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STOCHASTIC QUANTIZATION OF GENERAL RELATIVITY+

Helmut Rumpf
Institute for Theoretical Physics
University of Vienna
A-1090 Vienna, Austria

Abstract

Following an elementary exposition of the basic mathematical concepts used in the theory of stochastic relaxation processes the stochastic quantization method of Parisi and Wu is briefly reviewed. The method is applied to Einstein's theory of gravitation using a formalism that is manifestly covariant with respect to field redefinitions. This requires the adoption of Ito's calculus and the introduction of a metric in field configuration space, for which there is a unique candidate. Due to the indefiniteness of the Euclidean Einstein-Hilbert action stochastic quantization is generalized to the pseudo-Riemannian case. It is formally shown to imply the DeWitt path integral measure. Finally a new type of perturbation theory is developed.

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1. Introduction

The subject of this seminar is a new approach to the quantization of Einstein's theory of gravitation. Superficially, any such approach may appear outdated nowadays, after the advent of apparently finite superstring theories that contain gravitation (and, it is hoped, everything else) and imply a radical and very promising, from the quantum theoretical point of view, modification of the Einstein theory. But one should be aware that even if these hopes turn out to be correct the latter theory is not invalidated as an effective low energy limit: There is a vast energy regime in which the full nonlinearity of the classical theory (possibly augmented by higher order curvature terms) is relevant, yet the massive string excitations are not. Maybe this presumed effective character of the Einstein theory is indeed the reason why no entirely satisfactory quantum version of it has been found so far. We recall that each of the two main approaches, the so-called canonical and covariant quantizations, has its exclusive set of questions to which it can provide answers (though marred by mathematical ambiguities). There is a tendency, especially in the Russian school, to consider the canonical approach as fundamental, since the path integral formulation, in which the covariant quantization is expressed most conveniently, can be based on the Hamiltonian formalism. The stochastic quantization presented below does not fit into this philosophy. It should rather be viewed as an independent approach to quantum gravity providing an alternative foundation of the covariant quantization (indeed it is, in a sense to be detailed later, even "more covariant" than the latter).

The term "stochastic quantization" may give rise to wrong associations, as there exists a variety of different stochastic approaches to quantum theory. We mention only Nelson's stochastic mechanics and its field theoretical generalization [1], Boyer's stochastic electrodynamics [2], and quantum stochastic calculus [3] (in which a quantum analog of Brownian motion is constructed). All this we do not mean. We reserve the term "stochastic quantization" for the method developed by

Parisi and Wu [4] originally for non-abelian gauge theories. A brief outline of this method will be given in Section 3. In Sec. 2 the necessary mathematical apparatus is developed in an elementary way using Brownian motion as an example. In Section 4 the formalism is applied to the gravitational field, which requires the introduction of a metric on the manifold of space-time metrics and the adoption of Ito's calculus. Next, in Sec. 5, a generalization of the Parisi-Wu method to Lorentzian space-times is presented, which is enforced by the fact that the Euclidean Einstein-Hilbert action is not bounded from below. It is shown that stochastic quantization formally implies the DeWitt measure in the path integral of General Relativity and distinguishes a particular geometry in field configuration space. Finally, in Section 6 a stochastic perturbation theory for quantum gravity is developed.

2. The Einstein-Smoluchowski Theory

Stochastic quantization is related to ordinary quantization in much the same manner as non-equilibrium thermodynamics is related to equilibrium thermodynamics. The prototype of the processes studied in non-equilibrium thermodynamics is Brownian motion. Its mathematical description requires basically the same formalism that is also used in stochastic quantization. Here we restrict ourselves to the simplest theory of Brownian motion, which is due to Einstein and Smoluchowski. Consider the spatial coordinate $x(t)$ of a particle moving in one dimension and interacting with a medium which is at thermal equilibrium at a temperature T. It is natural to interpret $x(t)$ as a random function, for which "stochastic process" is just another word. It may be defined by a so-called stochastic differential equation,

$$\dot{x} = F(x) + D^{1/2}\,\xi(t) \qquad (2.1)$$

$$F(x) = K(x)/(m\alpha) \qquad (2.2)$$

$$D = kT/(m\alpha) \ . \qquad (2.3)$$

Here K(x) is an external force field, α is the friction coefficient appearing in the deterministic equation of motion

$$\ddot{x} = K(x)/m - \alpha\dot{x} \qquad (2.4)$$

and D is Einstein's diffusion constant. The last term on the right hand side of (2.1) represents the random collisions of the particle with the constituents of the medium. More precisely, $\xi(t)$ is a stochastic process of the simplest type, namely a so-called Gaussian white noise. It may be characterized by its "moments", i.e. the expectation values

$$\langle\xi(t)\rangle = 0 \qquad (2.5)$$

$$\langle\xi(t)\xi(t')\rangle = 2\delta(t-t') \qquad (2.6)$$

$$\langle\xi(t_1)\ \ldots\ \xi(t_{2n+1})\rangle = 0 \qquad (2.7)$$

$$\langle\xi(t_1)\ \ldots\ \xi(t_{2n})\rangle = \sum_{\text{comb.}} \prod_{\text{pairs}} \langle\xi(t_i)\xi(t_j)\rangle \ . \qquad (2.8)$$

