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# A TRANSIENT ANALYSIS OF A BUNCHED BEAM FREE ELECTRON LASER*

Jiunn-Ming Wang and Li-Hua Yu
National Synchrotron Light Source
Brookhaven National Laboratory
Upton, N. Y. 11973

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## Abstract

We studied the problem of the bunched beam operation of a free electron laser. Assuming the electron beam to be initially monoenergetic, the Maxwell-Vlasov equations describing the system reduce to a third order partial differential equation for the envelope of the emitted light. The Green's function corresponding to an arbitrary shape of the electron bunch, which describes the transient behavior of the system, is obtained. We use the Green's function to discuss the start up problem as well as the power output and the power spectrum of a self-amplified spontaneous emission.

## 1. Introduction

It has been suggested [1,2] that the FEL collective instability [2-6] provides a feasible source of producing high intensity coherent light. For such a self-stimulated FEL scheme, the understanding of the transient behavior of the radiation is a matter of utmost importance. A central question is: how does the initially uncorrelated noise which is inherent in the system evolve and produce coherent light? The answer to this question provides us a way for calculating the total power output and the final light packet shape. We attempt in this paper to answer these questions within a one-dimensional classical model [7,3].

We assume that the electron beam is a bunched beam, as is the case in a linac or a storage ring, instead of an infinitely long coasting beam. We ignore the effects of space charge, and work in the Coulomb gauge.



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The paper is organized as follows: In section 2, we derive a partial differential equation which must be satisfied by the slowly varying envelope function of the light packet which is evolving in a FEL. Then, in section 3, we derive the Green's function and the Green's theorem for the above equation. The Green's formalism provides us with a convenient method of analyzing the transient behavior. In section 4, we discuss the start up problem of a self-stimulated FEL and calculate its power spectrum and total output.

## 2. Envelope Wave Equation of a FEL

We demonstrate in this section that the slowly varying envelope function of the emitted light in a FEL satisfies a linear partial differential equation, third order in time and first order in space. The envelope function of the perturbed electron beam also satisfies the same equation. The basis of the derivation is a set of self-consistent linearized Vlasov-Maxwell equations.

Let us consider an initially monoenergetic highly relativistic bunched electron beam moving in the positive z direction through a periodic left-hand circularly polarized wiggler. The vector potential of the wiggler is given by $\vec{A}_w = A_w(\hat{e}_- e^{ik_w z} + c.c.)/\sqrt{2}$, where $\hat{e}_\pm = (\hat{x} + i\hat{y})/\sqrt{2}$, and a carat on a letter indicates a unit vector.

We denote the energy (normalized to $mc^2$) and the corresponding longitudinal velocity of the electron by $\gamma$ and v, respectively. Their initial values are $\gamma_o$ and $v_o$. From the conservation law of the transverse canonical momentum, we obtain the transverse velocity of the electron: $\vec{v}_T = -e\vec{A}_w(z)/mc\gamma$. Using this result it can be shown that $\gamma$ and v are related by, $c-v = c(1+K^2)/2\gamma^2 + O(1/\gamma^4)$, where the wiggler parameter $K = eA_w/mc^2$.

The wave number of an electron with energy $\gamma_o$ resonating with the ponderamotive potential is $k_r = k_w c/(c-v_o)$. Such an electron emits spontaneously left circularly polarized (positive helicity) photons with wave number $k_o = k_r - k_w$ and frequency $k_o c = k_r v_o$.

Describing the electric field by $\vec{\varepsilon} = \hat{e}_+\varepsilon(z,t)/\sqrt{2} + \text{c.c.}$ and the transverse electron current density by $\vec{j}(z,t) = \hat{e}_+ j(z,t)/\sqrt{2} + \text{c.c.}$, the energy gain of an electron is given by

$$\dot{\gamma} = e\vec{v}_T \cdot \vec{\varepsilon}/mc^2 = eK\,[e^{ik_w z}\,\varepsilon(z,t) + \text{c.c.}]/2mc\gamma = \Gamma(\zeta,t)/\gamma, \quad (1)$$

where, for later convenience, we have introduced a variable $\zeta = z - v_o t$, and the last part of eq. (1) defines the function $\Gamma(\zeta,t)$. Maxwell's wave equation relating $\vec{\varepsilon}$ and $\vec{j}$ is,

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varepsilon = -\frac{4\pi}{c}\,j \quad (2)$$

If the fractional change of the electron energy, $(\gamma-\gamma_o)/\gamma_o$, is small in one pass through the wiggler, the generated light of interest will propagate forward with frequency near $k_o c$. The modulation of this carrier frequency depends on the details of the photon-electron interaction. It is the purpose of this paper to explore this modulation.

