UWThPh-1986-4

QUANTIZATION IN PRESENCE OF EXTERNAL SOLITON FIELDS

H. Grosse and G. Karner*

Institut für Theoretische Physik

Universität Wien

Abstract

Quantization of a fermi field interacting with an external soliton potential is considered. Classes of interactions leading to unitarily equivalent representations of the canonical anticommutation relations are determined. Soliton-like potentials compared to trivial ones yield inequivalent representations.

*) Supported in part by "Fonds zur Förderung der wissenschaftlichen Forschung in Österreich", Projekt Nr. P5588.

18. Feb. 1986

INTRODUCTION

Quantization of fermions interacting with solitons has attracted people during the last few years, since fractional charged states may occur [1]. Different approaches have been advocated (for a review see [2]); the simplest situation with external soliton fields has been studied extensively too [3].

Similar problems describing the interaction of electrons with external electromagnetic fields have been treated in a rigorous way [4]. In particular scalar potentials decreasing rapidly enough at infinity allow the application of a Bogoliubov transformation which maps the free electron-Fock representation to the representation of the interacting field. For a large class of potentials, the Shale-Stinespring-Berezin [5] criterion shows that the Bogoliubov transformation yields a unitary mapping and the Furry picture holds.

We closely follow the above mentioned work dealing with electromagnetic interactions, but treat the one-dimensional case with potentials having either trivial asymptotics $v_I(x) \xrightarrow[x\to\pm\infty]{} m$ or nontrivial "solitonic" asymptotics $v_{II} \xrightarrow[x\to\pm\infty]{} \pm m$.

We study representations of the algebra of operators $a(f)$, $a^\dagger(f)$, which satisfy the canonical anticommutation relations

$$\{a(f),a(g)\} = 0 , \qquad \{a(f),a^\dagger(g)\} = (f,g)\, 1 , \qquad (1)$$

where f denotes a two component wave function $f \in H = L^2(\mathbb{R}) \otimes \mathbb{C}^2$ and (f,g) is the scalar product of f and g in H.

By comparing representations related to first quantized Dirac operators we determine classes of potentials belonging to unitarily equivalent representations. Comparing a problem with trivial asymptotics v_I to a soliton situation v_{II} yields inequivalent representations; therefore a discussion of charge quantum numbers (which will be discussed separately [6]) has to be done with care and a regularization procedure has to be used.

BOGOLIUBOV TRANSFORMATION

We start with the self-adjoint Dirac operator

$$H_0 = \alpha \frac{1}{i}\frac{d}{dx} + \beta m \,\mathrm{th}\, x = \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix} , \qquad A^\dagger = \frac{d}{dx} + m \,\mathrm{th}\, x \,, \tag{2}$$

acting on H and use as a representation for $\alpha = -\sigma_2$ and for $\beta = \sigma_1$, where σ_i's are Pauli matrices. There are two linear independent solutions to the Dirac equation (for fixed energy $|E_k| \geq m$), which correspond to particles moving from left to right and vice versa:

$$f_\pm^{(1)}(k,x) = \Theta(k) \frac{e^{ikx}}{\sqrt{4\pi}} \begin{pmatrix} 1 \\ \pm \dfrac{-ik + m \,\mathrm{th}\, x}{|E_k|} \end{pmatrix} \tag{3a}$$

$$f_\pm^{(2)}(k,x) = \Theta(k) \frac{e^{-ikx}}{\sqrt{4\pi}} \begin{pmatrix} 1 \\ \pm \dfrac{ik + m \,\mathrm{th}\, x}{|E_k|} \end{pmatrix} \tag{3b}$$

where $E_k^2 = k^2 + m^2$, and $\pm$ indicates positive and negative energy solutions. Beside the continuous spectrum there exists <u>one</u> zero energy bound state solution

$$f_s(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \dfrac{1}{\mathrm{ch}\, x} \end{pmatrix} . \tag{3c}$$

Spectral resolution of H_0 defines projection operators P_+^0, P_-^0 and P_s^0 onto the positive, negative respectively zero energy subspace of H. The CAR (1) may therefore be split into parts by defining

$$b(\hat{f}_+) = a(\hat{f}_+) \,, \qquad c(\hat{f}_s) = a(\hat{f}_s) \,, \qquad d(\hat{f}_-) = a^\dagger(\hat{f}_-) \,, \tag{4}$$

where $\hat{f}$ denotes the Dirac-Fourier transform of f, $\hat{f}_\pm \in P^0_\pm H$, $\hat{f}_s \in P^0_s H$ and the zero mode has been treated like positive energy states.