The sum in equation (2.8) is over all possible combinations of the 2n factors $\xi(t_i)$ into pairs. The "Gaussian" property consists in the fact that all correlation functions (which are defined in terms of the moments in an analogous way as the connected Green functions of field theory are defined in terms of the Green functions) of order higher than 2 vanish. Alternatively it is characterized by the quadratic exponent in the formal probability measure on the space of sample functions $\xi(t)$,

$$\rho(\mathcal{D}[\xi]) \propto \mathcal{D}[\xi]\ e^{-\frac{1}{4}\int dt\,\xi^2(t)} \ . \qquad (2.9)$$

This expression is purely formal, however, as the sample space consists of distributions rather than ordinary functions. Equation (2.1) is the prototype of a Langevin equation, and we have used the physicist's notation for it. For later reference we write it also in a different manner, which is preferred by mathematicians:

$$dx(t) = F(x)dt + D^{1/2}dw(t) \tag{2.10}$$

$$\langle w(t)\rangle = 0 \tag{2.11}$$

$$\langle w(t)w(t')\rangle = 2\min(t,t') \tag{2.12}$$

where w(t) is the Wiener process with

$$\dot{w}(t) = \xi(t) \tag{2.13}$$

$$\rho(\mathcal{D}[w]) \propto \mathcal{D}[w]\, e^{-\frac{1}{4}\int dt\,\dot{w}^2(t)} \,. \tag{2.14}$$

The expression (2.14) for the Wiener measure is again formal, as the sample functions w(t) are continuous, but not differentiable (with probability 1). The important point, however, is that there exists a well-defined measure implying the expectation values (2.11), (2.12).

Instead of using the stochastic differential equation (2.1), the stochastic process x(t) may be defined in a perfectly equivalent way in terms of the spatial probability density P(x,t) with which the particle may be found at the position x at time t (this is the description originally used by Einstein). This probability density allows to evaluate the expectation value of a function f(x(t)) according to

$$\langle f(x(t))\rangle = \int dx\, P(x,t)\, f(x) \,, \tag{2.15}$$

P(x,t) is determined by the so-called Fokker-Planck equation

$$\frac{\partial P}{\partial t} = [D\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}F(x)]P \,. \tag{2.16}$$

There is a unique Fokker-Planck equation associated to every Langevin equation (note that (2.16) reduces to the diffusion equation if the external force is set equal to zero). The Fokker-Planck equation can be used to show that the process x(t) is a relaxation process, i.e. there exists the equilibrium limit

$$P_{eq}(x) = \lim_{t\to\infty} P(x,t) \tag{2.17}$$

$$P_{eq}(x) \propto e^{-V(x)/kT} \tag{2.18}$$

if $K(x) = -\partial V/\partial x$. The equilibrium distribution is the Boltzmann distribution implied by the potential energy. The full Maxwell-Boltzmann distribution (containing in addition the kinetic energy term in the exponent in (2.18)) can be derived only from the more refined theory of Brownian motion due to Ornstein and Uhlenbeck.

3. Parisi-Wu Quantization

The equilibrium probability distribution (2.18) has an infinite-dimensional counterpart in Euclidean quantum field theory, where the expectation value of a functional $f[\phi]$ of the Euclidean quantum field ϕ is given by the path integral

$$\langle f[\phi]\rangle = \int D[\phi]\, f[\phi]\, P[\phi] \tag{3.1}$$

with

$$P[\phi] = e^{-\frac{1}{\hbar} S_E[\phi]} \Big/ \int d[\phi]\, e^{-\frac{1}{\hbar} S_E[\phi]} \quad . \tag{3.2}$$

As $P[\phi]$ is a positive probability distribution, the field ϕ may be viewed as an equilibrium process characterized by the Euclidean action $S_E[\phi]$ (generalizing the potential $V(x)$) and by the "temperature" $\hbar/k$. Parisi and Wu [4] have constructed a relaxation process $\phi(x,s)$ depending on a fictitious time parameter s such that the process $\phi(x)$ is obtained in the limit $s \to \infty$. The process $\phi(x,s)$ is defined by a Langevin equation analogous to (2.1),

$$\frac{\partial}{\partial s} \phi(x,s) = - \frac{\delta S_E[\phi]}{\delta \phi(x,s)} + \hbar^{1/2} \xi(x,s) \qquad (3.3)$$

where the noise $\xi(x,s)$ is Gaussian with

$$\langle \xi(x,s)\xi(x',s')\rangle = 2\delta(s - s')\delta^{(4)}(x - x') \ . \qquad (3.4)$$

