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ON THE SIMULTANEOUS DIAGONALIZABILITY OF MATRICES+

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Abstract

We prove two theorems on the simultaneous diagonalizability of a set of complex square matrices by a biunitary transformation.

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EINGANG

The problem of simultaneous diagonalization of a set of square matrices arises in the discussion of natural flavour conservation in Higgs induced neutral currents (e.g., Sartori (1979), Gatto, Morchio, Sartori and Strocchi (1980), Frère and Yao (1985)). In this context the following theorem was formulated:

Theorem 1: Let $\{A_1,\ldots,A_N\}$ be a set of complex $m \times m$ matrices. Then there exist unitary matrices U, V such that $U^\dagger A_i V$ is diagonal for all $i = 1,\ldots,N$ iff the sets $S_1 = \{A_i^\dagger A_j\}_{i,j=1,\ldots,N}$ and $S_2 = \{A_i A_j^\dagger\}_{i,j=1,\ldots,N}$ are abelian.

This is the original version of Sartori (1979) who, however, only sketched the idea of a proof in his paper. Theorem 1 was discussed again by Gatto, Morchio, Sartori and Strocchi (1980) who presented a proof under the additional assumption of e.g. $A_1 A_1^\dagger$ being nondegenerate. More recently, Frère and Yao (1985) referred to a similar theorem by Federbush where both the nondegeneracy of $A_1^\dagger A_1$ and the nonsingularity of the matrices A_i seem to play an essential rôle. The purpose of the present note is twofold: first we show that theorem 1 holds in its original form, i.e. no assumptions concerning nondegeneracy or nonsingularity of matrices need to be made. Secondly, if at least one of the matrices A_i is nonsingular it can be shown that the commutativity of either S_1 or S_2 is actually sufficient for the simultaneous diagonalization of $A_1,\ldots,A_N$ (theorem 2).

Let us first prove theorem 1. The commutativity of S_1 and S_2 is clearly necessary for the existence of unitary matrices U, V such that $U^\dagger A_i V$ are diagonal for all $i = 1,\ldots,N$. The more interesting part of the theorem concerns the opposite implication. If S_1 is abelian all elements of S_1 are normal matrices. Consequently, there exists a common orthonormal basis of eigenvectors $(x_\alpha)_{\alpha=1,\ldots,m}$ for all matrices of S_1:

$$A_i^\dagger A_j x_\alpha = \lambda_{ij}^\alpha x_\alpha \qquad (1)$$

with λ_{ij}^α ($\alpha = 1,\ldots,m$; $i,j = 1,\ldots,N$) the corresponding eigenvalues. Let $\{x_1,\ldots,x_k\}$ be those vectors for which $A_i x_\alpha = 0$ for all $i = 1,\ldots,N$.

Then for all $\alpha > k$ (for $k = m$ the theorem holds trivially because $A_i \equiv 0$) there exists an index i_α such that $A_{i_\alpha} x_\alpha \neq 0$. This allows one to define the normalized vectors

$$y_\alpha = A_{i_\alpha} x_\alpha / \| A_{i_\alpha} x_\alpha \| \; , \qquad \alpha = k+1, \ldots, m \; . \tag{2}$$

Using

$$\lambda^\alpha_{ii} = \| A_i \, x_\alpha \|^2 \tag{3}$$

one finds that the y_α form an orthonormal system since

$$(y_\alpha | y_\beta) = \lambda^\beta_{i_\alpha i_\beta} \, \delta_{\alpha\beta} / (\| A_{i_\alpha} x_\alpha \| \; \| A_{i_\beta} x_\beta \|) = \delta_{\alpha\beta} \; , \quad \alpha, \beta > k. \tag{4}$$