The quantum mechanical many body representation corresponding to the filled negative energy states is defined by

$$\omega(a(f)) = 0 , \qquad \omega(a(f),a^\dagger(g)) = (f,(P^0_+ + P^0_s)g), \;\ldots , \tag{5}$$

and is related via the GNS construction to the usual Fock space realization. Finally, the field operator can be expanded like

$$\phi(f) = b(\hat{f}_+) + c(\hat{f}_s) + d^\dagger(\hat{f}_-) . \tag{6}$$

We compare the above situation to another one starting from a "perturbed" Dirac operator

$$H = \alpha \frac{1}{i}\frac{d}{dx} + \beta\, v_{II}(x) , \qquad v_{II}(x) = m \,\mathrm{th}\, x + V(x) \tag{7}$$

assuming $\lim_{|x|\to\infty} V(x) = 0$. We obtain a different splitting of H according to projections onto continuous and discrete spectra of H. Finally, with an obvious notation, we may write a decomposition of the field operator of (6) like

$$\phi(f) = B(\tilde{f}_+) + C(\tilde{f}_s) + D^\dagger(\tilde{f}_-) . \tag{8}$$

To simplify notation let us choose an orthonormal base $\{\hat{f}_{\pm n}\}$ for $P^0_\pm H$ and similarly $\{\tilde{f}_{\pm n}\}$ for subspaces $P_\pm H$ corresponding to H, than (6) and (8) can be expressed as

$$B_n = (\tilde{f}_{+n},\hat{f}_{+m})b_m + (\tilde{f}_{+n},\hat{f}_s)c + (\tilde{f}_{+n},\hat{f}_{-m})d^\dagger_m \tag{9a}$$

$$C = (\tilde{f}_s,\hat{f}_{+m})b_m + (\tilde{f}_s,\hat{f}_s)c + (\tilde{f}_s,\hat{f}_{-m})d^\dagger_m \tag{9b}$$

$$D^\dagger_n = (\tilde{f}_{-n},\hat{f}_{+m})b_m + (\tilde{f}_{-n},\hat{f}_s)c + (\tilde{f}_{-n},\hat{f}_{-m})d^\dagger_m \tag{9c}$$

where $B_n = B(\tilde{f}_{+n}) \;\ldots$.

Next we follow standard procedures [4]: Let $\hat{\Omega}$ be the vacuum corresponding to the bare representation defined by $b_n\hat{\Omega} = d_n\hat{\Omega} = 0$, and $\tilde{\Omega}$ be the vacuum corresponding to the dressed representation defined by $B_n\tilde{\Omega} = D_n\tilde{\Omega} = 0$; assume both representations are unitarily equivalent; therefore there exists a dressing transformation with

$$\tilde{\Omega} = \mathcal{U}\hat{\Omega}\ , \quad B_n = \mathcal{U}\, b_n\, \mathcal{U}^{-1}\ , \quad C = \mathcal{U}\, c\, \mathcal{U}^{-1}\ , \quad D_n = \mathcal{U}\, d_n\, \mathcal{U}^{-1}\ . \tag{10}$$

It is not difficult to work out the explicit form of $\mathcal{U}$; an ambiguity resulting from the distinction between so-called weak and strong Bogoliubov transformations does not matter for our present purpose (see [6]). Normalizability of $\tilde{\Omega}$ yields necessary and sufficient conditions for the unitary implementability

$$\|P_+ P_-^o\|_{HS} < \infty\ , \qquad \|P_- P_+^o\|_{HS} < \infty\ , \tag{11}$$

where $\|\cdot\|_{HS}$ denotes the Hilbert Schmidt norm; note that contributions from the discrete spectrum do not matter, since P_s^o and P_s are assumed to finite dimensional.

Next we may state our first result from

<u>PERTURBING AROUND A KINK POTENTIAL</u>

<u>Theorem 1</u>: Let $H_o = \alpha \frac{1}{i}\frac{d}{dx} + m\,\beta\,\text{th}\,x$ and $H = H_o + \beta\, V(x)$, and assume that $\|V\|_p < \infty$ for $1 < p \leq 2$; the two representations of the CAR corresponding to H_o and H are unitarily equivalent.

<u>Proof</u>: Both conditions (11) are equivalent to finiteness of $\|P_+ - P_+^o\|_{HS}$, which we have to check; but projection operators can be expressed in terms of corresponding resolvents:

$$P_+ - P_+^o = \frac{1}{2\pi}\int_C dz\, [R(iz) - R^o(iz)] \tag{12}$$

$$R(z) = (H - z)^{-1}\ , \quad z \notin \sigma(H)\ ; \quad R^o(z) = (H_o - z)^{-1}\ , \quad z \notin \sigma(H_o)\ ,$$

where the path of integration C consists of lines $(-\infty,-\varepsilon]$ and $[\varepsilon,\infty)$, and a half circle connecting $-\epsilon$ to ϵ in the lower complex half plane. (We note that, since we are interested in unitarily equivalent representations, we may suppose that $\sigma(H) = \sigma(H_0)$).