In the following, $\hbar$ will be set equal to 1. Using the Fokker-Planck equation associated with (3.3),

$$\frac{\partial}{\partial s} P[\phi(x),s] = \int d^4x \frac{\delta}{\delta\phi(x)} \left(\frac{\delta}{\delta\phi(x)} + \frac{\delta S_E}{\delta\phi(x)}\right) P[\phi(x),s] \qquad (3.5)$$

it is possible to show formally that the Euclidean quantum field theoretical expectation values are given by

$$\langle f[\phi(x)]\rangle_{QFT} = \lim_{s\to\infty} \langle f[\phi(x,s)]\rangle \ . \qquad (3.6)$$

Although the analogy of the Parisi-Wu approach with non-equilibrium thermodynamics seems to be purely formal, it has physically interesting consequences: First, Langevin simulation on the lattice is an alternative to the Monte Carlo method used for numerical calculations in gauge theories. It has in fact a wider range of applicability [5], the fictitious time s being proportional to the computer time. Second, stochastic quantization offers the possibility of an invariant and non-perturbative regularization [6], which may be viewed as the ultimate version (now applicable also to gauge fields) of the proper time regularization of Schwinger [7]. Finally, if ϕ is a gauge field, no gauge-fixing and hence no Faddeev-Popov ghost fields have to be introduced. In particular, stochastic quantization provides a new type of perturbation theory. Let us indicate briefly how this comes about. Consider the Maxwell field A_a. In this case the Langevin equation (3.3) in momentum space reads

$$\dot{A}_a = - k^2 T_{ab} A_b + \xi_a \tag{3.7}$$

$$T_{ab} = \delta_{ab} - L_{ab} \tag{3.8}$$

$$L_{ab} = k_a k_b / k^2 \tag{3.9}$$

and the explicit form of (3.4) is

$$\langle \xi_a(k,s)\xi_b(k',s')\rangle = 2(2\pi)^4 \delta_{ab} \delta^{(4)}(k+k')\delta(s-s') \; . \tag{3.10}$$

Equation (3.7) may be solved for A_a with the help of the heat kernel

$$H = e^{-k^2 Ts} = e^{-k^2 s} \, T + L \; . \tag{3.11}$$

The solution is

$$A_a(k,s) = \int_0^s ds' \, H_{ab}(k,s-s')\xi_b(k,s') \tag{3.12}$$

if A(k,0) = 0. Equation (3.10) then implies

$$\langle A_a(k,s)A_b(k',s')\rangle = 2(2\pi)^4 \delta^{(4)}(k+k') \int_0^{\min(s,s')} d\sigma H_{ab}(k,s+s'-2\sigma) \tag{3.13}$$

and

$$\lim_{s\to\infty} \langle A_a(k,s)A_b(k',s)\rangle = (2\pi)^4 \delta^{(4)}(k+k')\left(\frac{T_{ab}}{k^2} + \infty^2 L_{ab}\right) \; . \tag{3.14}$$

The quadratic divergence in (3.14) does not contribute to gauge-invariant expectation values, which hence may be computed without gauge-fixing. This holds true also in the non-abelian case.

4. Stochastic Euclidean Quantum Gravity

In equations (3.3) and (3.4) we have suppressed any indices that the field ϕ and the source ξ may bear. As long as we are dealing with tensor fields on the Euclidean background space, the Euclidean metric may always be used to define the additional structure necessary, as was done in equations (3.7) and (3.10). However if we want to quantize the metric tensor field $g_{\alpha\beta}(x)$ itself, the form of the Langevin equation is less obvious. The same problem is also encountered when a non-gravitational field is to be quantized upon a transformation of the field variable, e.g. $\phi \to e^{\phi}$ for the scalar field. In both cases the principle of general covariance with respect to field redefinitions is powerful enough to determine the general structure of the Langevin equation. In the following we shall denote the field variable by ϕ^A, where the index A comprises also the space-time argument x, the most interesting example being of course $\phi^A = g_{\alpha\beta}(x)$. In field configuration space the ϕ^A have the status of coordinates, hence $\partial\phi^A/\partial s$ form a "vector". Since the action $S_E[\phi]$ is a scalar, the general form of the Langevin equation is

$$\frac{\partial\phi^A}{\partial s} = - G^{AA'} \frac{\delta S_E[\phi]}{\delta\phi^{A'}(s)} + \xi^A(s) \tag{4.1}$$

where $G^{AA'}[\phi]$ is a metric in field configuration space. A covariant definition of the stochastic source ξ^A is suggested by writing (4.1) in a form analogous to (2.10):