Keeping only the forward propagating component of $\varepsilon$, eq. (2) becomes

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\varepsilon = -\frac{2\pi}{c}\,j \quad (3)$$

We are now ready to define the envelope functions associated with $\varepsilon$ and $j$. They are given, respectively, by $E(\zeta,t) = \exp[-ik_o(z-ct)]\,\varepsilon(z,t)$, and $J(\zeta,t) = \exp[-ik_o(z-ct)]j(z,t)$. In terms of these functions, eq. (3) becomes

$$\left[\frac{\partial}{\partial t} + (c-v_o)\frac{\partial}{\partial \zeta}\right] E(\zeta,t) = -2\pi J(\zeta,t) \quad (4)$$

We discuss next the Vlasov equation describing the electron propagation. Letting $\Psi(z,\gamma,t)$ be the electron distribution function and noting that $\dot{z} = v-v_o$, (a dot on a letter indicates a differentiation with respect to time), we can write

$$\frac{\partial\Psi}{\partial t} + (v-v_o)\frac{\partial\Psi}{\partial z} + \dot{\gamma}\frac{\partial\Psi}{\partial\gamma} = 0 \; . \tag{5}$$

This equation is equivalent to the following integral equation,

$$\Psi(\zeta,\gamma,t) = f_o(\zeta)\,\delta(\gamma-\gamma_o) - \frac{1}{\gamma}\int_o^t dt'\;\Gamma[\zeta-(v-v_o)/(t-t'),\; t']\frac{\partial\Psi[\zeta-(v-v_o)(t-t'),\gamma,t']}{\partial\gamma} \tag{6}$$

where $f_o(\zeta)\delta(\gamma-\gamma_o)$ is the initial distribution, and $\Gamma$ is defined in eq. (1). We adopt the normalization that the peak value of the unperturbed electron bunch shape function $f_o(\zeta)$ is equal to unity, and denote the peak volume density of the unperturbed electron bunch by $n_o$. The transverse electron current density is related to $\Psi$ by $\vec{j} = en_o\int d\gamma\vec{v}_T\Psi(\zeta,\gamma,t)$, or

$j(\zeta,t) = -ecn_oK\exp(-ik_wz)\int d\gamma\Psi(\zeta,\gamma,t)/\gamma.$

We solve eq. (6) to the first order in perturbation theory. To do this, iterate eq. (6) once, and then operate with $\int d\gamma/\gamma$ on the resulting equation. Now express the result in terms of E and J and then simplify using the identity $k_wz + k_o(z-ct) = k_r\zeta$. We obtain

$$J(\zeta,t) = -\frac{ecn_oK}{\gamma_o}e^{-ik_r\zeta}f_o(\zeta) - \frac{\alpha}{2\pi}f_o(\zeta)\int_o^t dt'[1+i\omega_w(t-t')]\,E(\zeta,t), \tag{7}$$

where $\alpha = 2\pi\, e^2n_oK^2/m\gamma_o^3$, and $\omega_w = k_wc$. It follows from eq. (7) that

$$\dot{J}(\zeta,t) = -\frac{\alpha}{2\pi}f_o(\zeta)[E(\zeta,t) + i\omega_w\int_o^t dt'\;E(\zeta,t')], \text{ and} \tag{8}$$

$$\ddot{J}(\zeta,t) = -\frac{\alpha}{2\pi}f_o(\zeta)\;[\dot{E}(\zeta,t) + i\omega_wE(\zeta,t)]. \tag{9}$$