In contrast to Ref. [4], both H and H_0 have a zero eigenvalue; we therefore define

$$\tilde{R}(z) = (1-P_s)R(z)\ , \qquad \tilde{R}^0(z) = (1-P_s^0)R_0(z) \tag{13}$$

and rewrite (12), using the second resolvent identity as

$$P_+ - P_+^0 = \frac{1}{2\pi}\int_C dz[-\frac{1}{iz}P_s^0 + \tilde{R}^0(iz)]\beta V[-\frac{1}{iz}P_s + \tilde{R}(iz)]\ . \tag{14}$$

We note that both, P_s and P_s^0 are 2×2 matrix operators with only the two-two component nonvanishing, therefore $P_s^0\,\beta\,P_s$ gives no contribution and only three terms in (14) have to be estimated.

Since $\|\tilde{R}(z)\| = \mathrm{dist}(z, \mathrm{spec}(1-P_s)H)^{-1}$ and both $\tilde{R}^0$ and $\tilde{R}$ have no zero energy bound state, the last contribution of (14) is estimated as

$$\|\int_C dz\ \tilde{R}^0(iz)\beta V\ \tilde{R}(iz)\|_{HS} \leq \int_{-\infty}^{\infty} d\eta\|\tilde{R}_0(i\eta)\beta V\|_{HS}\frac{1}{\sqrt{1+\eta^2}}\ , \tag{15}$$

where we have put $m = 1$.

In order to proceed, we need the explicite form of the free resolvent; this is easy to get since $A^\dagger A = -d^2/dx^2 + 1$, with A given in (2). Such a supersymmetric quantum mechanical situation allows to write down the resolvent as

$$R^0(z) = \begin{pmatrix} z(A^\dagger A - z^2)^{-1} & (A^\dagger A - z^2)^{-1}A^\dagger \\ A(A^\dagger A - z^2)^{-1} & \frac{1}{z}\{A(A^\dagger A - z^2)^{-1}A^\dagger - 1\} \end{pmatrix} \tag{16}$$

with the explicit kernel

$$(A^\dagger A - z^2)^{-1}(x,y) = \frac{i}{2\sqrt{z^2-1}}\, e^{i\sqrt{z^2-1}\,|x-y|},\ \text{for } \mathrm{Im}\sqrt{z^2-1} > 0\ . \tag{17}$$

Next we have to find conditions on V such that the Hilbert Schmidt norm of $\tilde{R}_o(i\eta)\beta V$ in (15) is finite and $o(1/\eta)$ for $|\eta| \to \infty$. We estimate the norms of all four matrix elements separately; for example

$$\|[\tilde{R}_o(i\eta)\beta V]_{11}\|^2_{HS} \le \frac{1}{4}\,\frac{\eta^2}{\eta^2+1}\int dx \int dy \; e^{-2\sqrt{\eta^2+1}|x-y|}\; |V(x)||V(y)| \;, \qquad \eta \in \mathbb{R} . \tag{18}$$

Hölder and Young's inequality yields

$$\|[\tilde{R}_o(i\eta)\beta V]_{11}\|^2_{HS} \le \|V\|^2_p \,\| e^{-2\sqrt{\eta^2+1}|\cdot|} \|_r \;, \quad \frac{2}{p} = 2 - \frac{1}{r}\,, \qquad 1 \le r < \infty . \tag{19}$$

Since we need at least some decay for $|\eta| \to \infty$, the allowed range of r is restricted to $[1,\infty)$, which turns into a range for $p \in (1,2]$ as imposed in theorem 1.

The other matrix elements can be estimated in a similar way; the difference between $[R_o]_{22}$ and $[\tilde{R}_o]_{22}$, given by the zero energy bound state contribution, has to be taken into account. Again finiteness is implied by the assumption on V.