$$d\phi^A(s) = - G^{AA'} \frac{\delta S_E}{\delta\phi^{A'}} + \delta\omega^A(s) \tag{4.2}$$

where $\delta\omega^A$ is an anholonomic stochastic differential, i.e. there need not exist a process ω^A of which it is the increment. The covariant generalization of the Wiener measure (2.14) is contained in the ansatz

$$\langle F[\phi]\rangle = \int |G|^{1/2} \prod_{A,s} \delta\omega^A(s) \exp[-\frac{1}{4}\int ds\, G_{BC}[\phi] \frac{\delta\omega^B}{ds}\frac{\delta\omega^C}{ds}] \tag{4.3}$$

for the expectation value of an arbitrary functional F of ϕ, G denoting the determinant of $G_{AA'}$. Note that in general G_{BC} depends on ϕ and hence on $\delta\omega^A/ds = \xi^A$ so that ξ^A is not Gaussian. It may however be related to a Gaussian process

$$\xi^{(o)M} = dW^M/ds \tag{4.4}$$

by the transformation

$$\xi^A = E_M{}^A[\phi]\ \xi^{(o)M} \tag{4.5}$$

where $E_M{}^A[\phi]$ is a stochastic vielbein functional obeying

$$G_{AB}[\phi]\ E_M{}^A[\phi]\ E_N{}^B[\phi] = G^{(o)}_{MN} \tag{4.6}$$

with $G^{(o)}_{MN}$ a reference metric <u>independent</u> of ϕ. Equations (4.3) - (4.5) imply

$$\langle F[\phi]\rangle = \int |G^{(o)}|^{1/2} \prod_{L,s} dW^L(s)\ \exp[-\tfrac{1}{4}\int ds\ G^{(o)}_{MN}\ \frac{dW^M}{ds}\frac{dW^N}{ds}] \tag{4.7}$$

so that W^M is indeed of the Wiener type.

There is a mathematical ambiguity in the multiplicative noise ansatz

$$\delta\omega^A = E_M{}^A[\phi]\ dW^M \tag{4.8}$$

implied by (4.4) and (4.5). As it stands, it can be interpreted in an infinite number of ways:

$$\delta\omega^A(s) = \lim_{\Delta s\to 0+} \{(1-\alpha)E_M{}^A[\phi(s)] + \alpha E_M{}^A[\phi(s+\Delta s)]\}[W^M(s+\Delta s) - W^M(s)] \tag{4.9}$$

$$0 \leq \alpha \leq 1\ . \tag{4.10}$$

It can be shown [8] that if $\alpha \neq 0$ the process ϕ^A depends on the choice of vielbein field $E_M{}^A$, which is physically unacceptable (this holds in particular for the Stratonovich value $\alpha = 1/2$). Therefore we have to set $\alpha = 0$, i.e. we have to adopt Ito's stochastic calculus. This brings about a new complication, however: In Ito's calculus $\partial\phi^A/\partial s$ is not a vector, since it transforms as

$$\dot{\phi}'^A = \frac{\delta\phi'^A}{\delta\phi^A}\,\dot{\phi}^B + G^{BC}\,\frac{\delta^2\phi'^A}{\delta\phi^B\delta\phi^C}\,. \tag{4.11}$$

Thus our original Langevin ansatz (4.1) is not covariant. This can be easily remedied, however, by covariantizing: The Langevin equation

$$\dot{\phi}^A - \Delta_G\,\phi^A = -\,G^{AA'}\,\frac{\delta S_E\,\phi}{\delta\phi^{A'}} + \xi^A \tag{4.12}$$

with Δ_G the Laplace-Beltrami operator with respect to the field metric $G_{AA'}$ is manifestly covariant, as

$$\Delta_G\,\phi^A = |G|^{-1/2}\,\frac{\delta}{\delta\phi^B}\,(|G|^{1/2}\,G^{AB}) \tag{4.13}$$

transforms as

$$\Delta_{G'}\,\phi'^A = \Delta_G\,\phi^A + G^{BC}\,\frac{\delta^2\phi'^A}{\delta\phi^B\delta\phi^C}\,. \tag{4.14}$$

In order to show formally that the Langevin equation (4.12) implies the desired equilibrium limit, we derive the associated Fokker-Planck equation. Consider an arbitrary functional $F[\phi]$. We have

$$\langle dF[\phi]\rangle = \langle\frac{\delta F}{\delta\phi^A}\,d\phi^A + \frac{1}{2}\,\frac{\delta^2\,F}{\delta\phi^A\delta\phi^B}\,d\phi^A\,d\phi^B\rangle = \tag{4.15}$$

$$= \langle\frac{\delta F}{\delta\phi^A}\,(-G^{AA'}\,\frac{\delta S_E}{\delta\phi^{A'}} + \Delta_G\,\phi^A)ds + \frac{\delta^2 F}{\delta\phi^A\delta\phi^B}\,G^{AB}[\phi]ds\rangle\,. \tag{4.16}$$