We can now eliminate E or J from equations (4) and (9). The results are

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial}{\partial t} + (c-v_o)\frac{\partial}{\partial \zeta}\right] E(\zeta,t) - \alpha f_o(\zeta)\left[\frac{\partial}{\partial t} + i\omega_w\right] E(\zeta,t) = 0, \text{ and} \tag{10}$$

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial}{\partial t} + (c-v_o)\frac{\partial}{\partial \zeta}\right] D(\zeta,t) - \alpha f_o(\zeta)\left[\frac{\partial}{\partial t} + i\omega_w\right] D(\zeta,t) = 0, \tag{11}$$

where $D(\zeta,t) = J(\zeta,t)/f_o(\zeta)$.

Equation (10) for the envelope function of the laser light is the foundation of the following discussion.

## 3. Green's Function and the Initial Value Problem

In this section we use the Green's function formalism to investigate the transient behavior of our FEL model. We construct the Green's function corresponding to equation (10), and then use the Green's theorem to express $E(\zeta,t)$ in terms of the appropriate initial conditions. The functions E, $\dot{E}$, $\ddot{E}$ and J evaluated at t = 0 will be denoted by $E_o(\zeta)$, $\dot{E}_o(\zeta)$, $\ddot{E}_o(\zeta)$ and $J_o(\zeta)$.

The Green's function $G(\zeta,\zeta',t-t')$ of eq. (10) is defined by

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial}{\partial t} + (c-v_o)\frac{\partial}{\partial \zeta}\right] G - \alpha f_o(\zeta)\left[\frac{\partial}{\partial t} + i\omega_w\right) G = \delta(\zeta-\zeta')\,\delta(t-t') \tag{12}$$

To solve this equation, we first Fourier decompose $G(\zeta,\zeta',t)$ in time. The resulting Fourier components satisfy an ordinary first order differential equation is $\zeta$, which can readily be solved. The resulting expression for the Green's function is

$$G(\zeta,\zeta' t) = -\frac{\theta(\zeta-\zeta')}{2\pi(c-v_o)}\int_{-\infty}^{\infty}\frac{d\Omega}{\Omega^2}\, e^{-i\Omega t}\exp\left[\frac{i}{c-v_o}\left\{\Omega(\zeta-\zeta') + \frac{\alpha}{\Omega^2}(\Omega-\omega_w)\int_{\zeta'}^{\zeta}dx\, f_o(x)\right\}\right] \tag{13}$$

The presence of the Heaviside step function $\theta(\zeta-\zeta')$ in this equation is a reminder that the light can only propagate in the forward direction. Causality demands that the contour of integration in eq. (13) be above all singularities in the $\Omega$ plane.

Assuming that the function $f_o(x)$ is symmetric in x, $f_o(-x) = f_o(x)$, the following reciprocity relation follows directly from eq. (13):

$$G(-\zeta',-\zeta,t) = G(\zeta,\zeta',t). \tag{14}$$

From eqs. (10), (12) and (14), we obtain the Green's theorem:

$$E(\zeta,t) = \int d\zeta' \; [\ddot{E}_o(\zeta') + \dot{E}_o(\zeta')\{\frac{\partial}{\partial t} - (c-v_o)\frac{\partial}{\partial \zeta'}\} + E_o(\zeta')\{\frac{\partial^2}{\partial t^2} - (c-v_o)\frac{\partial^2}{\partial t \partial \zeta'} - \alpha f_o(\zeta')\}] \; G(\zeta,\zeta',t) \tag{15}$$

This equation can be simplified by using eqs. (4) and (8). One obtains,

$$E(\zeta,t) = \int d\zeta' \; [-2\pi J_o(\zeta')\frac{\partial}{\partial t} + E_o(\zeta')\frac{\partial^2}{\partial t^2}] \; G(\zeta,\zeta',t). \tag{16}$$

This is the final form of the Green's theorem we shall use in the next section. We emphasize that this equation is equivalent to the linearized Vlasov-Maxwell equations.

Let us end this section by elaborating on the Green's function corresponding to a coasting beam.