Furthermore, we obtain

$$(y_\alpha | A_i x_\beta) = \lambda^\beta_{i_\alpha i} \, \delta_{\alpha\beta} / \sqrt{\lambda^\alpha_{i_\alpha i_\alpha}} \tag{5}$$

which implies that each vector $A_i x_\beta$ has only a y_β component and a possible component in the space orthogonal to all y_α ($\alpha = k+1, \ldots, m$). To show that $A_i x_\beta$ is in fact proportional to y_β it is sufficient to prove

$$| (y_\beta | A_i \, x_\beta) | = \| A_i \, x_\beta \| \; . \tag{6}$$

At this point the commutativity of S_2 enters. From

$$A_i \, A_i^\dagger \, A_i \, A_j^\dagger \, A_j \, x_\beta = A_i \, A_j^\dagger \, A_i \, A_i^\dagger \, A_j \, x_\beta \tag{7}$$

we get the relation

$$\lambda^\beta_{ii} \, \lambda^\beta_{jj} = \lambda^\beta_{ji} \, \lambda^\beta_{ij} = | \lambda^\beta_{ij} |^2 \tag{8}$$

using (1). With the help of (8) we can derive (6)

$$|(y_\beta|A_i x_\beta)| = |\lambda^\beta_{i_\beta i}|/\sqrt{\lambda^\beta_{i_\beta i_\beta}} = \sqrt{\lambda^\beta_{ii}} = \|A_i\, x_\beta\| \ .$$

Completing $\{y_{k+1},\ldots,y_m\}$ to an orthonormal basis $\{y_1,\ldots,y_k,y_{k+1},\ldots,y_m\}$ we can write*

$$A_i\, x_\alpha = \rho^\alpha_i\, y_\alpha \ , \qquad i = 1,\ldots,N;\ \alpha = 1,\ldots,m \qquad (9)$$

and

$$A_i = \sum_{\alpha=1}^{m} \rho^\alpha_i\, y_\alpha\, x^\dagger_\alpha = (y_1,\ldots,y_m) \begin{pmatrix} \rho^1_i & & 0 \\ & \ddots & \\ 0 & & \rho^m_i \end{pmatrix} \begin{pmatrix} x^\dagger_1 \\ \vdots \\ x^\dagger_m \end{pmatrix} \ . \qquad (10)$$

Thus we have obtained unitary matrices

$$U = (y_1,\ldots,y_m)$$
$$V = (x_1,\ldots,x_m) \qquad (11)$$

which diagonalize all A_i completing the proof of theorem 1.

The commutativity of both S_1 and S_2 was crucial for the proof. If, however, one of the matrices A_i $(i = 1,\ldots,N)$ is nonsingular it is already sufficient for the simultaneous diagonalizability of the A_i that either S_1 or S_2 is abelian. This is the content of

Theorem 2: Let A_1 be nonsingular and $S_1 = \{A^\dagger_i A_j\}_{i,j=1,\ldots,N}$ be an abelian set. Then there exist unitary matrices U, V such that $U^\dagger A_i V$ is diagonal for all $i = 1,\ldots,N$ and thus the set $S_2 = \{A_i A^\dagger_j\}_{i,j=1,\ldots,N}$ is also abelian.

In order to prove theorem 2 we choose vectors $\{x_\alpha\}_{\alpha=1,\ldots,m}$ as before and we define

$$y_\alpha = A_1\, x_\alpha / \|A_1\, x_\alpha\| \ , \qquad \alpha = 1,\ldots,m \ . \qquad (12)$$

*) Of course, $\rho^\alpha_i = 0$ for $\alpha = 1,\ldots,k$ and all i.

The y_α exist for all α because A_1 is nonsingular. As in (5) we obtain

$$(y_\alpha | A_i \, x_\beta) = \lambda^\beta_{1i} \, \delta_{\alpha\beta} / \sqrt{\lambda^\alpha_{11}} \; . \tag{13}$$

Since now $\{y_1, \ldots, y_m\}$ is a complete orthonormal basis of C^m it follows immediately from (13) that $A_i x_\beta$ is proportional to y_β. As for theorem 1 U and V are given by (11).

In conclusion we want to note that the nonsingularity of A_1 in theorem 2 is essential. For instance, it is not sufficient to demand only nondegeneracy of $A_1^\dagger A_1$. Finally, we emphasize once again that no assumptions concerning nondegeneracy were needed to prove both theorems making their actual application much simpler in most cases.

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References

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