The last term in (4.16) stems from

$$\langle\delta\omega^A(s)\delta\omega^B(s)\rangle = \langle E_M{}^A[\phi(s)]E_N{}^B[\phi(s)]\rangle\,\langle dW^M(s)\,dW^N(s)\rangle = \tag{4.17}$$

$$= 2\,\langle E_M{}^A\,E_N{}^B\rangle\,G^{(o)\,MN}\,ds = 2G^{AB}[\phi(s)]ds\,. \tag{4.18}$$

Note that the factorization on the right hand side of (4.17) occurs only in the Ito interpretation. (4.16) implies

$$<\frac{dF[\phi]}{ds}> = <\frac{\delta F}{\delta\phi^A}\left(- G^{AA'}\frac{\delta S_E}{\delta\phi^{A'}} + \Delta_G\,\phi^A\right) + \frac{\delta^2 F}{\delta\phi^A\delta\phi^B}G^{AB}[\phi]> . \qquad (4.19)$$

In terms of the Fokker-Planck probability density $P[\phi,s]$ this may be written as

$$\int d[\phi]\,\frac{\partial P}{\partial s}F[\phi] = \int d[\phi]P\left[\frac{\delta F}{\delta\phi^A}\left(- G^{AA'}\frac{\delta S_E}{\delta\phi^{A'}} + \Delta_G\,\phi^A\right) + \frac{\delta^2 F}{\delta\phi^A\delta\phi^B}G^{AB}\right] . \qquad (4.20)$$

Integrating by part and using the fact that F is arbitrary we obtain

$$\frac{\partial P}{\partial s} = \frac{\delta}{\delta\phi^A}\left[\frac{\delta}{\delta\phi^B}G^{AB} - \Delta_G\,\phi^A + G^{AA'}\frac{\delta S_E}{\delta\phi^{A'}}\right]P . \qquad (4.21)$$

Introducing the scalar

$$Q[\phi,s] = |G|^{-1/2}\,P[\phi,s] \qquad (4.22)$$

and using (4.13) the Fokker-Planck equation (4.21) becomes

$$\frac{\partial Q}{\partial s} = |G|^{-1/2}\frac{\delta}{\delta\phi^A}|G|^{1/2}\,G^{AB}\left(\frac{\delta}{\delta\phi^B} + \frac{\delta S_E}{\delta\phi^B}\right)Q = \qquad (4.23)$$

$$= \nabla^B\left(\frac{\delta Q}{\delta\phi^B} + \frac{\delta S_E}{\delta\phi^B}Q\right) . \qquad (4.24)$$

This equation is manifestly covariant, ∇_A denoting the covariant derivative implied by the Christoffel connection of the metric G_{BC}. It is obvious from (4.24) that the Fokker-Planck equation has the stationary solution

$$Q_{eq} \sim e^{-S_E[\phi]} \qquad (4.25)$$

as desired.

We remark that the above derivation of the Fokker-Planck equation is strictly valid only if the field metric G_{AB} is Riemannian, i.e.

positive. In the pseudo-Riemannian case the process $\xi^{(0)M}(s)$ is necessarily complex, and so is ϕ. It may however be possible to replace ϕ by a real process ϕ_R with complex probability [9] such that the expectation values are unchanged. The corresponding complex Fokker-Planck probability (if it exists) still has to obey (4.24).

We now specialize to the gravitational field in the standard parametrization of Riemannian metrics, $\phi^A = g_{\alpha\beta}(x)$. It is natural to require that the field metric (i) be local and (ii) that the actions of diffeomorphisms on $g_{\alpha\beta}(x)$ be isometries with respect to the field metric. The properties (i) and (ii) are satisfied by the one-parameter (λ) family [10]

$$G_{AA'} \equiv G^{\alpha\beta,\alpha'\beta'}(x,x') = \frac{C}{2}\, g^{1/2}(g^{\alpha\alpha'}g^{\beta\beta'} + g^{\alpha\beta'}g^{\beta\alpha'} + \lambda g^{\alpha\beta}g^{\alpha'\beta'})\delta^{(4)}(x-x') \tag{4.26}$$

$$\lambda \neq -1/2 \tag{4.27}$$

where the constant C is not important and may be chosen to be

$$C = 4\kappa \tag{4.28}$$

for convenience. We note that $G_{AA'}$ is Riemannian for $\lambda > -1/2$ and that its determinant

$$G = \prod_x C^{10}(1 + 2\lambda) \tag{4.29}$$

is independent of g (this happens only in 4 dimensions). For later reference we write down also the inverse field metric

