALISM[3)]

Consider again the simple case of $e^+e^- \to \mu^+\mu^-\gamma$, with the photon being radiated from the muon line only. We can explicitly construct two photon polarizations orthogonal to the photon momentum k and to each other:

$$\begin{aligned} \varepsilon^{\parallel}_{\mu} &= N\,[(q_+k)q_{-\mu} - (q_-k)q_{+\mu}] , \\ \varepsilon^{\perp}_{\mu} &= N\,\varepsilon_{\mu\alpha\beta\gamma}\, q^{\alpha}_{+}\, q^{\beta}_{-}\, k^{\gamma}, \end{aligned} \tag{14}$$

where the normalization factor is

$$N = [2(q_+q_-)(q_+k)(q_-k)]^{-1/2}. \tag{15}$$

These two linear polarizations can be combined into circular polarizations

$$\varepsilon^{\pm}_{\mu} = 2^{-1/2}\,[\,\varepsilon^{\parallel}_{\mu} \pm i\varepsilon^{\perp}_{\mu}\,] . \tag{16}$$

But, in QED, only the combination $\not\epsilon^{\pm}$ appears in helicity amplitudes. It can effectively be written as

$$\not\epsilon^{\pm} = -2^{-3/2}\, N\, [\, \not k\, \not q_- \not q_+ (1 \pm \gamma_5) - \not q_- \not q_+ \not k\, (1 \mp \gamma_5)] \quad , \qquad (17)$$

where we have dropped terms proportional to $\not k\gamma_5$, which vanish in this case because of axial current conservation for massless muons.

Suppose we want to calculate the helicity amplitude $M(+,-,+,-,+)$, where the arguments indicate the helicities of the e^+, e^-, μ^+, μ^-, and γ. Because of the $\frac{1}{2}(1-\gamma_5)$ helicity projection operator for the spinor $v(q_+)$, only the second term of $\not\epsilon^+$ can contribute. Inserting $\not\epsilon^+$ in our previous expression for M_1 [Eq.(4)] will give zero because of the Dirac equation $\bar{u}(q_-)\not q_- = 0$. Hence, only M_2 contributes, and

$$M(+,-,+,-,+)$$
$$= -\frac{ie^3}{2s} N_q\, \bar{v}(p_+)\gamma^{\mu}\, (1-\gamma_5)u(p_-)\bar{u}(q_-)\gamma_{\mu}(\not p_+ + \not p_-)\not q_-(1-\gamma_5)v(q_+), \qquad (18)$$

with

$$N_q^{-1} = 4\, [(q_+q_-)(q_+k)(q_-k)\,]^{1/2}. \qquad (19)$$

Note that our choice of polarization vectors eliminated one diagram in this example and that the fermion propagator pole $(q_+k)^{-1}$ was cancelled. It reappeared, together with (q_-k), in the overall normalization factor N_q, but with the right power in view of the result (7) of Sec. II.

Formula (18) can further be simplified by eliminating the repeated index μ. To this end, we observe that, for any light-like vector q,

$$(1 \mp \gamma_5)\, v(q)\bar{v}(q)(1 \pm \gamma_5) = (1 \mp \gamma_5) \sum_{\text{hel}} v(q)\bar{v}(q)(1 \pm \gamma_5)$$
$$= 2(1 \mp \gamma_5)\not q. \qquad (20)$$

Hence,

$$
\begin{aligned}
&M(+,-,+,-,+) \\
&= -\frac{ie^3}{2s} N_q \frac{\bar{v}(p_+)\gamma^\mu(1-\gamma_5)u(p_-)\bar{u}(p_-)\not{a}(1-\gamma_5)u(q_-)\bar{u}(q_-)\gamma_\mu(\not{p}_+ + \not{q}_-)\not{q}_-(1-\gamma_5)v(q_+)}{\bar{u}(p_-)\not{a}(1-\gamma_5)u(q_-)} \\
&= -\frac{2ie^3}{s} N_q \frac{\bar{v}(p_+)\gamma^\mu \not{p}_- \not{a}\ \not{q}_-\ \gamma_\mu(\not{p}_+ + \not{p}_-)\not{q}_-(1-\gamma_5)v(q_+)}{\bar{u}(p_-)\ \not{a}(1-\gamma_5)u(q_-)} \qquad (21) \\
&= \frac{4ie^3}{s} N_q \frac{\bar{v}(p_+)\not{q}_-\not{a}\ \not{p}_-\not{p}_+\not{q}_-(1-\gamma_5)v(q_+)}{\bar{u}(p_-)\not{a}(1-\gamma_5)u(q_-)} .
\end{aligned}
$$

Choosing $a = p_+$ yields

$$
\begin{aligned}
M(+,-,+,-,+) &= 4ie^3 N_q u \frac{\bar{v}(p_+)\not{q}_-(1-\gamma_5)v(q_+)}{\bar{u}(p_-)\not{p}_+(1-\gamma_5)u(q_-)} \qquad (22) \\
&\doteq 2^{1/2} e^3 \left[\frac{s'}{(q_+k)\ (q_-k)} \right]^{1/2} \frac{u}{(ss')^{1/2}} ,
\end{aligned}
$$

where in the last line we neglected an irrelevant overall phase factor.