From translational invariance, the Green's function for a coasting beam is of the form $G(\zeta,\zeta',t) = G(\zeta-\zeta,t)$. The Green's function can be obtained by simply setting $f_o(x) = 1$ in equation (13). After a little manipulation, the resulting formula can be expressed as,

$$G(\zeta,t) = -\frac{1}{2\pi}\frac{1}{(2\rho\omega_w)^2}\int dk \; e^{ik\zeta}\; [\frac{e^{-i\lambda_1 \cdot 2\rho\omega_w t}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + (\text{permutations})] \tag{17}$$

where $\lambda_{1,2,3} = \lambda_{1,2,3}$ (k) are the solutions of the following well-known cubic equation in the form of ref. [2],

$$\lambda^3 - \delta(k)\,\lambda^2 + 2\rho\lambda - 1 = 0. \tag{18}$$

In eq. (18), $\rho$ is the Pierce gain parameter defined by $(2\rho\omega_w)^3 = \alpha\omega_w$, and the detuning parameter is given by $\delta(k) = (c-v_o)k/2\rho\omega_w$.

## 4. Start Up Problem in the Self-Stimulated High Gain FEL

It is our purpose in this section to calculate the power spectrum and the total power output one can obtain in a self-stimulated FEL. In a self-stimulated FEL [1], the initial spontaneous radiation emitted by the randomly distributed electrons is amplified through FEL coherent instabilities.

The starting point is the Green's relation given in eq. (16). Let us assume that the FEL does not have an externally stimulating laser, and therefore $E_o(\zeta) = 0$. We also assume that the FEL is started purely from the shot noise inherent in the electron beam. The shot noise is represented as a sum of random delta functions (see eq. (25)).

The contents of this section is restricted to the case of a coasting electron beam. The bunched beam case will be discussed elsewhere.

Defining the Fourier components of $E(\zeta,t)$, $J_o(\zeta)$ and $G(\zeta,t)$ by $E(\zeta,t) = \int dk\, \tilde{E}(k,t)\, e^{ik\zeta}/\sqrt{2\pi}$, $J_o(\zeta) = \int dk\, \tilde{J}_o(k)\, e^{ik\zeta}/\sqrt{2\pi}$ and $G(\zeta,t) = \int dk\, \tilde{G}(k,t)\, e^{ik\zeta}/\sqrt{2\pi}$, equation (16) can be written as, (recall that we set $E_o = 0$).

$$\tilde{E}(k,t) = -2\pi\, \tilde{J}_o(k)\, \frac{\partial}{\partial t}\, \tilde{G}(k,t)\ . \tag{19}$$

The power spectrum and the total power density are given, respectively, by

$$P(k,t) = \frac{c}{4\pi}\left|\tilde{E}(k,t)\right|^2 = c\pi\left|\tilde{J}_o(k)\right|^2\left|\frac{\partial}{\partial t}\,\tilde{G}(\zeta,t)\right|^2 \ , \text{ and} \tag{20}$$

$$P_{tot}(t) = \int dk\ P(k,t). \tag{21}$$

Let us suppose that $\rho << 1$, and thererore that the linar term in $\lambda$ can be ignored in eq. (18). One can then find the following approximate solutions for the equation: $\lambda_1 = \exp(i2\pi/3) + \delta/3 + \delta^2 \exp(-i2\pi/3)$, $\lambda_2 = \lambda_1^*$ and $\lambda_3 = 1$. $\lambda_1$ is the normalized eigenvalue of an unstable mode, the $\lambda_2$-mode is damped and $\lambda_3$-mode is stationary. $\lambda_1$-mode is the only mode which can give rise to high gain. We, therefore, approximate eq. (17) by keeping only the term with $\lambda_1$ in the exponent. As a result, one has

$$\left|\frac{\partial}{\partial t} G(k,t)\right|^2 = \frac{1}{4\pi^2} \cdot \frac{1}{9} \frac{1}{(2\rho\omega_w)^2} e^{2\sqrt{3}\,\rho\omega_w t} e^{-k^2/2\sigma_k^2} , \tag{22}$$

where the factor 1/9 comes from $\left|\lambda_1/\{\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\}\right|^2 = 1/9$, and $\sigma_k$ is the r.m.s bandwidth of the Green's function given by