$$G^{AA'} \equiv G_{\alpha\beta,\alpha'\beta'}(x,x') = \frac{1}{2Cg^{1/2}}(g_{\alpha\alpha'}g_{\beta\beta'} + g_{\alpha\beta'}g_{\beta\alpha'} + \mu g_{\alpha\beta}g_{\alpha'\beta'})\delta^{(4)}(x-x') \tag{4.30}$$

$$\mu = -\lambda/(2\lambda + 1)\ . \tag{4.31}$$

The explicit form of the covariant Langevin equation (4.12) for the Euclidean gravitational field follows from the Euclidean Einstein-Hilbert action:

$$S_E[g] = -\frac{1}{2\kappa}\int d^4x\, g^{1/2}\, R[g_{\alpha\beta}] \tag{4.32}$$

$$\frac{\delta S_E}{\delta g_{\alpha\beta}} = \frac{1}{2\kappa}\, g^{1/2}(R^{\alpha\beta} - \frac{1}{2}\, g^{\alpha\beta}\, R)\ . \tag{4.33}$$

Therefore we obtain with the choice (4.28)

$$\dot{g}_{\alpha\beta} - \frac{9}{2}(1+\mu)\delta^{(4)}(0) g^{-1/2}\, g_{\alpha\beta} = -2R_{\alpha\beta} + \frac{\lambda+1}{2\lambda+1}\, g_{\alpha\beta}R + \xi_{\alpha\beta}\ . \tag{4.34}$$

The divergent term on the left hand side vanishes for $\mu = \lambda = -1$. This is the parameter value for which the field coordinate $g_{\alpha\beta}(x)$ is harmonic with respect to G_{AB}. In Section 6 we shall give an independent argument for the preferred status of this parameter value.

It follows from (4.22), (4.25) and (4.29) that the formal equilibrium limit $\lim_{s\to\infty} g_{\alpha\beta}(x,s)$ is characterized by the partition function

$$Z = \int \mathcal{D}[g]\, e^{-S_E[g]} \tag{4.35}$$

with the DeWitt path integral measure

$$\mathcal{D}[g] = \prod_x \prod_{\alpha\leq\beta} dg_{\alpha\beta}(x)\ . \tag{4.36}$$

Different measures can be obtained from the non-covariant Langevin equations lacking the correction term on the left hand side of (4.12). These measures are

$$\mathcal{D}[g] = \prod_x \prod_{\alpha\leq\beta} g^{\gamma}\, dg_{\alpha\beta} \tag{4.37}$$

$$\gamma = -\frac{9}{2}(1+\lambda)\ . \tag{4.38}$$

A measure of this more general type has been proposed by Misner [11] and others. Obviously the non-covariant measure with weight $g^{-5/2}\ {}^{(3)}g$ implied by the Hamiltonian formalism [12,13] cannot arise in stochastic quantization.

5. Stochastic Quantization in Physical Space-Time

The partition function (4.35) does not exist even formally, because the Euclidean Einstein-Hilbert action (4.32) is not bounded from below. We consider this a serious drawback of the Euclidean quantization program. Another drawback is that even in field theory on a fixed curved background the complexified background manifold does in general not possess a Riemannian section. In view of this a generalization of the Parisi-Wu method to the case of Lorentzian metric signature is highly desirable. Such a generalization has indeed been shown to exist [14]. The Langevin equation (3.3) generalizes to

$$\frac{\partial}{\partial s}\phi(x,s) = i\frac{\delta S[\phi]}{\delta\phi(x,s)} + \xi(x,s) \tag{5.1}$$

where S is the action in the pseudo-Riemannian space-time. The stochastic source ξ is again Gaussian, but its covariance matrix (assumed to be unity in (3.4)) will in general be indefinite, forcing ξ and hence ϕ to become complex, as was already remarked in the preceding Section. But even if ξ is real in (5.1), ϕ has to be complex because the "drift term" of the generalized Langevin equation is complex. For the same reason the process $\phi(x,s)$ will not possess an equilibrium limit in the ordinary sense. However it can be shown that in Minkowski space $\lim_{s\to\infty} \langle\phi(x_1,s) \ldots \phi(x_n,s)\rangle$ exists in the sense of tempered distributions of the Cartesian coordinates $x_1,\ldots,x_n$ [14]. Alternatively we may consider the Langevin equation

$$\frac{\partial\phi}{\partial s} = i\frac{\delta S}{\delta\phi} - \varepsilon\phi + \xi \tag{5.2}$$

where $\varepsilon > 0$ ensures the existence of the equilibrium limit and take the limit $\varepsilon \to 0$ in the expectation values after having $s \to \infty$. The equivalence of these two versions of stochastic quantization with standard quantization has been proved perturbatively for non-gauge fields in Minkowski space [14].