The evaluation of the remaining non-zero helicity amplitudes proceeds along the same lines, merely resulting in a replacement of u by u', t, or t' in Eq. (22). The previous result (7) for the total squared matrix element is then easily obtained by squaring the helicity amplitudes and by adding.

If, in addition to radiation from the muon line, one includes the radiation from the electron line, a slight complication has to be faced. For diagrams M_3 and M_4 one wants to use an expression for $\not{\epsilon}^{\pm}$ in terms of the momenta p_+ and p_- instead of q_+ and q_- in order to have similar simplifications. Since a photon can only have two polarizations, the polarization vectors must be related by a phase (up to gauge terms):

$$\varepsilon_\mu^\pm(q_+,q_-) = e^{\pm i\phi}\, e_\mu^\pm\,(p_+,p_-) + A_\pm k_\mu\ . \tag{23}$$

The quantities $A_\pm$ are irrelevant because of separate electron and muon current conservation. Also,

$$\begin{aligned} e^{\pm i\phi} &= -(\varepsilon_p^\mp \varepsilon_q^\pm) = N_p N_q \,\mathrm{Tr}\,[\not p_+ \not p_- \not k \not q_- \not q_+ \not k (1 \mp \gamma_5)]\ , \\ N_p^{-1} &= 4\,[(p_+p_-)(p_+k)(p_-k)]^{1/2}. \end{aligned} \tag{24}$$

Including M_3 and M_4 then yields

$$M(+,-,+,-,+) \triangleq 4e^3\,[s'\,N_q + s\,N_p e^{i\phi}]\ \frac{u}{(ss')^{1/2}}\ , \tag{25}$$

and the squared absolute value of the bracketed expression is easily seen to be proportional to the complete infrared factor S(k) [Eq. (13)].

This example shows that the use of explicit photon polarization vectors leads, in a covariant way, to simple expressions for helicity amplitudes. The factorization property, which was found in Sec. II for bremsstrahlung cross sections, is seen to be present already at the level of the amplitude itself. It is, of course, much easier to find at the amplitude level than to dig it out of a lengthy cross section formula!

IV. MULTIPLE BREMSSTRAHLUNG [4)]

Two key features of single bremsstrahlung cross sections, which are largely responsible for their simplicity, are factorization and the absence of double poles in the invariants. A detailed analysis of multiple bremsstrahlung processes shows that this is still the case for certain helicity amplitudes. E.g. , for

$$e^+(p_+) + e^-(p_-) \to \mu^+(q_+) + \mu^-(q_-) + \gamma(k_1) + \ldots + \gamma(k_n)\ , \tag{26}$$

one finds that

$$|M(+,-,+,-,+,\ldots,+)|^2 = S(k_1)\ldots S(k_n)\, 2e^4 \frac{u^2}{ss'}, \tag{27}$$

with $S(k_i)$ given by Eq. (13). In this case, the simplicity is due to the fact that all photons have the same helicity.

When this is not the case, the results are more complicated. Consider, e.g., the process

$$e^+(p_+) + e^-(p_-) \to \bar{q}(k_3) + q(k_4) + g(k_1) + g(k_2), \tag{28}$$

which is one of the subprocesses for $e^+e^- \to 4$ jets. It is still adventageous to introduce explicit polarization vectors for the gluons and to eliminate the repeated index in the various helicity amplitudes. Nevertheless, for the $M(-,+,-,+,-,+)$ amplitude where the arguments refer to the $e^+, e^-, \bar{q}, q, g(k_1)$, and $g(k_2)$ helicities, a relatively long expression results. We find that the most efficient way to evaluate this expression is to introduce explicit forms for the spinors and to compute the helicity amplitude for a given point in phase space as a complex number.