$$\sigma_k(t) = k_r[3\sqrt{3}\,\rho/\omega_w t]^{1/2} \tag{23}$$

Since the bandwidth of the shot noise power $\langle|J_o(k)|^2\rangle$ is infinite, it can be seen from eq. (20) that the bandwidth of the emitted light power, $\langle P(k,t)\rangle$, is equal to $\sigma_k$. Here the notation $\langle\ldots\rangle$ is used to indicate an average over the shot noise. The correlation length of the emitted light is given, from Wiener-Khintchin theorem, by

$$\sigma_\zeta(t) = 1/\sigma_k(t) . \tag{24}$$

Notice that the bandwidth decreases in time like $1/\sqrt{t}$ in contrast to the case of a spontaneous radiation where it decreases like $1/t$.

We calculate next the shot noise power.

Taking into account the discrete nature of the electron, the initial transverse current density is given by

$$J_o(\zeta) = -ecn_oK\, e^{-ik_r\zeta}\, \frac{1}{\gamma_o}\frac{1}{N} \sum_{\ell=1}^{N} \delta(\zeta-\zeta_\ell) \tag{25}$$

where N is the total number of particles in the electron beam, and $\zeta_\ell$ is the initial position of the $\ell$-th electron. Let us denote by L the electron beam length, $L \to \infty$, and by $n_\zeta = N/L$ the line density. The $\zeta_\ell$ in eq. (25) is, for each $\ell$, a random number with uniform probability over the beam length L. $\zeta_\ell$'s corresponding to different $\ell$ are assumed to be uncorrelated. Now we Fourier expand eq. (25) in the interval L, using the statistical properties of $\zeta_\ell$ mentioned above, and using the standard method of mode number density counting, while letting $L \to \infty$ and keeping $n_\zeta$ fixed, we obtain

$$\langle |J_o(k)|^2 \rangle = \frac{2\pi}{n_\zeta \gamma_o^2} \cdot (ecn_oK)^2 \tag{26}$$

Taking the average of eq. (20) and using (22) and (26), we obtain

$$\langle P(k,t) \rangle = \frac{1}{18\pi}\, \frac{\rho}{n_\zeta}\, P_e\, e^{2\sqrt{3}\,\rho\omega_w t}\, e^{-k^2/2\sigma_k^2} \tag{27}$$

where the electron power density is given by $P_e = (n_o mc^2 \gamma_o)c$. The average total power density can be obtained by integrating (27) over k. The result is

$$\langle P_{tot} \rangle = \frac{1}{9}\, \frac{1}{\sqrt{2\pi}\, n_\zeta \sigma_\zeta}\, \rho\, P_e\, e^{2\sqrt{3}\,\rho\omega_w t} \tag{28}$$

## 5. Conclusions

We have presented a method of treating the transient behavior of a FEL within the context of a linearized one-dimensional model. Our treatment consists of converting the Vlasov-Maxwell equations, which describe the evolution of the electromagnetic field and the electron current, into a single partial differential equation satisfied by the slowly varying envelope fucntion of the emitted light. The impulse response of the FEL device is represented by the Green's function of the partial differential equation. The bandwidth $\sigma_k$ of the FEL device is given in eq. (23).

We then applied this formalism to study the power output of a self-stimulated FEL which starts from the electron beam shot noise as the input. Since the bandwidth of the shot noise is $\infty$, the bandwidth of the output light is determined entirely by the FEL to be $\sigma_k$. The correlation length of the light is, then, given by $\sigma_\zeta = 1/\sigma_k$.

Notice that in eq. (28), $\langle P_{tot} \rangle / P_e$ is inversely proportional to $n_\zeta \sigma_\zeta$, the number of electrons in a correlation length, instead of the number of electrons in a wavelength as suggested in ref. [1,2]. This has an important implication for the determination of the optimal wiggler length of a self-stimulated FEL. An interesting analogy to this result is that of stochastic cooling device [8]. Where, assuming the beam particles to be uncorrelated, the cooling rate per turn around the storage ring is proportional to the bandwidth of the cooling device divided by the line density.

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