Tempered distributions possess a natural analog on pseudo-

Riemannian manifolds (being defined via spectral properties of the minimally coupled d'Alembertian [15]). Therefore at least perturbatively (5.1) implies a well-defined equilibrium limit. Remarkably, it even distinguishes two preferred quantum states and thus provides a definite answer to a well-known problem of quantum field theory in curved space-time. For a linear quantum field ϕ with

$$\frac{\delta S[\phi]}{\delta\phi} = V\phi \tag{5.3}$$

and V self-adjoint the modified stochastic quantization implies a unique Feynman propagator (independent of the special choice within a natural class of initial distributions for $\phi(x,0)$)

$$K(x,x') = -i \lim_{s\to\infty} \langle\phi(x,s)\phi^{\dagger}(x',s)\rangle \tag{5.4}$$

$$K = -i \int_0^{\infty} ds\; e^{iVs} = (V + i0)^{-1} \; . \tag{5.5}$$

The propagator K defines two states $|in\rangle$ and $|out\rangle$ via

$$K(x,x') = -i \frac{\langle out|T[\phi(x)\phi^{\dagger}(x')]|in\rangle}{\langle out|in\rangle} \tag{5.6}$$

(T denoting chronological ordering). In the absence of singularities these states turn out to be the vacua of Fock spaces giving rise to a description of particle creation that complies with physical intuition [16].

The Lorentzian space-time version of stochastic quantization is easily implemented in the general development of Section 4 by replacing S_E by $-iS$ everywhere. (4.29) has to be replaced by

$$G = \prod_x C^{10}(-1 - 2\lambda) \tag{5.7}$$

and $G_{AA'}$ is never Riemannian. For this reason, and of course because of the complex drift term in (5.1), the "analytically continued" ($S_E \to -iS$) Fokker-Planck equation (4.24) pertains to the fictitious real process ϕ_R introduced in Section 4. Although this need not exist, the formal equilibrium limit (4.25) and the path integral measures (4.36) and (4.37) are significant also in the pseudo-Riemannian case.

6. Perturbation Theory

In order to develop a perturbation theory for the evaluation of stochastic averages we first treat the stochastic quantization of the linearized gravitational field defined by

$$g_{\alpha\beta} = \eta_{\alpha\beta} + 2\kappa^{1/2}\, \psi_{\alpha\beta} \tag{6.1}$$

$$S[g] = S^{(0)}[\psi] + S^{(int)}[\psi] \tag{6.2}$$

$$S^{(0)}[\psi] = \frac{1}{2} \int d^4x\, \psi_{\alpha\beta}\, V^{\alpha\beta\gamma\delta}\, \psi_{\gamma\delta} \,. \tag{6.3}$$

The self-adjoint differential operator V is most compactly expressed in terms of a complete orthogonal set of spin projection operators [17] P^0, $P^{0'}$, P^1, P^2 obeying

$$P^0 + P^{0'} + P^1 + P^2 = id \tag{6.4}$$

$$P^i\, P^j = \delta^{ij}\, P^{ij} \tag{6.5}$$

where id denotes the identity on the space of symmetric tensor fields. One finds

$$V = - \Box(P^2 - 2P^{0'}) \,. \tag{6.6}$$

The linearized Langevin equation (4.34) in Minkowski space is well-defined in the variable ψ only if $\lambda = \mu = -1$ and reads

$$\dot{\psi}_{\alpha\beta} = i\, G_{\alpha\beta,\gamma\delta}\, V^{\gamma\delta\mu\nu}\, \psi_{\mu\nu} + \xi^{(0)}_{\alpha\beta} \tag{6.7}$$

$$G_{\alpha\beta,\gamma\delta} = \frac{1}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \eta_{\alpha\beta}\eta_{\gamma\delta}) \tag{6.8}$$

$$\langle\tilde{\xi}^{(0)}_{\alpha\beta}(x,s)\tilde{\xi}^{(0)}_{\gamma\delta}(x',s')\rangle = 2G_{\alpha\beta,\gamma\delta}\,\delta^{(4)}(x-x')\delta(s-s')\ . \tag{6.9}$$