Thus it remains to estimate I_1 and I_2:

$$I_1 = \|\int_C dz\, \frac{1}{z}\, P^o_s \beta V\, \tilde{R}(iz)\|_{HS}\,, \qquad I_2 = \|\int_C dz\, \frac{1}{z}\, \tilde{R}_o(iz)\beta V P_s\|_{HS}\,. \tag{20}$$

We note that $P^o_s\beta V$ is a Hilbert Schmidt operator for the class of potentials above. Therefore

$$I_1 = \|\lim_{\varepsilon\to 0}\Big(\int_{-\infty}^{-\varepsilon}\frac{dz}{z} + \int_{\varepsilon}^{\infty}\frac{dz}{z} + \int_{\text{half circle}}\frac{dz}{z}\Big)P^o_s\beta V\tilde{R}(iz)\|_{HS} \le$$
$$\le \|P^o_s\beta V\|\,\Big(\pi\,\|\tilde{R}(0)\| + 2\int_0^\infty \frac{d\eta}{\eta^2+1}\Big) < \infty \tag{21}$$

where the estimate $\|\tilde{R}(i\eta) - \tilde{R}(-i\eta)\| \le 2\eta(\eta^2+1)^{-1}$, $\eta \in \mathbb{R}$ has been used. Finally we have

$$I_2 \leq \pi \, \|\tilde{R}(0)\beta V\|_{HS} + 2 \lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} d\eta \, \|\tilde{R}_o(i\eta)\tilde{R}_o(-i\eta)\beta V P^o_s\|_{HS} \leq$$

$$\leq \pi \, \|\tilde{R}(0)\beta V\|_{HS} + 2 \int_0^{\infty} \frac{d\eta}{\sqrt{1+\eta^2}} \, \|\tilde{R}_o(i\eta)\beta V\|_{HS} < \infty \tag{22}$$

which proves theorem 1.

REMARKS: As we expect, a large class of perturbations of a typical soliton potential like $m\,\mathrm{th}\,x$ do not change the field theory representation.

The same actually results if one perturbs around a potential with trivial asymptotics; the technique to prove this is similar to the above, therefore we state only the result for

PERTURBING AROUND A CONSTANT POTENTIAL

Theorem 2: Let $H_o = \alpha \frac{1}{i}\frac{d}{dx} + \beta m$ and $H = H_o + \beta V(x)$ and assume that $\|V\|_p < \infty$ for $1 < p \leq 2$; the two representations of the CAR corresponding to H_o and H are unitarily equivalent.

REMARK: The most essential question concerns a comparison of two problems: one potential with trivial asymptotics v_I to another one with soliton asymptotics v_{II}. To check this, it is only necessary to take one example out of each class and compare both.

This leads to

Theorem 3: Let $H_o = \alpha \frac{1}{i}\frac{d}{dx} + \beta v_I$ with $\lim_{|x|\to\infty} |v_I - m| = 0$ and and $H = \alpha \frac{1}{i}\frac{d}{dx} + \beta v_{II}$ with $\lim_{|x|\to\infty} |v_{II} - m\,\mathrm{th}\,x| = 0$. The corresponding representations of the CAR are not equivalent.

Proof: If one takes $v_I = m$ and $v_{II} = m\,\mathrm{th}\,x$ themselves, one may use the explicit solutions for the free case and eqs. (3a) and (3b). The appropriate norm turns out to be infinite

$$\|P_+ P^o_-\|_{HS} = \int_{-\infty}^{\infty} dx \int_0^{\infty} \frac{dq\,(1 - \mathrm{th}\,x)^2}{1 + q^2} = \infty . \tag{23}$$

REMARK: Due to the above facts a rigorous discussion of the occurrence of fractional charges has to be done with care. We are presently studying such questions following a constructive approach.

REFERENCE

[1] R. J

[2] A.J.

[3] R. F
J. E
R. N
R. E
A.Z.

[4] M. M
G.

[5] S.

F.A.

[6] H.

REFERENCES

[1] R. Jackiw and J.R. Schrieffer, Nucl. Phys. B190 [FS3] (1981) 253.

[2] A.J. Niemi and G.W. Semenoff, "Fermion Number Fractionization in Quantum Field Theory" (Princeton preprint 1985).

[3] R. Rajaraman and J. Bell, Phys. Lett. 116B (1982) 151;
J. Bell and R. Rajaraman, Nucl. Phys. B220 [FS2] (1983) 1;
R. Mac Kenzie and F. Wilcek, Phys. Rev. D30 (1984) 2194;
R. Blankenbecler and D. Boyanovsky, Phys. Rev. D31 (1985) 2089;
A.Z. Capri, R. Ferrari and L.E. Picasso, Phys. Rev. D30 (1984) 2136.

[4] M. Klaus and G. Scharf, Helv. Phys. Acta 50 (1977) 779;
G. Scharf and H.P. Seipp, Phys. Lett. 108B (1982) 196.

[5] S. Shale and W.F. Stinespring, Journ. of Math. and Mech. 14 (1965) 315;
F.A. Berezin, The Method of Second Quantization (Academic Press, New York 1966)

[6] H. Grosse and G. Karner (to be published).