Suppose that we go to e^+e^- c.m. frame with the z-direction along $\vec{p}_+$, and that we introduce the notation

$$k_\pm = k_o \pm k_z,\; k_\perp = k_x + ik_y = |k_\perp| e^{i\phi_k}, \tag{29}$$

for any light-like vector k. Choosing a representation for the γ-matrices for which

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{30}$$

we can take

$$u_+(k) = v_-(k) = \begin{pmatrix} \sqrt{k_+} \\ \sqrt{k_-}\, e^{i\phi_k} \\ 0 \\ 0 \end{pmatrix},\quad u_-(k) = v_+(k) = \begin{pmatrix} 0 \\ 0 \\ -\sqrt{k_-}\, e^{-i\phi_k} \\ \sqrt{k_+} \end{pmatrix}. \tag{31}$$

Clearly, the first spinor is an eigenstate of $1+\gamma_5$, and the second one of $1-\gamma_5$.

With these formulae, it is easily seen that

$$\bar{u}(k_i)(1+\gamma_5)u(k_j) = [\bar{u}(k_j)(1-\gamma_5)u(k_i)]^* = 2\,\frac{k_{j\perp}Z^*_{ij}}{k_{j-}\sqrt{k_{i+}k_{j+}}}\,, \tag{32}$$

where

$$Z_{ij} = k_{i+}k_{j-} - k^*_{i\perp}k_{j\perp}\,, \quad i,j=1,2,3,4. \tag{33}$$

By making repeated use of the relation (20), any spinorial expression not containing repeated indices can be reduced to expressions of the type (32). Hence, the helicity amplitudes can be expressed in terms of the quantities Z_{ij}, which are simple functions of the various momenta.

In this way, we find that

$$|M(-,+,-,+,-,+)|^2$$

$$= \frac{N^2-1}{128N}\,\frac{Q_f^2e^4g^4}{A}\,\frac{(k_3k_4)}{E^2(k_1k_2)k^2_{1+}k^2_{2-}k_{3+}k_{4-}}\,[(N^2-2)|c_1+c_2|^2 + N^2|c_1-c_2|^2], \tag{34}$$

with

$$A = (k_1k_2)(k_1k_3)(k_1k_4)(k_2k_3)(k_2k_4)(k_3k_4),$$

$$c_1 = |\alpha|^2 + \frac{(k_2k_4)}{2E(E-k_{30})}\alpha\beta + \frac{(k_1k_3)}{2E(E-k_{40})}\alpha^*\gamma\,,$$

$$c_2 = \beta\gamma + \frac{(k_1k_4)}{2E(E-k_{30})}\alpha\beta + \frac{(k_2k_3)}{2E(E-k_{40})}\alpha^*\gamma\,, \tag{35}$$

$$\alpha = Z_{12}Z_{34} - Z_{14}Z_{32} ,$$

$$\beta = Z_{32}^{*} [Z_{12}^{*} + Z_{14}^{*}] ,$$

$$\gamma = Z_{14} [Z_{12} + Z_{32}] .$$

In this formula, E denotes the beam energy, g the SU(N) gauge coupling constant, and Q_f the fractional quark charge.

All other non-zero helicity amplitudes for $e^+e^- \to q\bar{q}gg$ and $e^+e^- \to q\bar{q}\,q\bar{q}$ can be calculated in this way.

Because of the relative simplicity of our formulae, we were able to establish that the 4-jet cross section is proportional to $R = \sum_f Q_f^2$. The terms proportional to $(\sum_f Q_f)^2$ dropped out after momentum symmetrization.

It is our hope that the above procedure can be systematically applied to all multiple bremsstrahlung processes, especially to those for which the standard manipulations of covariant summation over polarization degrees of freedom would become prohibitively lengthy.

V. CONCLUSIONS

We have shown that the introduction of explicit polarization vectors for the gauge particles in the calculation of helicity amplitudes for bremsstrahlung processes at high energies has many advantages:

1) only a limited number of Feynman diagrams contribute to a given helicity configuration;
2) the procedure is manifestly covariant;
3) ghost contributions associated with unphysical polarization degrees of freedom do not have to be considered;
4) many helicity amplitudes are easily found to factorize, exhibiting "infrared factors";

5) repeated indices can be eliminated;

6) multiple bremsstrahlung processes become calculable by the introduction of explicit spinors.

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REFERENCES

1) D.R. Yennie, S.C. Frautschi, and H. Suura, Ann.Phys. (NY) 13 (1961) 379.

2) F.A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, and Tai Tsun Wu, Phys. Lett. 103B (1981) 124.

3) P. De Causmaecker, R. Gastmans, W. Troost, and Tai Tsun Wu, Phys. Lett.105B (1981)215, and DESY preprint 81-050 (to be published in Nucl.Phys.B); F.A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, W. Troost,and Tai Tsun Wu, Leuven preprint KUL-TF-81/18 (to be published in Nucl. Phys.B).

4) D. Danckaert, P. De Causmaecker, R. Gastmans, W. Troost, and Tai Tsun Wu, Leuven preprint KUL-TF-82/1 (to be published in Phys. Lett. B).