The Feynman propagator is obtained from the so-called stochastic propagator

$$D_{\alpha\beta,\alpha'\beta'}(x,s;x',s') = \langle\psi_{\alpha\beta}(x,s)\psi_{\alpha'\beta'}(x',s')\rangle \tag{6.10}$$

in the limit $s \to \infty$:

$$\lim_{s\to\infty} D_{\alpha\beta,\alpha'\beta'}(k,s;k',s) = i(2\pi)^4\delta^{(4)}(k+k')K_{\alpha\beta,\alpha'\beta'}(k) \tag{6.11}$$

$$K_{\alpha\beta,\alpha'\beta'} = \frac{1}{k^2+i0}(\eta_{\alpha\alpha'}\eta_{\beta\beta'} + \eta_{\alpha\beta'}\eta_{\beta\alpha'} - \eta_{\alpha\beta}\eta_{\alpha'\beta'}) + L_{\alpha\beta,\alpha'\beta'} \tag{6.12}$$

$$L_{\alpha\beta,\alpha'\beta'} = \frac{1}{k^2+i0}(\eta_{\alpha\alpha'}k_\beta k_{\beta'} + \eta_{\alpha\beta'}k_\alpha k_{\beta'} + \eta_{\beta\beta'}k_\alpha k_{\alpha'}) - i\infty^2(P^1 + 2P^0)\ . \tag{6.13}$$

By L we have denoted that part of the propagator that does not contribute to gauge-invariant expectation values. The gauge-independent part of K coincides with the standard propagator in the harmonic gauge. In particular it is causal, a property that holds only in the parameter range $-2 < \lambda < -1/2$ if a more general field metric is allowed for in (6.7) and (6.9). We note that the denumerator of the gauge-invariant part of K is the same for all values of λ and is identical to (6.8). This shows that the field metric with $\lambda = -1$ is not only kinematically associated with the metric tensor field, but governs also the linearized Einstein dynamics.

We remark in passing that the propagator (6.12) may be represented as

$$K = \lim_{m^2\to 0}\ [V - (m^2 - i0)G]^{-1}\ . \tag{6.14}$$

The massive extension of linearized gravity implied by (6.14) contains only modes of physical mass m. The only other massive extension of linearized gravity with this property is the Fierz-Pauli extension describing a massive spin-2 particle, while exhibiting the van Dam-Veltman [18] mass discontinuity. All other massive extensions contain a tachyon.

In order to set up rules for perturbation theory we make now a definite choice of the reference field metric

$$G^{(0)AA'} = G^{AA'}[g^{(0)}] \tag{6.15}$$

$$g^{(0)}_{ab} = \delta_{ab} \tag{6.16}$$

and of the stochastic vielbein,

$$E_M{}^A \equiv E^{mn}{}_{\alpha\beta} = |g|^{-1/4}\, e^m{}_\alpha\, e^n{}_\beta \tag{6.17}$$

where $e^m{}_\alpha$ is a stochastic tetrad field obeying

$$g_{\alpha\beta} = e^m{}_\alpha\, e^n{}_\beta\, g^{(0)}_{mn} \; . \tag{6.18}$$

Note that $e^m{}_\alpha$ is complex. We fix it to be

$$e^a{}_\alpha(x,s) = (g_+)^{1/2}_{a\alpha} \tag{6.19}$$

$$g_+^{1/2} = \eta_+^{1/2}(1 + 2\kappa^{1/2}\eta\psi)^{1/2} = \eta_+^{1/2}(1 + \kappa^{1/2}\eta\psi - \tfrac{\kappa}{2}(\eta\psi)^2 + \ldots) \tag{6.20}$$

where $\eta_+^2 = \eta$ and has one positive imaginary eigenvalue.

The complete Langevin equation for gravity may be written in the form

$$\dot{\psi} - iW\psi = iI(\psi,\partial\psi) + J(\psi)\xi^{(0)} + \xi^{(0)} \; . \tag{6.21}$$

From this and the initial condition $\psi(0) = 0$ we obtain the integral equation

$$\psi(s) = \int_0^\infty d\sigma H(s-\sigma)[iI(\psi(\sigma),\partial\psi(\sigma)) + J(\psi(\sigma))\tilde{\xi}^{(0)}(\sigma) + \tilde{\xi}^{(0)}(\sigma)] \qquad (6.22)$$

where

$$H(s) = e^{iWs} . \qquad (6.23)$$

The integral equation may be solved iteratively and the solution be represented graphically as a series of tree diagrams. The term $J\tilde{\xi}^{(0)}$ in (6.22) gives rise to the appearance of a new type of vertex in these diagrams, which is not encountered in the stochastic quantization of non-gravitational fields. These "stochastic vertices" appear also in the diagrams corresponding to the perturbative expansion of the average of products of ψ fields.

At finite fictitious time s, these diagrams will suffer from the same types of divergences as standard perturbation theory. Thus a so-called "stochastic regularization" is needed, unless the stochastic theory is finite. In addition, all diagrams will diverge as $s \to \infty$, because the Feynman propagator is divergent. But these divergences cancel in gauge-invariant expectation values. For practical reasons it may be advantageous to use the technique of stochastic gauge-fixing [19], which yields a finite propagator.

The question of the equivalence of the stochastic perturbation theory with standard perturbation theory and of the finiteness of one or both of them must be left to future investigations.

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